E- and M-functions

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Based on joint works with Colin Faverjon

Reminder

A complex number ξ is transcendental if

$$P(\xi) \neq 0\,.$$

for every nonzero $P \in \mathbb{Z}[X]$.

Complex numbers $\xi_1, \ldots, \xi_r$ are algebraically independent (over the field of algebraic number $\overline{\mathbb{Q}}$) if

$$P(\xi_1, \ldots, \xi_r) \neq 0$$

for every nonzero $P \in \mathbb{Z}[X_1, \ldots, X_r]$.

The power series $f_1(z), \ldots, f_r(z) \in \overline{\mathbb{Q}}[[z]]$ are algebraically independent over $\overline{\mathbb{Q}}(z)$ if

$$P(z, f_1(z), \ldots, f_r(z)) \neq 0$$

for every nonzero $P \in \mathbb{Z}[z, X_1, \ldots, X_r]$.

Preamble

Definition (Siegel 1929)

$f(z) = \sum_{n \geq 0} \frac{a_n}{n!} z^n \in \overline{\mathbb{Q}}[[z]]$ is an *E-function* if there exist $p_0(z), \ldots, p_m(z) \in \overline{\mathbb{Q}}[z]$, not all zero, such that

$$p_0 f(z) + p_1 f'(z) + \cdots + p_m f^{(m)}(z) = 0$$

+ height growth condition on $(a_n)_{n \geq 0}$.

Examples. Important functions occurring in geometry and physics: $e^z, \sin z, \cos z$,

$$J_k(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+k)!} \left(\frac{z}{2}\right)^{2n+k},$$

and (some) more general hypergeometric functions.

Definition (inspired by Mahler 1929)

$f(z) = \sum_{n \geq 0} a_n z^n \in \overline{\mathbb{Q}}[[z]]$ is an *M_q-function* if there exist $p_0(z), \ldots, p_m(z) \in \overline{\mathbb{Q}}[z]$, not all zero, such that

$$p_0 f(z) + p_1 f(z^q) + \cdots + p_m f(z^{q^m}) = 0.$$

The parameter $q \geq 2$ is an integer.

Examples. Functions related to numeration and computer science:

$$\sum_{n=0}^{\infty} z^{q^n}, \prod_{n=0}^{\infty} \frac{1}{1 - z^{q^n}}, \sum_{n=0}^{\infty} s_q(n) z^n,$$

and the generating series of the sequences produced by finite automata.

Mahler (Fifty Years as a Mathematicians II, 1971):

While I was ill at home in 1927, I succeeded in proving the transcendency of

$$z + z^2 + z^4 + z^8 + \cdots$$

for algebraic z *satisfying* $0 < |z| < 1$. *The method was new and depended on the functional equation*

$$f(z) = z + f(z^2)$$

for the series.
[...] E. Landau did not show much interest in these results. So I next turned to a closer study of the approximation properties of the transcendental numbers e *and* π.

Linear systems and singularities

One studies linear systems of the form:

$$\boldsymbol{Y}'(z) = A(z)\boldsymbol{Y}(z)$$

with $A(z) \in \mathcal{M}_n(\overline{\mathbb{Q}}(z))$.

One studies linear systems of the form:

$$\boldsymbol{Y}(z^q) = A(z)\boldsymbol{Y}(z)$$

with$A(z) \in \mathrm{GL}_n(\overline{\mathbb{Q}}(z))$.

A point α is regular if the matrix $A(z)$ is well-defined at α.

A point α is regular if, for all $n \geq 0$, the matrix $A(z)$ is well-defined and invertible at α^{q^n}.

Siegel–Shidlovskii and Nesterenko–Nishioka

Theorem

Let $f_1(z), \ldots, f_m(z) \in \overline{\mathbb{Q}}[[z]]$ be $\star$-functions that form the entries of a solution vector of a linear $\star$-system. Let $\alpha \in \overline{\mathbb{Q}} \setminus \{0\}$ be a regular point. Then

$$\mathrm{degtr}_{\overline{\mathbb{Q}}}(f_1(\alpha), \ldots, f_m(\alpha)) = \mathrm{degtr}_{\overline{\mathbb{Q}}(z)}(f_1(z), \ldots, f_m(z)) .$$

- First proof by Shidlovskii (1956) using Siegel's method.
- Second proof by André (2000) using the theory of G-functions.

