Algebraicity and transcendence of power series: combinatorial and computational aspects

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Exercises

Exercises, Part II

1. Let $M_{n,k}$ be the number of $\{(1,1),(1,-1)\}$-walks in $\mathbb{N}^2$ of length n that start at $(0,0)$ and end at vertical altitude k. Let $M(x,y) = \sum_{n,k} M_{n,k} x^n y^k$.

   (a) Show that $(y - x(1+y^2)) \cdot M(x,y) = y - x \cdot M(x,0)$

   (b) Deduce that $M(x,y) = \dfrac{\sqrt{1-4x^2} + 2xy - 1}{2x(y - x(1+y^2))}$

2. Prove that Gessel's generating function for excursions

$$G(t;0,0) = \frac{{}_2F_1\left(\left.\begin{matrix}-\frac{1}{2} & -\frac{1}{6} \\ \frac{2}{3} & \end{matrix}\right| 16\,t\right) - 1}{2t} = 1 + 2t + 11t^2 + 85t^3 + 782t^4 + \cdots$$

   is algebraic, using the Beukers-Heckman and Schwarz's theorems.

3. Prove that the power series $\sum_n \binom{2n}{n}^r t^n$ is algebraic if and only if $r = 1$.

Solution, ex. 1(a)

$$S = \{(1,1),(1,-1)\}$$

$$M_{n+1,k} = M_{n,k-1} + M_{n,k+1}, \quad M_{0,0} = 1,\ M_{-1,k} = M_{n,-1} = 0 \text{ for } k,n \geq 0$$

Multiply by $y^{k+1}x^{n+1}$, and sum over $n,k \in \mathbb{N}$

$$\implies \qquad y \cdot \left(M(x,y) - \underbrace{\sum_{k\geq 0} M_{0,k} y^k}_{M(0,y)=1} \right) = y^2 x \cdot M(x,y) + x \cdot \left(M - \underbrace{\sum_{n\geq 0} M_{n,0} x^n}_{M(x,0)} \right)$$

$$\implies \qquad (y - x(1+y^2)) \cdot M(x,y) = y - x \cdot M(x,0) \qquad (\text{kernel equation})$$

Solution, ex. 1(b)

$$S = \{(1,1),(1,-1)\}$$

$$(y - x(1+y^2)) \cdot M(x,y) = y - x \cdot M(x,0) \qquad \text{(kernel equation)}$$

Kernel method: let $y_0 \in \mathbb{Q}[[x]]$ the power series root of $K = y - x(1+y^2)$

$$y_0 = \frac{1-\sqrt{1-4x^2}}{2x} = x + x^3 + 2x^5 + \cdots \in \mathbb{Q}[[x]]$$

Plugging $y = y_0$ in the (kernel equation) $\quad \Longrightarrow \quad E(x) = M(x,0) = \dfrac{y_0}{x}$

$$\Longrightarrow \qquad M(x,y) = \frac{y - y_0}{K(x,y)} = \frac{\sqrt{1-4x^2} + 2xy - 1}{2x(y - x(1+y^2))}$$