Height gaps for coefficients of D-finite power series and related results

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Height

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For most of this talk, we only need the absolute logarithmic Weil height

$$h : \bar{\mathbb{Q}} \to \mathbb{R}_{\geq 0}.$$

In general, one needs to combine the contributions from all absolute values to define height functions. But for the above h on $\bar{\mathbb{Q}}$, there is an alternative explicit formula, as follows.

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Fix an embedding $\bar{\mathbb{Q}} \subset \mathbb{C}$. For $\alpha \in \bar{\mathbb{Q}}$, express its minimal polynomial over $\mathbb{Z}$ as:

$$c(x - \alpha_1) \cdots (x - \alpha_d).$$

Then $h(\alpha) = \dfrac{1}{d}\left(\log|c| + \sum_{i=1}^{d} \log\max\{|\alpha_i|, 1\}\right)$.

Example: $\alpha \in \mathbb{Q}$, express $\alpha = \dfrac{a}{b}$ with $a, b \in \mathbb{Z}$ and $\gcd(a, b) = 1$, then $h(\alpha) = \log\max\{|a|, |b|\}$. For the talk, it's perfectly fine to think of only power series with rational coefficients.

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D-finite series

$\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, K is a field, and $m \in \mathbb{N}$.

Let $\mathbf{n} = (n_1, \ldots, n_m) \in \mathbb{N}_0^m$ and let $\mathbf{x} = (x_1, \ldots, x_m)$ be the vector of the indeterminates $x_1, \ldots, x_m$. Write $\mathbf{x}^{\mathbf{n}}$ to denote the monomial $x_1^{n_1} \ldots x_m^{n_m}$ having the total degree

$$\|\mathbf{n}\| := n_1 + \ldots + n_m.$$

A power series $f(\mathbf{x}) \in K[[\mathbf{x}]]$ is said to be D-finite (over $K(\mathbf{x})$) if the partial derivatives (of all orders) span a finite-dimensional vector space over $K(\mathbf{x})$.

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Problem: $f(\mathbf{x}) = \sum_{\mathbf{n}\in\mathbb{N}_0^m} a_{\mathbf{n}}\mathbf{x}^{\mathbf{n}} \in \bar{\mathbb{Q}}[[\mathbf{x}]]$ is D-finite, study the growth of $h(a_{\mathbf{n}})$ with respect to $\|\mathbf{n}\|$.

It is helpful to think of the univariate case ($m = 1$), here $f(x) = \sum_{n=0}^{\infty} a_n x^n \in \bar{\mathbb{Q}}[[x]]$ is D-finite iff it satisfies a linear differential equation with coefficients in $\bar{\mathbb{Q}}[x]$.

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D-finite series

Equivalently, the coefficients (eventually) satisfy a linear recurrence relation with polynomial coefficients: there exist $d \in \mathbb{N}$ and $P_0(x), \ldots, P_d(x) \in \bar{\mathbb{Q}}[x]$ with $P_d \neq 0$ such that

$$P_d(n)a_{n+d} + \ldots + P_0(n)a_n = 0$$

for all sufficiently large n.

Sad fact: when the P_i's are constant polynomials, Skolem (1933), Mahler (1935), and Lech (1953) proved that $\{n : a_n = 0\}$ is the union of a finite set and finitely many arithmetic progressions, yet we still don't know if the same holds for general polynomials P_i's.

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Examples with different growth behaviors

Example 1: exponential function

$$f(x) = \sum \frac{x^n}{n!}, \; h(a_n) = \log(n!) \sim n \log n.$$

Example 2: rational function with at least one pole not a root of unity

$$f(x) = \frac{1}{1-2x} = \sum 2^n x^n, \; h(a_n) = n \log 2.$$

Example 3: logarithmic function

$$f(x) = \log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \dots, \; h(a_n) = \log n.$$

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A height gap result in 2019

Here $f(\mathbf{x}) = \sum_{\mathbf{n}\in\mathbb{N}_0^m} a_{\mathbf{n}}\mathbf{x}^{\mathbf{n}} \in \bar{\mathbb{Q}}[[\mathbf{x}]]$ is a D-finite power series in m variables.