- First proof by Ku. Nishioka (1990) using Nesterenko's approach.
- Second proof by A. & Faverjon (2020) using the pioneering ideas of Mahler.
- Proofs also work in p-adic settings and over $\mathbb{F}_q(t)$ (Fernandes, 2018).

Motivation for outsiders

Theorem HLW

Let $\alpha_1, \ldots, \alpha_r$ be $\mathbb{Q}$-linear independent algebraic numbers. Then the numbers $e^{\alpha_1}, \ldots, e^{\alpha_r}$ are algebraically independent over $\overline{\mathbb{Q}}$.

The case $r = 1$, combines with Euler's identity $e^{i\pi} + 1 = 0$, implies that π is transcendental, which proves the impossibility of squaring the circle.

Proof. Let us consider the E-functions $e^{\alpha_1 z}, \ldots, e^{\alpha_r z}$. They form a vector solution to the linear E-system:

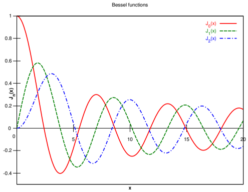
$$\boldsymbol{Y}'(z) = \begin{pmatrix} \alpha_1 & & \\ & \ddots & \\ & & \alpha_r \end{pmatrix} \boldsymbol{Y}(z)\,.$$

Since 1 is a regular point, the Siegel-Shidlovskii theorem implies that

$$\mathrm{degtr}_{\overline{\mathbb{Q}}}(e^{\alpha_1}, \ldots, e^{\alpha_r}) = \mathrm{degtr}_{\overline{\mathbb{Q}}(z)}(e^{\alpha_1 z}, \ldots, e^{\alpha_r z})\,.$$

The assumption on the α_i's implies that the right-hand side equal r, which ends the proof. $\square$

E2: Bourget's Hypothesis



Bourget's hypothesis (1866): the Bessel functions $J_0(z), J_1(z), J_2(z), \ldots$ have no common zeros other than the origin.

Theorem (Siegel, 1929)

Let $\alpha \in \overline{\mathbb{Q}}^*$ and $n \in \mathbb{N}$. Then $J_n(\alpha)$ and $J_n'(\alpha)$ are algebraically independent.

By the classical theory of Bessel functions, the putative common zeros were known to be algebraic. Hence, Siegel's theorem implies Bourget's hypothesis.

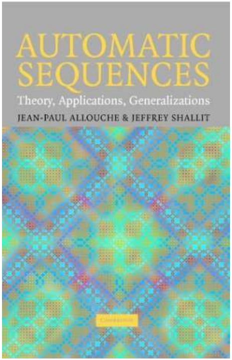
Proof. The E-functions $J_n(z)$ and $J_n'(z)$ form a vector solution to the E-system:

$$\boldsymbol{Y}'(z) = \begin{pmatrix} 0 & 1 \\ \frac{z^2-n^2}{z^2} & \frac{1}{z} \end{pmatrix} \boldsymbol{Y}(z)\,.$$

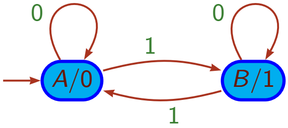
Since α is a regular point, the Siegel-Shidlovskii theorem implies that

$$\operatorname{degtr}_{\overline{\mathbb{Q}}}(J_n(\alpha), J_n'(\alpha)) = \operatorname{degtr}_{\overline{\mathbb{Q}}(z)}(J_n(z), J_n'(z)) = 2\,. \quad \square$$

Automatic sequences



A sequence $a := (a_n)$ with values in a finite set is q-automatic if there exists a finite automaton that takes as input the base-q expansion of n and produces as output the symbol a_n.



This 2-automaton computes the Thue–Morse sequence $\mathrm{t} := (t_n)$ defined by $t_n = 1$ if the sum of binary digits of n is odd, and $t_n = 0$ otherwise.

Link with M-functions

In 1968, Cobham noticed the following fundamental connection between automatic numbers and M-functions: If $(a_n) \in \overline{\mathbb{Q}}^{\mathbb{N}}$ is q-automatic, then the generating series

$$f(z) := \sum_{n=0}^{\infty} a_n z^n \in \overline{\mathbb{Q}}[[z]]$$

is an M_q-function.

