Dr Flajolet's elixir or Mellin transform and asymptotics

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Philippe Flajolet (abbreviated PF in the sequel) greatly developed the use of Mellin transform in the asymptotic evaluation of some combinatorial sums that appear in the average case analysis of algorithms. In fact, the Mellin transform runs throughout PF's work from the beginning [74] to the very end [13].

Mellin transform and fundamental strip

The Mellin transform is an integral transform, like the Laplace transform or the Fourier transform. It takes as input the original $f(x)$, which is a function of a real variable defined on the real positive half-line. It produces the image $f^*(s)$, which is a function of a complex variable, defined by

$$f^*(s) = \int_0^{+\infty} f(x) x^{s-1}\, dx.$$

It is not clear that the formula actually defines anything, but the kernel x^{s-1} leads us to a comparison with the powers of x. It is readily seen that the assumption $f(x) =_{x\to 0} O(x^\alpha)$ guarantees the convergence of the lower part of the integral, say from 0 to 1, for complex numbers s whose real part is greater than $-\alpha$, that is for the s which are on the right of $-\alpha$. We can make a similar assumption about the behavior at infinity. In this way, the image $f^*(s)$ is defined within the intersection of a right half-plane and a left half-plane. This is a strip, called the fundamental strip.

Certainly the most basic example of a Mellin transform is the gamma function

$$\Gamma(s) = \int_0^{+\infty} e^{-x} x^{s-1}\, dx.$$

It is the Mellin transform of the exponential e^{-x}. In that case the original is $O(1)$ at 0, so the left abscissa of the fundamental strip is 0. It decreases as x tends to infinity more rapidly than every power of x and the right abscissa is $+\infty$. Hence the fundamental strip in this case is the positive right half-plane. But the gamma function extends to the whole complex plane as a meromorphic function, and the extension has poles at

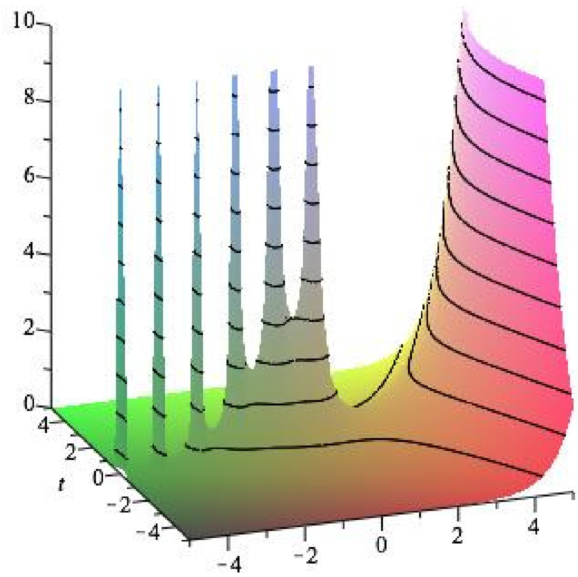

Figure 9.1. Absolute value of the gamma function.

all the nonpositive integers. The extension has poles at all the nonpositive integers, hence the peaks on the left-hand side of Figure 1.

Symbolic analysis

Imagine a flat landscape, something like a flat sand desert. This country is the complex plane. A track straight through it. This is the real axis. But you are thinking of a meromorphic function and suddenly this changes the landscape. Some hills or even prodigious mountains appear and at the top of these mountains some placards are fixed on poles. Each vertex is located above a pole of the meromorphic function and on the placard you can read the singular part of the function at the pole. For example, if you are thinking of the gamma function, you see an infinity of chimneys aligned on the negative part of the real axis, which disappear at the horizon in a haze of heat. On the placard at abscissa $-k$, you read $(-1)^k/(x+k)$. Your fantasy knows no limits and if you thought only for a moment about Stirling's formula a placard appears on the other side, at the end of the real positive axis, with its wording $\Gamma(x) \sim_{x\to+\infty} (x/e)^x \sqrt{2\pi/x}$. The more you direct your gaze toward a part of the landscape, the more details spring up.

There is no doubt that PF had such a mental image of analytic functions, certainly in a more subtle way, refined by more than thirty years of practice. Undisputable evidences are the introduction to the saddle-point method of [115, Chap. VIII], the discussion about coalescing saddle-points from [4], or the picture of [55] illustrating the application of the saddle-point method to a generalized exponential integral.