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Theorem (Bell-N.-Zannier)

Suppose $\lim_{\|\mathbf{n}\|\to\infty} \frac{h(a_{\mathbf{n}})}{\log \|\mathbf{n}\|} = 0$. *Then:*

(a) *f is a rational function.*

(b) *If f is not a polynomial, its denominator, up to scalar multiplication, has the form*

$$\prod_{i=1}^{\ell}(1 - \zeta_i \mathbf{x}^{\mathbf{n}_i})$$

where $\ell \geq 1$, ζ_i is a root of unity, $\mathbf{n}_i \in \mathbb{N}_0^m \setminus \{0\}$ for $1 \leq i \leq \ell$, and the $1 - \zeta_i \mathbf{x}^{\mathbf{n}_i}$'s are ℓ distinct irreducible polynomials.

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Key observation after the above result: there's a "gap" in the possible growth of $h(a_{\mathbf{n}})$. More precisely if $h(a_{\mathbf{n}})$ is dominated by $\log \|\mathbf{n}\|$ then it is $O(1)$.

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Many great results motivated by the above work

From the previous 5 examples, it's natural to ask whether we can completely classify the growth of $h(a_n)$ as $O(n \log n)$, $O(n)$, $O(\log n)$, or $O(1)$ when f is D-finite.

Right after our work in 2019, we have some idea for further results toward the above classification. But its release was delayed until June 2022 (my fault)! In the meantime, there are many great results motivated by our work.

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- Results by Bell, Hu, Ghioca, Satriano on a height gap phenomenon in arithmetic dynamics.
- A complete classification for the possible height growth of coefficients of Mahler functions by Adamczewski, Bell, and Smertnig.
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Our result in 2022

A set $S \subseteq \mathbb{N}$ is said to have positive upper density if

$$\limsup \frac{|S \cap [1, n]|}{n} > 0,$$

otherwise S is said to have zero density.

For algebraic numbers $\alpha_1, \ldots, \alpha_n$, the denominator $\mathrm{den}(\alpha_1, \ldots, \alpha_n)$ is the smallest positive integer D such that every $D\alpha_i$ is an algebraic integer. This is the lcm of the individual $\mathrm{den}(a_i)$ for $1 \leq i \leq n$.

In the next theorem: $K \subset \mathbb{C}$ is a number field, $f(x) = \sum a_n x^n \in K[[x]]$ is D-finite, $r \in [0, \infty]$ is the radius of convergence of f.

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(a) *If $r \in \{0, \infty\}$ and f is not a polynomial then $h(a_n) = O(n \log n)$ for every large n and $h(a_n) \gg n \log n$ on a set of positive upper density.*

(b) *If $r \notin \{0, \infty\}$ then at least one of the following holds:*

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  (ii) *$\mathrm{den}(a_n) \gg n$, and hence $h(a_n) > (\log n)/[K : \mathbb{Q}] + O(1)$ on a set of positive upper density.*

  (iii) *f is a rational function whose poles are roots of unity, hence the a_n's belong to a finite set.*

Some open problems

Roughly speaking, the previous theorem says that $n \log n$, n, $\log n$, and the constant function are the possible lower bounds for $h(a_n)$.

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Question

$f(x) = \sum a_n x^n \in \bar{\mathbb{Q}}[[x]]$ is D-finite. Is it true that one of the following holds?

(i) $h(a_n) = O(n \log n)$ *for every* n *and* $h(a_n) \gg n \log n$ *on a set of positive upper density.*

(ii) $h(a_n) = O(n)$ *for every* n *and* $h(a_n) \gg n$ *on a set of positive upper density.*

(iii) $h(a_n) = O(\log n)$ *for every* n *and* $h(a_n) \gg \log n$ *on a set of positive upper density.*

(iv) $h(a_n) = O(1)$ *for every* n.

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Here's a weaker version of the above.

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Part (ii) above is analogous to a long standing open problem in the theory of Siegel E-functions. Instead of $h(a_n)$, the below problem considers the (affine) height of a tuple of algebraic numbers.

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$f(x) = \sum a_n x^n \in \bar{\mathbb{Q}}[[x]]$ is D-finite. Assume that $h(a_0, \ldots, a_n) = o(n \log n)$. Is it true that $h(a_0, \ldots, a_n) = O(n)$?