Example. The Thue-Morse sequence (defined by $t_n = 1$ if the sum of binary digits of n is odd, and $t_n = 0$ otherwise) satisfies $t_{2n} = t_n$ and $t_{2n+1} = 1 - t_n$. It follows that

$$\begin{aligned} f_{\mathrm{t}}(z) &= \sum_{n\geq 0} t_{2n} z^{2n} + \sum_{n \geq 0} t_{2n+1} z^{2n+1} \\ &= f_{\mathrm{t}}(z^2) - z f_{\mathrm{t}}(z^2) + \frac{z}{1-z^2} \,. \end{aligned}$$

Hence

$$z - (1-z^2) f_{\mathrm{t}}(z) + (1-z)(1-z^2) f_{\mathrm{t}}(z^2) = 0 \,.$$

Automatic real numbers

A real number ξ is said to be automatic in base b if its base-b expansion can be computed by a finite automaton (i.e., q-automatic for some q).

Example. The binary Thue–Morse number

$$\langle\tau\rangle_2 = 0.011\,010\,011\,001\,011\,010\,010\,110\,011\,010\,011\,001\,011\cdots,$$

is automatic in base 2.

Proposition

Automatic real numbers are values at rational points of M-functions with rational coefficients.

Proof. If $x = 0.a_1a_2\cdots$ is q-automatic in base b, then the generating series

$$f(z) := \sum_{n=0}^{\infty} a_n z^n \in \mathbb{Q}[[z]]$$

is an M_q-function and $x = f(1/b)$. □

M1: base-b expansions of mathematical constants

While $\sqrt{2}$ and π have very simple geometric descriptions, their decimal expansions

$$\langle\sqrt{2}\rangle_{10} = 1.414\,213\,562\,373\,095\,048\,801\,688\,724\,209\,698\,078\,569\cdots$$

and

$$\langle\pi\rangle_{10} = 3.141\,592\,653\,589\,793\,238\,462\,643\,383\,279\,502\,884\,197\cdots$$

remain totally mysterious.

Different directions have been envisaged so far to formalize and express the expected *complexity* of such expansions:

- probability theory (Borel 1909),
- symbolic dynamics (Morse and Hedlund 1938),
- computer science (Turing 1936, Hartmanis and Stearns 1965, Cobham 1968).

Theorem AF1 (2017)

Let $f(z)$ be an M-function with rational coefficients and $\alpha \in \mathbb{Q} \setminus \{0\}$ be such that $f(\alpha)$ is well-defined. Then either $f(\alpha) \in \mathbb{Q}$ or $f(\alpha)$ is transcendental.

Corollary

The base-b expansion of an algebraic irrational real number cannot be generated by a finite automaton.

Proof. Let us assume that $x \in (\mathbb{R} \cap \overline{\mathbb{Q}}) \setminus \mathbb{Q}$ is automatic in base b. Then

$$\langle x \rangle_b = 0.a_1 a_2 \cdots$$

$x = f(1/b)$ where $f(z) = \sum a_n z^n$ is an M-function with rational coefficients. Theorem AF1 implies that x is either rational or transcendental, a contradiction. $\square$

M2: base conversion

While the binary expansion of the binary Thue–Morse number has a simple description

$$\langle \tau \rangle_2 = 0.011\,010\,011\,001\,011\,010\,010\,110\,011\,010\,011\,001\,011\cdots,$$

its decimal expansion

$$\langle \tau \rangle_{10} = 0.412\,454\,033\,640\,107\,597\,783\,361\,368\,258\,455\,283\,089\cdots$$

remains totally mysterious.

This problem was first tackled by Furstenberg in 1969 using the language of dynamical systems.

M2: base conversion

Let T_b denote the map defined on $\mathbb{R}/\mathbb{Z}$ by $x \mapsto bx$.

Let $\mathcal{O}_b(x)$ denote the forward orbit of x under T_b, that is

$$\mathcal{O}_b(x) := \{x, T_b(x), T_b^2(x), \ldots\}.$$

Dichtionary.

