Hypertranscendence of solutions of iterated functional equations and Galois theory

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Definition 1.1

1. *A formal power series $f(z) \in \mathbb{C}[[z]]$ is **hypertranscendental or differentially transcendental** over $\mathbb{C}(z)$ if there is no non-zero polynomial $P(z, X_0, \ldots, X_n)$ with coefficients in $\mathbb{C}$ such that:*

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2. *A formal power series $f(z) \in \mathbb{C}[[z]]$ is D-**finite** over $\mathbb{C}(z)$ if it satisfies a linear differential equation with coefficients in $\mathbb{C}(z)$:*

$$a_0(z)f(z) + \cdots + a_n(z)f^{(n)}(z) = 0,$$

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- Nature of the fixed point of $R \rightsquigarrow$ convergence of $(R^n(z))_n$ near α.

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In the sequel, we assume that $\alpha = 0$.

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Except in some cases, solutions of Equations (S), (B) and (A) are hypertranscendental over $\mathbb{C}(z)$.

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Theorem 1.3 (B. Adamczewski, T. Dreyfus, C. Hardouin, 2019)

If $R(z) = z + h$ or qz or z^d, the solutions of

$$\sum_{i=0}^{n} a_i(z) f\left(R^i(z)\right) = 0$$

are either hypertranscendental or in the "base field".

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- Coefficients $a_i(g(z)) \in \mathbb{C}(z)$ if R is a Möbius transformation but not in general.

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- Endomorphism

$$\begin{aligned} \Phi_R : \mathbb{F} &\to \mathbb{F} \\ f(z) &\mapsto f(R(z)). \end{aligned}$$

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Then either $f \in \mathbb{K}$ or f is hypertranscendental over $\mathbb{K}$.

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Théorème 2.1 (For example, M. Aschenbrenner and W. Bergweiler)

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- **New proof** of Ritt's result with L. Di Vizio.
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  - Less technical arguments.
  - Could be adapted to the problem of knight walks.

Thank you for your attention!