Equivalent version following the terminology in Rivoal's talk: is every E-function a strict E-function?

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Main ingredients for the proof

Let's recall the statement of our result. Here $K \subset \mathbb{C}$ is a number field, $f(x) = \sum a_n x^n \in K[[x]]$ is D-finite, $r \in [0, \infty]$ is the radius of convergence of f.

Theorem (Bell-N.-Zannier)

(a) *If $r \in \{0, \infty\}$ and f is not a polynomial then $h(a_n) = O(n \log n)$ for every large n and $h(a_n) \gg n \log n$ on a set of positive upper density.*

(b) *If $r \notin \{0, \infty\}$ then at least one of the following holds:*

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  (ii) *$\mathrm{den}(a_n) \gg n$, and hence $h(a_n) > (\log n)/[K : \mathbb{Q}] + O(1)$ on a set of positive upper density.*

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Main ingredients for the proof

After some simple arguments, We reduce to the following case:

- $f(x) \in \mathbb{Q}[[x]]$ is D-finite with rational coefficients, and
- the radius of convergence $r = 1$.

And we need to prove that at least one of the following holds:

A. $\mathrm{den}(a_n) \gg n$ on a set of positive upper density.

B. f is rational.

Suppose A is not true. This means that for a large N, there is a "thin" exceptional subset E of $\{1, \ldots, N\}$ such that $\mathrm{den}(a_n)$ is small vs n for every $n \in \{1, \ldots, N\} \setminus E$. We need to prove that f is rational.

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First ingredient: Hankel determinant and rational approximation.

Let $g(x) = \sum b_n x^n$ and $m \geq 0$, define

$$\Delta_m(g) = \det \begin{pmatrix} b_0 & b_1 & \dots & b_m \\ b_1 & b_2 & \dots & b_{m+1} \\ \dots & & & \\ b_m & b_{m+1} & \dots & b_{2m} \end{pmatrix}.$$

Main ingredients for the proof

First ingredient: Hankel determinant and rational approximation.

Let $g(x) = \sum b_n x^n$ and $m \geq 0$, define

$$\Delta_m(g) = \det \begin{pmatrix} b_0 & b_1 & \dots & b_m \\ b_1 & b_2 & \dots & b_{m+1} \\ \dots & & & \\ b_m & b_{m+1} & \dots & b_{2m} \end{pmatrix}.$$

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Facts:

- If $\Delta_m(g) = 0$ for many consecutive values of m then g can be "well" approximated by rational functions.
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Second ingredient: Polya's inequality.

Suppose $g(x) = \sum b_n x^n \in \mathbb{C}[[x]]$ converges in the open unit disk and can be continued analytically beyond the open unit disk. Then there exists $\rho < 1$ such that

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Third ingredient: construction of an auxiliary polynomial.

Recall that we assume Property A does not hold. This means for a large N, there's a thin subset E of $\{1, \ldots, N\}$ such that $\mathrm{den}(a_n)$ is small vs n for $n \in \{1, \ldots, N\} \setminus E$.

Construct an integer-valued polynomial P such that $P(n) = 0$ for $n \in E$. Hence although $\mathrm{den}(a_n)$ for $n \in E$ might be large, we simply have $P(n)a_n = 0$.

Then consider:

$$g(x) := \sum P(n)a_n x^n$$

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Therefore $\Delta_m(g) = 0$ for $m \leq N$. Then g can be well approximated by rational functions. Then g is rational and it's not hard to prove rationality of f from here.

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Further comments

To be honest, we did not use any advanced tools that are very specific for D-finite series.

The essential properties we need are:

- f can be extended analytically beyond its disk of convergence.
- If g (which is a linear combination of derivatives of f) is well approximated by a rational function then it is rational.

In fact we can adapt the above method to prove a more general/flexible criterion for the "Pólya-Carlson dichotomy" and apply this criterion to a certain dynamical zeta function.

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Dynamical zeta functions

Let φ be a map from a set X to itself.

For $k \geq 1$, let $N_k(\varphi)$ denote the number of fixed points of the k-th fold iterate $\varphi^k := \varphi \circ \cdots \circ \varphi$ (k times).