- Finite orbit $\iff$ eventually periodic base-b expansion $\iff$ $x \in \mathbb{Q}$,
- Dense orbit $\iff$ every block of digits occurs infinitely often in $\langle x \rangle_b$,
- Uniformly distributed orbit $\iff$ x is normal in base b.

Conjecture (Furstenberg, 1969)

Let p and q be two multiplicatively independent (i.e., $\log p / \log q \notin \mathbb{Q}$) natural numbers, and let $x \in [0, 1)$ be an irrational real number. Then

$$\dim_H \overline{\mathcal{O}_p(x)} + \dim_H \overline{\mathcal{O}_q(x)} \geq 1.$$

The set of putative exceptions has Hausdorff dimension zero (Shmerkin and Wu, 2019).

Theorem AF2 (2020)

Let α and β be two multiplicative independent algebraic numbers, and let f and g be two M-functions such that $f(\alpha)$ and $g(\beta)$ are well-defined and transcendental. Then $f(\alpha)$ and $g(\beta)$ are algebraically independent over $\overline{\mathbb{Q}}$.

Corollary

An irrational real number cannot be generated by a finite automaton in two multiplicatively independent bases.

Proof. Let us assume that $x \in \mathbb{R} \setminus \mathbb{Q}$ is automatic in two independent bases b_1 and b_1. Then $x = f(1/b_1) = g(1/b_2)$ where f and g are M-functions with rational coefficients. Since $x \notin \mathbb{Q}$, Theorem AF1 implies that x is transcendental, but then Theorem AF2 implies that $f(1/b_1)$ and $g(1/b_2)$ are algebraically independent, a contradiction. $\square$

M3: Pascal's paper *De Numeris Multiplicibus...* 1654

DE NUMERIS MULTIPLICIBUS
EX SOLA CHARACTERUM NUMERICORUM
ADDITIONE AGNOSCENDIS

Monitum.

Nihil tritius est apud arithmeticos quàm numeros numeri 9 multiplices constare characteribus quorum aggregatum est quoque ipsius 9 multiplex. Si enim ipsius v. g. dupli, 18, characteres numericos, 1 + 8, jungas, aggregatum erit 9. Ita ut ex solâ additione characterum numericorum numeri cujuslibet liceat agnoscere utrum sit ipsius 9 multiplex ; v. g. si numeri 1719 characteres numericos jungas 1 + 7 + 1 + 9, aggregatum 18 est ipsius 9 multiplex ; unde certo colligitur & ipsum 1719 ejusdem 9 esse multiplicem. Vulgata sanè illa observatio est ; verùm ejus demonstratio à nemine quod sciam data est, nec ipsa notio ulteriùs provecta. In hoc autem Tractatulo non solùm istius, sed et variarum aliarum observationum generalissimam demonstrationem dedi, ac methodum universalem agnoscendi, ex solâ additione characterum numericorum propositi cujusvis numeri, utrum ille sit alterius propositi numeri multiplex. Et non solum in progressione denariâ, quâ numeratio nostra procedit, (denaria enim ex instituto hominum, non

Nothing in arithmetic is better known than the proposition according to which any multiple of 9 is composed of digits whose sum is itself a multiple of 9. [...] In this little treatise [...], I shall also set out a general method which allows one to discover, by simple inspection of its digits, whether a number is divisible by an arbitrary other number; this method applies not only to our decimal system of numeration (which system rests on a convention, an unhappy one besides, and not on a natural necessity, as the vulgar think), but it also applies without fails to every system of numeration having for base whatever number one wishes, as may be discovered in the following pages.

M3: modern formulation

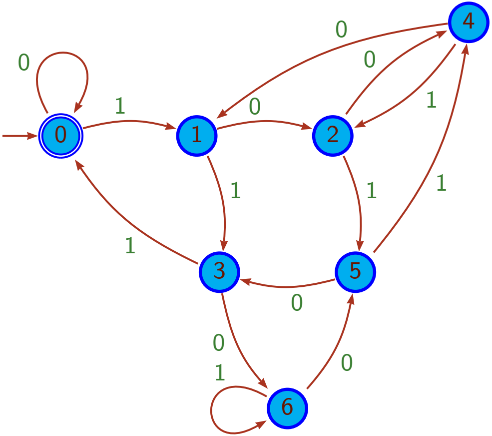
Are the binary numbers

$100\,011\,110\,011\,101$

and

$10\,011\,001$

divisible by 7?