De Bruijn, 1948	$\sum_{k=0}^{+\infty} \ln \frac{1}{1-e^{-xr^k}}$	
De Bruijn, Knuth, Rice, 1972	$\sum_{k\geq 1} k^b d(k) e^{-k^2 x}$	$x = 1/\sqrt{n}$
Knuth, 1973	$\sum_{j\geq 1} 2^j (e^{-x/2^j} - 1 + x/2^j)$	$x = n$
Sedgewick, 1978	$\sum_{k\geq 1} \nabla\Delta F(k) e^{-k^2 x}$	$x = 1/\sqrt{j}$
Kemp, 1979	$\sum_{k\geq 1} k^a v_2(2k) e^{-16k^2 x}$	$x = 1/\sqrt{n}$

TABLE 1. The first uses of the Mellin transform approach related to the analysis of algorithms, with their authors and date, and the harmonics sums therein.

Similarly, the search for summatory formulas is reduced to a purely formal handling [98]; the asymptotic study of divide-and-conquer type sums is reduced to picking residues [58, 62]. Always he was defining rules that provide an automatic treatment of issues and reduce mathematical analysis to algebra and rewriting systems [100, 51]. Here we try to mimic this attitude and concentrate on the ideas, neglecting the mathematical assumptions.

Fundamental result

The fundamental result about Mellin transform is the following. There is a strong correspondence between the behavior of the original at 0 and the poles of the image in the left half-plane with respect to the fundamental strip. Similarly, the behavior at infinity of $f(x)$ is related to the poles of $f^*(s)$ in the half-plane to the right of the fundamental strip. The correspondence is explicit and given by the following formulas: a term $x^\xi \ln^k x$ in the expansion at 0 corresponds exactly to a singular term $(-1)^k k!/(s+\xi)^k$ for a pole $-\xi$ at the left of the fundamental strip. The formula is the same for the expansion at $+\infty$ and the poles on the right-hand side of the strip, but with an opposite sign: to $(-1)^k k!/(s+\xi)^k$ corresponds $-x^\xi \ln^k x$. Particularly, for simple poles (that is $k=0$), it is very simple: on the left-hand side the coefficients of the expansion at 0 are the residues of the poles, and similarly on the right-hand side. Clearly, the correspondence works very well in case of the gamma function,

$$(1) \qquad \Gamma(s) \underset{\mathrm{Re}(s)\leq 0}{=} \sum_{k=0}^{+\infty} \frac{(-1)^k}{k!} \frac{1}{s+k}, \qquad e^{-x} \underset{x\to 0}{=} \sum_{k=0}^{+\infty} \frac{(-1)^k}{k!} x^k.$$

Note that in (1) we do not claim that the series converges or even that the sum is the gamma function. This equation is only a formal writing, in line with symbolic analysis.

Harmonic sums

The second component of the story is the notion of harmonic sum. We start with a base function and its Mellin transform. To the base function, we associate a harmonic sum

$$F(x) = \sum_k \lambda_k f(\lambda_k x). \tag{2}$$

This is a linear combination of some dilations of the base function. We merge both ideas and we obtain a very simple result about the Mellin transform $F^*(s)$ of the harmonic sum. It is the product of the Mellin transform $f^*(s)$ of the base function and some generalized Dirichlet series $\Lambda(s)$, which depends only on the coefficients involved in the harmonic sum,

$$F^*(s) = f^*(s) \sum_k \frac{\lambda_k}{\mu_k^s}. \tag{3}$$

Zigzag method

We understand the power of the previous results by playing with the zigzag method, going back and forth between originals and images. We start with our favorite example $f_1(x) = e^{-x}$ and its Mellin transform, the gamma function. We consider an alternative function

$$f_2(x) = \frac{x}{e^x - 1} = \sum_{k=1}^{+\infty} x e^{-kx},$$

which is actually a harmonic sum. We compute its Mellin transform and we collect its poles

$$f_2^*(s) = \Gamma(s+1)\zeta(s+1) \underset{\mathrm{Re}(s)\leq 0}{=} \frac{1}{s} + \sum_{j=0}^{+\infty} \frac{(-1)^j \zeta(-j)}{j!} \frac{1}{s+j+1}.$$

They are 0 and the negative integers. But we know that

$$f_2(x) = 1 - \frac{1}{2}x + \sum_{n=1}^{+\infty} \frac{B_{2k}}{(2k)!} x^{2k}$$

is the generating function of the Bernoulli numbers. Moreover these numbers are zero for odd integers starting at 3, hence the writing

$$f_2^*(s) \underset{\mathrm{Re}(s)\leq 0}{=} \frac{1}{s} - \frac{1}{2}\frac{1}{s+1} + \sum_{k=1}^{+\infty} \frac{B_{2k}}{(2k)!} \frac{1}{s+2k}.$$