Definition

Assume that $N_k(\varphi) < \infty$ for every k, then we can define the dynamical or Artin-Mazur zeta function:

$$\zeta_\varphi(x) = \exp\left(\sum_{k=1}^{\infty} \frac{N_k(\varphi)}{k} x^k\right).$$

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Problem (Artin-Mazur, Smale,...): in interesting situations, determine whether ζ_φ is rational, algebraic, or transcendental.

Example: V is an algebraic variety defined over a finite field $\mathbb{F}_q$, $X = V(\overline{\mathbb{F}_q})$, and φ is the Frobenius. Then $N_k(\varphi)$ is $|V(\mathbb{F}_{q^k})|$ and ζ_φ is the Hasse-Weil zeta function of V.

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The Pólya-Carlson dichotomy

Terminology: $A(x) \in \mathbb{C}[[x]]$ with radius of convergence $r \in (0, \infty)$ is said to admit the circle of radius r as a natural boundary if it cannot be extended to an analytic function beyond the disk of radius r.

Fact: $A(x)$ with a natural boundary as above, then A is transcendental. More generally, A is not D-finite.

Theorem (Pólya-Carlson)

A power series with integer coefficients and radius of convergence 1 *is either rational or has the unit circle as a natural boundary.*

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After work of many people, in 2014, Bell, Miles, and Ward state their observation that in many interesting situations, the dynamical zeta function satisfies the Pólya-Carlson dichotomy.

We studied the zeta function for the dynamics on a so called "positive characteristic torus" and aimed to establish the Pólya-Carlson dichotomy.

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The Pólya-Carlson dichotomy

The power series we encounter has the form $B(x) = \sum b_k x^k$ where:

- The b_k's belong to a number field K.
- For every embedding σ of K into $\mathbb{C}$, the series $\sigma(B)$ converges in the open unit disk.
- $\mathrm{den}(b_k) = e^{o(k)}$ for "most" k. But occasionally, we get a "bad" k where $\mathrm{den}(b_k)$ can be exponential in k.

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A criterion for the Pólya-Carlson dichotomy

The above motivates the following more general/flexible criterion for the Pólya-Carlson dichotomy:

Theorem (Bell, Gunn, N., and Saunders, 2023)

Let $\mathcal{E} \subset \mathbb{N}$ such that $|\mathcal{E} \cap [1, n]| = o(n/\log n)$ as $n \to \infty$. Let K be a number field, $A(x) = \sum a_n x^n \in K[[x]]$ such that $\sigma(A)(x)$ converges in the open unit disk for every embedding σ of K into $\mathbb{C}$. Suppose that for every given $c > 1$, we have:

$$\mathrm{den}(a_i : \ 1 \leq i \leq n, \ i \notin \mathcal{E}) < c^n$$

for all sufficiently large n. Then either $A(x)$ admits the unit circle as a natural boundary or there exists $\sum u_n x^n$ that is a rational function and $a_n = u_n$ for every $n \notin \mathcal{E}$.

A criterion for the Pólya-Carlson dichotomy

Some remarks:

(i) The function $n/\log n$ (in $|\mathcal{E} \cap [1, n]| = o(n/\log n)$) is best possible.

(ii) Although I said we did not use anything too specific for D-finite series in the earlier theorem, if assuming D-finiteness then we can allow the weaker requirement $|\mathcal{E} \cap [1, n]| = o(n)$ in this criterion.

(iii) Back to our series $\sum b_k x^k$, we can let $\mathcal{E}$ be the set of bad k where $\mathrm{den}(b_k)$ is large. In our situation, we can have $\mathcal{E}$ with $|\mathcal{E} \cap [1, n]| = O((\log n)^2)$, much less than the required $o(n/\log n)$ in the criterion.

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(iv) Byszewski, Cornelissen, and Houben successfully apply our criterion for the zeta functions of the dynamical systems that they study.

Main ingredients in the proof of the criterion

- Construct a sequence of auxiliary polynomials P_m for $m = 1, 2, \ldots$
- Consider $\sum_{n=0}^{\infty} P_m(n)a_nx^n$, use Hankel determinant, and Pólya's inequality as before to prove that this can be well approximated by rational functions.
- Use the above well aproximation property to relate the different $\sum_{n=0}^{\infty} P_m(n)a_nx^n$ as m varies.

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THANK YOU!