Proposition (divisibility rules are automatic)

A periodic set $\mathcal{E} \subset \mathbb{N}$ can be recognized by a finite automaton in all bases.

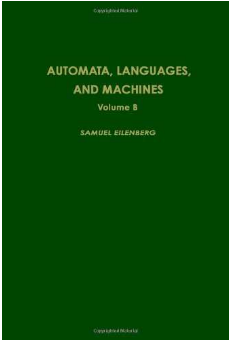
1	2
10	4
100	8
1000	16
10000	32
100000	64
1000000	128
10000000	256
$\vdots$	$\vdots$
100 000 000 000 000 000 000	2 097 152
1 000 000 000 000 000 000 000	4 194 304
$\vdots$	$\vdots$

M3: Cobham's theorem

Theorem (Cobham, 1969)

If a set $\mathcal{E} \subset \mathbb{N}$ can be recognized by a finite automaton in two multiplicatively independent bases, then it is periodic.

Consequence: the powers of 2 are not automatic in base 3 or 10.



In his book *Automata, Languages, and Machines*, S. Eilenberg states Cobham's theorem without proof and makes the following comment:

The proof is correct, long and hard. It is a challenge to find a more reasonable proof of this fine theorem.

M3: New proof and generalization

Theorem AF3 (2020)

Let f be an M_p-function and g be an M_q-function, p and q independent, and let α and β be two algebraic numbers such that $f(\alpha)$ and $g(\beta)$ are well-defined and transcendental. Then $f(\alpha)$ and $g(\beta)$ are algebraically independent.

Corollary

Let f be an M_p-function and g be an M_q-function, p and q independent. If none of them is a rational function, then $f(z)$ and $g(z)$ are algebraically independent over $\overline{\mathbb{Q}}(z)$.

Proof. Since f and g are not rational, they are transcendental. Hence there exists α such that $f(\alpha)$ and $g(\beta)$ are well-defined and transcendental. By Theorem AF3, $f(\alpha)$ and $g(\beta)$ are algebraically independent. $\square$

Remark. Cobham's theorem is implied by $f(z) \neq g(z)$ under the assumption of the corollary. The corollary has been independently proved in collaboration with Dreyfus, Hardouin and Wibmer using Galois theoretic arguments.

Transcendental number theory??

The (F)-questions

Let $\Xi \subset \mathbb{C}$.

(F.1) Given $\xi \in \Xi$, is ξ irrational? Transcendental?

$$\forall (p,q) \in \mathbb{Z}^2 \setminus \{(0,0)\}, \quad q\xi - p \neq 0,$$

$$\forall P(z) \in \mathbb{Q}[z] \setminus \{0\}, \quad P(\xi) \neq 0.$$

(F.2) Given $\xi_1, \ldots, \xi_r \in \Xi$, are these numbers linearly independent over $\mathbb{K}$?

$$\forall a_1, \ldots, a_r \in \mathbb{K}, \quad a_1\xi_1 + \cdots a_r\xi_r \neq 0.$$

(where $\mathbb{K}$ is usually $\mathbb{Q}$, a number field, or $\overline{\mathbb{Q}}$).

(F.3) Given $\xi_1, \ldots, \xi_r \in \Xi$, are these numbers algebraically independent over $\overline{\mathbb{Q}}$?

$$\forall P \in \overline{\mathbb{Q}}[X_1, \ldots, X_r] \setminus \{0\}, \quad P(\xi_1, \ldots, \xi_r) \neq 0.$$

(Q.1–3) If the answer to one of the questions F.(1–3) is positive, we want to quantify the proof, that is to obtain *measures* of irrationality, transcendence, linear independence, or algebraic independence.

$$\left|\xi - \frac{p}{q}\right| > \psi(q)\,, \quad \forall p/q \in \mathbb{Q}\,.$$

For instance, ξ is not a Liouville number if there exists μ such that

$$\left|\xi - \frac{p}{q}\right| > \frac{c}{q^{\mu}}\,, \quad \forall p/q \in \mathbb{Q}\,.$$

$$\forall P \in \mathbb{Z}[X_1, \dots, X_r] \setminus \{0\}\,, \quad |P(\xi_1, \dots, \xi_r)| > \psi(H(P), \deg P)\,.$$

Reminder

The naive height $H(P)$ of a polynomial $P \in \mathbb{Z}[X_1, \dots, X_r]$ is equal to the maximum of the modulus of its coefficients. The height and the degree of P measure its *complexity*.