As a consequence the zeta function vanishes for the negative even integers. We now bring into the game a new harmonic sum

$$f_3(x) = \sum_{k=1} e^{-k^2 x^2}$$

and compute its Mellin transform

$$f_3^*(s) = \frac{1}{2}\Gamma\left(\frac{s}{2}\right)\zeta(s).$$

The vanishing of zeta at all the even negative integers removes almost all the poles of the gamma function. There remain only two poles, at 0 and 1. With

$$f_3^*(s) \underset{\mathrm{Re}(s)\le 1}{=} \frac{\sqrt{\pi}}{2}\frac{1}{s-1} - \frac{1}{2}\frac{1}{s}$$

(again, the writing is purely formal), we readily obtain the expansion of the function at 0

$$f_3(x) \underset{x\to 0}{=} \frac{\sqrt{\pi}}{2}\frac{1}{x} - \frac{1}{2} + O(x^{+\infty}).$$

Once we have understood the trick, it is not difficult to deal with other examples, like

$$f_4(x) = \sum_{k=1} d(k)e^{-k^2x^2}.$$

Here, d is the divisor function and the Mellin transform is $f_4^*(s) = \Gamma(s/2)\zeta(s)^2/2$. Again we obtain the expansion at 0 easily (the double pole at 1 makes a logarithm arise)

$$f_4(x) \underset{x\to 0}{=} \frac{\sqrt{\pi}}{2}\frac{\ln x}{x} + \frac{\sqrt{\pi}}{4}(3\gamma - \ln 4)\frac{1}{x} + \frac{1}{4} + O(x^{+\infty}).$$

Average-case analysis of algorithms and harmonic sums

The application of the previous ideas to the analysis of algorithms began in the seventies, with a 1972 article [27] of Nicolaas G. De Bruijn, Donald E. Knuth, and Stephen O. Rice about the height of rooted plane trees (Table 1). Knuth [150, p. 132-124] refers to the method of the gamma function and credits De Bruijn with first having this idea. De Bruijn had used it in a 1948 paper [26] about the asymptotic evaluation of the binary partitions number. Next we encounter the study of radix-exchange sorting by Donald Knuth [150], of the odd-even merging by Robert Sedgewick [179], and of the register allocation for binary trees by Rainer Kemp [144]. In every case, a harmonic sum comes out (in the expressions of Table 1, d is the divisor function, ∇ and Δ are the backward and forward difference operators, v_2 is the dyadic valuation function). The first sum is a generating function, while the others ones are combinatorial sums, but they are all amenable to the same treatment.

According to [71], PF learned the Mellin transform from Rainer Kemp around 1979. In a 1977 work about register allocation [86, 87], PF and his coauthors follow an elementary way *à la* Delange, but in a 1978/1979 talk [42] at the *Séminaire Delange-Pisot-Poitou* PF gives an explanation about the "Mellin-Fourier transform", with only words but in a totally clear way. PF has systematized the idea starting in the eighties and completely defined the method in the early nineties. This led him to write first [88], a very illuminating presentation of the Mellin transform in the context of the analysis of algorithms, and next [60], a more comprehensive version. He returned to the topic in [108], in which the reader can find not only examples but even exercises.

PF, Odlyzko, 1981 [73, 74]	average height of simple trees	$\sum_{n\geq 1} \sigma_r(n) e^{-nu}$
PF, Puech, 1983 [82, 84]	retrieval of multi-dimensional data	$\sum_{\ell}^{k-1} \varepsilon_\ell \sum_{j\geq 0} 2^{j(k-s)} \times (1 - e^{-x\alpha_{i,j}} - \beta_{j,\ell} e^{-x\alpha_{j,\ell}})$
Fayolle, PF, Hofri, 1986 [36, 37]	multi-access broadcast channel	$\sum_{\sigma\in H} e^{-\mu}\ (\ (1+K\mu) \times (e^{-rx} - 1 - rx) + Krx(e^{-rx} - 1)\ )$
PF, Richmond, 1992 [92, 93]	generalized digital trees	$\sum_{k=0}^{+\infty} 2^k \frac{(1+2^k t)^{b-1}}{Q(2^k t)^b}, \quad Q(u) = \prod_{j=0}^{+\infty} \left(1 + \frac{u}{2^j}\right)$
Mahmoud, PF, Jacquet, Régnier, 2000 [160, 161]	bucket selection and sorting	$z \sum_{k=0}^{+\infty} (1 - e^{-z/B^k})$ and relatives
PF, Fusy, Gandouet, Meunier, 2007 [54]	cardinality estimation	$\sum_{k=1}^{+\infty} (e^{-x/2^k} - e^{-2x/2^k}) e^{-xu/2^k}$
Broutin, PF, 2010 [13]	height in non-plane binary trees	$\sum_{h\geq 1} h^r \frac{e^{-ht}}{(1-e^{-ht})^2}$

TABLE 2. Some of PF's contributions with their authors, date, and references, next the topics under consideration, and finally the harmonic sums that appear in these papers.