The (A)-questions

(A.1–3) If the answer to one of the questions (F.1–3) is negative, can we *explicitly find* the corresponding relations?

For instance, we would like to find a basis of the $\mathbb{K}$-vector space formed by the linear relations over $\mathbb{K}$ between $\xi_1, \ldots, \xi_r$, or a set of generators of the ideal of $\overline{\mathbb{Q}}[X_1, \ldots, X_r]$ formed by the algebraic relations over $\overline{\mathbb{Q}}$ between $\xi_1, \ldots, \xi_r$?

(M) Can we determine the *raison d'être* of the algebraic relations, if any, between the elements of Ξ?

Examples

(1) The Kontsevich–Zagier conjecture predicts that all algebraic relations among periods should follow from the fundamental rules of integral calculus – additivity, change of variables and the Stokes formula.

(2) The Rohrlich–Lang conjecture predicts that all algebraic relations among the elements of

$$\Xi := \left\{ \frac{1}{2\pi}\Gamma(r) : r \in \mathbb{Q} \right\}$$

would follow from specializations of standard functional relations associated with the Euler Γ function.

Sine and cosine

In their trigonometry course, all high school students learn the Pythagorean identity

$$\cos^2\alpha + \sin^2\alpha = 1\,,$$

as well as the angle addition formulae

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta \quad \text{and} \quad \sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta.$$

These geometric identities can be used recursively to produce various polynomial relations between the values of the trigonometric functions sine and cosine, such as

$$\cos(2\alpha)\cos(\alpha)^2 + 4\sin(2\alpha)\sin\left(\frac{\alpha}{2}\right)\cos\left(\frac{\alpha}{2}\right)\cos(\alpha) - \sin(\alpha)^2\cos(2\alpha) - 1 = 0\,.$$

Vague Theorem

Any algebraic relation over $\overline{\mathbb{Q}}$ between the values of sine and cosine at algebraic points can be derived from the Pythagorean identity and the angle addition formulae.

Set

$$\mathcal{T} := \overline{\mathbb{Q}}[\cos(\alpha), \sin(\alpha) : \alpha \in \overline{\mathbb{Q}}] \subseteq \mathbb{C}\,.$$

Then we introduce the ring of polynomials with algebraic coefficients in countably many variables X_α, Y_α, $\alpha \in \overline{\mathbb{Q}}$:

$$\mathcal{A} := \overline{\mathbb{Q}}[(X_\alpha, Y_\alpha)_{\alpha\in\overline{\mathbb{Q}}}]\,.$$

Let ev denote the evaluation map from $\mathcal{A}$ to $\mathcal{T}$ defined by

$$\mathrm{ev}(X_\alpha) = \cos(\alpha) \quad \text{and} \quad \mathrm{ev}(Y_\alpha) = \sin(\alpha)\,, \quad \alpha \in \overline{\mathbb{Q}}.$$

Let $\mathfrak{I}$ denote the ideal of $\mathcal{A}$ spanned by the polynomials

$$X_\alpha^2 + Y_\alpha^2 - 1, \quad X_{\alpha+\beta} - X_\alpha X_\beta + Y_\alpha Y_\beta\,, \quad \text{and} \quad Y_{\alpha+\beta} - Y_\alpha X_\beta - X_\alpha Y_\beta\,,$$

where α and β run along $\overline{\mathbb{Q}}$.

Since $\mathrm{ev}(\mathfrak{I}) = \{0\}$, the map ev allows to define a homomorphism of $\overline{\mathbb{Q}}$-algebras $\overline{\mathrm{ev}}$ from the quotient algebra $\mathcal{A}/\mathfrak{I}$ to $\mathcal{T}$.

Theorem AD

The map $: \mathcal{A}/\mathfrak{I} \to \mathcal{T}$ is an isomorphism.