Table 2 provides a small sample of PF's contributions. The third example [36, 37] is impressive: the sum is over the affine transforms $\sigma(z) = \mu + rz$ in a semi-group $H = \{\sigma_1, \sigma_2\}^*$ generated by two affine transforms $\sigma_1(z) = \lambda + pz$, $\sigma_2(z) = \lambda + qz$ with $p + q = 1$. The scope of application is quite broad and we refer to [60, p. 5] for a list of relevant fields.

Exponentials in harmonic sums

It is remarkable how often the exponential function appears in the harmonic sums. The reason is the following. In the process of analyzing an algorithm, we are faced with combinatorial sums, which generally are *not* harmonic sums. But there may be a suitable approximation which is a harmonic sum. There are essentially two rules. The first is the approximation of a large power by an exponential,

$$(1-a)^n \underset{n\to+\infty}{=} e^{-na}\left(1 + O(na^2)\right) \quad \text{with} \quad na \underset{n\to+\infty}{=} n^{\varepsilon}, \qquad 0 < \varepsilon < 1/2$$

Here is an example related to the analysis of the radix-exchange sort algorithm [150, p. 131]

$$\sum_{k=0}^{+\infty}\left(1-\left(1-\frac{1}{2^k}\right)^n\right) \underset{n\to+\infty}{=} F(n)+O\left(\frac{1}{\sqrt{n}}\right), \qquad F(x)=\sum_{k=0}^{+\infty}(1-e^{-x/2^k}).$$

Others examples can be found in [70, p.188], [84, p. 230], [76, p. 388], [13, p.131], or [125, p. 71].

The second rule is the approximation of the binomial distribution by a Gaussian distribution

$$\frac{\binom{2n}{n-k}}{\binom{2n}{n}} \underset{n\to+\infty}{=} e^{-w^2}\left(1+O\left(\frac{1}{n}\right)\right) \quad \text{with} \quad k=w\sqrt{n}, \quad k \underset{n\to+\infty}{=} o(n^{3/4}).$$

It appears in the study of the expected height of plane (Catalan) trees [27, p. 20]

$$\sum_{k=1}^{n} d(k)\frac{\binom{2n}{n-k}}{\binom{2n}{n}} \underset{n\to+\infty}{=} G\left(\frac{1}{\sqrt{n}}\right)+o(1) \qquad G(x)=\sum_{k=1}^{+\infty} d(k)e^{-k^2x^2}.$$

or in the study of odd-even merging [179], [85, p. 153] resumed in [46, p. 286] and [194, p. 478].

Technical point

For the benefit of the reader who wants to apply the Mellin transform to his/her own problem, we leave for a while the formal style and enter into analysis. Frequently, the original $f(x)$ is not only defined on the real positive axis, but on a sector $|\arg x| < \omega$ of the complex plane. This constraints the image strongly. In this case, it satisfies

$$|f^*(s)| \underset{s\to\pm i\infty}{=} e^{-\omega|\,\mathrm{Im}(s)|}. \tag{4}$$

Such an inequality (not necessarily of exponential type) is the key point allowing to use the inverse formula

$$f(x)=\frac{1}{2\pi i}\int_{(c)} f^*(s)x^{-s}\,ds \tag{5}$$

(see the proof of Theorem 4 in [60]). In this formula, (c) is the vertical line at abscissa c taken in the fundamental strip. It is noteworthy that x can be a complex variable and not only a positive real variable, contrary to what we started with ([31] or [140] for a brief account). This is of practical importance since the application of the Mellin transform to a generating function is frequently the first step of an analytical process. It provides the local behaviour of the function at a distinguished point, and can be followed by the use of the Cauchy formula, for example with the saddle-point method [26, 35], or followed by singularity analysis [48, p. 397], [76, p. 238], [13, p. 24].