E- and M-values

E- and M-values

In the sequel, we consider the case where Ξ is one of the following sets:

$$\mathcal{E} := \{f(1) : f(z) \text{ is an } E\text{-functions}\}$$

and

$$\mathcal{M}_{q,\alpha} := \{f(\alpha) : f(z) \text{ is an } M_q\text{-function}\}\,.$$

Remark

If $f(z)$ is an E-function and α is algebraic, then $f(\alpha z)$ is also an E-function. Thus, it does not matter to evaluate at 1 or at any other nonzero algebraic point in the case of E-functions.

Things are drastically different with M_q-functions!

Hypothesis A (Shidlovskii 1994)

Let $f_1(z), \ldots, f_m(z) \in \overline{\mathbb{Q}}[[z]]$ be linearly independent E-functions that form the entries of a solution vector of a linear E-system and $\alpha \in \overline{\mathbb{Q}} \setminus \{0\}$ be a regular point. Then $f_1(\alpha), \ldots, f_m(\alpha)$ are linearly independent over $\overline{\mathbb{Q}}$.

The lifting theorems

Theorem

Let $f_1(z), \ldots, f_m(z) \in \overline{\mathbb{Q}}[[z]]$ be $\star$-functions that form a solution vector of a linear $\star$-system and let α be a nonzero algebraic regular point. Let $P \in \overline{\mathbb{Q}}[X_1, \ldots, X_m]$ be a homogeneous polynomial such that

$$P(f_1(\alpha), \ldots, f_m(\alpha)) = 0\,.$$

Then there exists a homogeneous polynomial $Q \in \overline{\mathbb{Q}}[z, X_1, \ldots, X_m]$ such that

$$Q(z, f_1(z), \ldots, f_m(z)) = 0 \quad \text{and} \quad Q(\alpha, X_1, \ldots, X_m) = P(X_1, \ldots, X_m)\,.$$

- First proof by Beukers (2006) using the theory of E-operators.
- Second proof by André (2014) derived from the quantitative statement.

- First proof by A. & Faverjon (2017) following Philippon (2015) and derived from the quantitative statement (see also Nagy and Szamuely (2020)).
- Second proof by A. & Faverjon (2020) using the pioneering ideas of Mahler.

Existence of good equations

Theorem

Let $f(z)$ be a $\star$-function. Then $f(z)$ satisfies a $\star$-equation whose only singularities at finite distance are 0 and poles of $f(z)$.

- The existence of such an equation for $f(z) = \sum \frac{a_n}{n!} z^n$ follows, using the Fourier-Laplace transform, from the fact that the minimal equation satisfying by the corresponding G-function $\sum a_n z^n$ is Fuschian (Chudnovsky).
- Follows from the lifting theorem and uses the Mahler denominator associated with f (A. & Faverjon, 2023).

Example

The E-function $f(z) = (z-1)e^z$ is solution to the differential equations

$$zf(z) - (z-1)f'(z) = 0$$

and

$$zf(z) + 2f'(z) - zf''(z) = 0\,.$$

A system-free theory

System without singularity

Let $f_1, \ldots, f_r$ be E-functions and set $\delta := d/dz$. Since each of them satisfies a good equation, for each i, there exist a positive integer m_i and a matrix $A_i(z) \in \mathcal{M}_{m_i}(\overline{\mathbb{Q}}[z, 1/z])$ such that

$$\begin{pmatrix} \delta(f_i) \\ \vdots \\ \delta^{m_i}(f_i) \end{pmatrix} = A_i(z) \begin{pmatrix} f_i \\ \vdots \\ \delta^{m_i-1}(f_i) \end{pmatrix},$$

Set

$$A(z) := \begin{pmatrix} A_1(z) & & \\ & \ddots & \\ & & A_r(z) \end{pmatrix}.$$

The functions $f_{i,j}(z) = \delta^j(f_i)$, $1 \leq i \leq r$, $0 \leq j \leq m_i - 1$, form a vector solution to the differential system associated with $A(z)$.

Since $A(z)$ has coefficients in $\overline{\mathbb{Q}}[z, 1/z]$, all $\alpha \in \overline{\mathbb{Q}}^*$ are regular!

Answer to question (M)

A δ-algebraic relation between $f_1(z), \ldots, f_r(z)$ is an algebraic relation over $\overline{\mathbb{Q}}(z)$ between these functions and their successive derivatives.