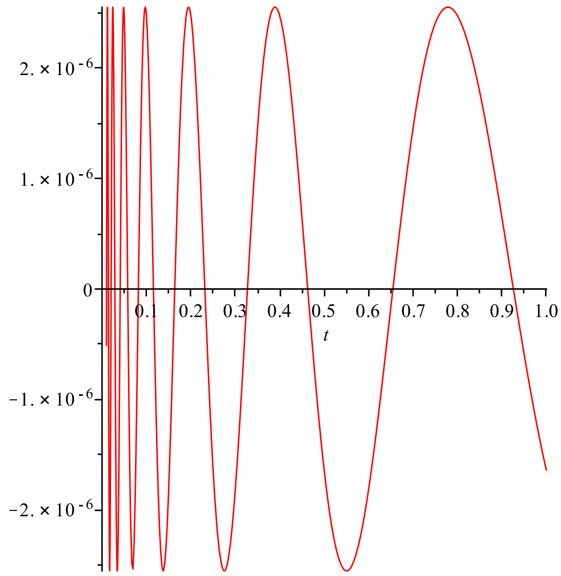

Figure 9.2. The Mellin transform captures oscillating behavior of a very small amplitude.

Oscillations

The study of the number of registers necessary to evaluate an expression represented by a binary tree is perhaps the most classical example which provides a harmonic sum. PF and Helmut Prodinger dealt with a variant of the problem in a 1986 paper [80]. They begin by revisiting the case of a binary tree. They need to know the local behaviour of the function

$$E(z) = \frac{1-u^2}{u} \sum_{k \geq 1} v_2(k) u^k$$

in the neighborhood of $1/4$. For this, they perform some changes of variables

$$z = \frac{u}{(1+u)^2}, \qquad u = \frac{1-r}{1+r}, \qquad r = \sqrt{1-4z}, \qquad u = e^{-t}$$

($z = 1/4$ corresponds to $u = 1$ and $t = 0$) and a harmonic sum $V(t)$ comes out. They then compute its Mellin transform,

$$V(t) = \sum_{k \geq 1} v_2(k) e^{-kt}, \qquad V^*(s) = \frac{\zeta(s)}{2^s - 1} \times \Gamma(s).$$

They collect the coefficients of the asymptotic expansion of $V(t)$ at 0 by a process which seems now to be routine. Because of the denominator $2^s - 1$, there is a line of poles $\chi_k = 2k\pi i/\ln 2$ on the imaginary axis. These poles are regularly spaced and contribute a trigonometric series $C + P(\log_2 t)$ with respect to $\log_2 t$,

$$C = \frac{1}{4} + \frac{\gamma - \ln(2\pi)}{2\ln 2}, \qquad P(\log_2 t) = \sum_{k\neq 0} \frac{\Gamma(\chi_k)\zeta(\chi_k)}{\ln 2} e^{-2k\pi i \log_2(t)}.$$

In this way they obtain the expansion they are looking for

$$V(t) \underset{t\to 0}{=} \frac{1}{t} + \frac{1}{2\ln 2}\ln(t) + C + P(\log_2 t) + O(t),$$

$$E(z) \underset{z\to 1/4}{=} 2 + 2r\log_2 r + (2C+1)r + 4rP(\log_2 r) + O(r^2), \qquad r = \sqrt{1-4z}.$$

The key point is the occurrence of a function which is 1-periodic with respect to $\log_2 t$. The gamma function decreases very rapidly on the imaginary axis and this periodic function therefore has a very small amplitude. (Figure 2 displays the graph of $P(\log_2 t)$). More precisely, the magnitudes are $C \simeq -0.66$, $|\Gamma(\chi_1)| \simeq 5.5\ 10^{-7}$, $|\Gamma(\chi_2)| \simeq 2.5\ 10^{-13}$, $|\Gamma(\chi_3)| \simeq 1.4\ 10^{-19}\ldots$ One could say that this function is so small as to be of no importance. But it emphasizes the difficulty to obtain such an asymptotic expansion by elementary arguments. This point in particular delighted PF [163, Comment 5, p. 226], [**?**, p. 8], [70, p. 206].

Related topics

In this small introduction to the Mellin transform, we have neglected many issues. Among them, the Rice formula permits to study high order differences of a sequence. Also, the Poisson-Mellin-Newton cycle relates the Poisson generating function of a sequence, its Mellin transform, and the Rice integral. A good reference about these topics is PF and Robert Sedgewick's 1995 article [107]. Another topic omitted here is the Mellin-Perron formula. It is presented in the next chapter of this volume. Mellin transform is a pivotal ingredient in *depoissonization* [140, 54]. It is also related with the Lindelöf representation, which appears in [50, Formula (11)], [51, p. 565], itself connected with the magic duality. PF often spoke about this topic, but he has written very little about it. It is alluded to in [55] and developed briefly in a note of *Analytic combinatorics* [115, p. 238]. The right reference is [154, Chap. V].