A σ_q-algebraic relation between $f_1(z), \ldots, f_r(z)$ is an algebraic relation over $\overline{\mathbb{Q}}(z)$ between these functions and their successive images by $\sigma_q : z \to z^q$.

General Lifting Theorem

Let $f_1(z), \ldots, f_r(z)$ be $\star$-functions. Let $\alpha \in \overline{\mathbb{Q}} \setminus \{0\}$ be such that these functions are well-defined. Then any homogeneous algebraic relation over $\overline{\mathbb{Q}}$ between $f_1(\alpha), \ldots, f_r(\alpha)$ is the specialization of a homogeneous $\star$-algebraic relation over $\overline{\mathbb{Q}}(z)$ between $f_1(z), \ldots, f_r(z)$.

Example

For the E-function $f(z) := (z-1)e^z$, the linear relation $f(1) = 0$ cannot be obtained by specialization of a relation satisfied by $f(z)$. It is now explained as the degeneracy of the δ-linear relation

$$zf(z) - (z-1)f'(z) = 0\,,$$

as predicted by the theorem.

Consequence

Corollary

Let $f(z)$ be a $\star$-function with coefficients in a number field K. Let $\alpha \in K \setminus \{0\}$ be a point where this function is well-defined. Then either $f(\alpha) \in K$ or $f(\alpha)$ is transcendental.

Proof. Let us assume that $f(\alpha) \in \overline{\mathbb{Q}}$. This means that 1 and $f(\alpha)$ are linearly dependent over $\overline{\mathbb{Q}}$. Then the general lifting theorem implies that there exists a $\star$-linear relation over $\overline{\mathbb{Q}}(z)$ between 1 and $f(z)$ whose specialization provides a $\overline{\mathbb{Q}}$-linear relation between 1 and $f(\alpha)$. Since $K((z))$ and $\overline{\mathbb{Q}}(z)$ are linearly disjoint over $K(z)$, we deduce that there exist a $\star$-linear relation over $K(z)$ between 1 and $f(z)$ whose specialization at α provides a nontrivial linear relation over K between 1 and $f(\alpha)$. Hence $f(\alpha) \in K$. $\square$

Theorem

Let $f_1(z), \ldots, f_r(z)$ be $\star$-functions and $\alpha \in \overline{\mathbb{Q}} \setminus \{0\}$ be such that these functions are well-defined. Then there exists an algorithm that allows to compute a set of generators of the ideal $\mathcal{I}_\alpha$ of the algebraic relations between $f_1(\alpha), \ldots, f_r(\alpha)$.

Step 1. Find a good equation for each of the function $f_i(z)$ (related to minimization, see Bostan-Salvy-Rivoal and A-Faverjon) and let us denote by $A_i(z)$ the corresponding companion matrix. Then consider the linear system associated with the direct sum of the $A_i(z)$.

Step 2. Let $\mathcal{I}_z$ denote the ideal of algebraic relations over $\overline{\mathbb{Q}}(z)$ between $f_1(z), \ldots, f_r(z)$ and their successive derivatives. Find a set of generator of the ideal $\mathcal{I}_z$ (Hrushosvki-Feng algorithm).

Step 3. Specialize at α the set of generators of $\mathcal{I}_z$ and use a Gröbner basis elimination algorithm to find a set of generators of $\mathcal{I}_\alpha$.

Answer to question (Q)

Theorem

Let $f_1(z), \ldots, f_r(z)$ be $\star$-functions and $\alpha \in \overline{\mathbb{Q}} \setminus \{0\}$ be such that these functions are well-defined. Let $P \in \mathbb{Z}[X_1, \ldots, X_r]$. Then either

$$P(f_1(\alpha), \ldots, f_r(\alpha)) = 0$$

or

$$P(f_1(\alpha), \ldots, f_r(\alpha)) > c_0 H^{c_1 \deg(P)^t},$$

where t is at most equal to the sum of the order of the minimal equations of the f_i's. In particular, a $\star$-value cannot be a Liouville number.

Remark

Proofs were first obtained (Lang, Nesterenko, Nishioka...) with the following three additional assumptions: $f_1(z), \ldots, f_r(z)$ form a vector solution of a $\star$-system, they are algebraically independent, and $\alpha \in \overline{\mathbb{Q}} \setminus \{0\}$ is regular.