congruences modulo p, algebraic independence and monodromy

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If $f(z)$ is algebraic modulo p, $deg(f_{|p})$ is the degree of the minimal polynomial of $f_{|p}$.

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- If $f(z) \in \mathbb{Q}[[z]]$ is p-Lucas then $f(z)$ is algebraic modulo p.
- Most of the power series that are p-Lucas for infinitely many prime numbers p are G-functions.

Hypergeometric series

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$$ {}_nF_{n-1}(\underline{\alpha}, \underline{\beta}; z) = \sum_{j \geq 0} \frac{(\alpha_1)_j \cdots (\alpha_n)_j}{(\beta_1)_j \cdots (\beta_{n-1})_j j!} z^j \in 1 + z\mathbb{Q}[[z]], $$

where for a real number x, $(x)_0 = 1$ and $(x)_j = \prod_{i=0}^{j-1}(x+i)$ for $j > 0$.

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where for a real number x, $(x)_0 = 1$ and $(x)_j = \prod_{i=0}^{j-1}(x+i)$ for $j > 0$. For example, the hypergeometric series

$$ {}_2F_1((1/5, 1/5), 2/7; z) = \sum_{j \geq 0} \frac{(1/5)_j^2}{(2/7)_j j!} z^j $$

is p-Lucas for all $p \equiv 1 \bmod 35$.

Diagonals

Let $\Delta_d : \mathbb{Q}[[z_1, \ldots, z_d]] \cap \mathbb{Q}(z_1, \ldots, z_d) \to \mathbb{Q}[[z]]$,

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4. **Q**: Are there algebraic relations between $\{\mathfrak{f}_r\}_{r \geq 2}$ and $\mathfrak{t}(z)$?

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If $f(z) \in 1 + z\mathbb{Q}[[z]]$ is p-Lucas for all $p \in \mathcal{S}$ then $f(z) \in \mathcal{L}(\mathcal{S})$.

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Let $f_1(z), \ldots, f_r(z) \in \mathcal{L}(\mathcal{S})$, $\mathcal{S}$ infinite. Then, $f_1(z), \ldots, f_r(z)$ are algebraically dependent over $\mathbb{Q}(z)$ if and only if there exist $m_1, \ldots, m_r \in \mathbb{Z}$ not all zero, such that $f_1(z)^{m_1} \cdots f_r(z)^{m_r} \in \mathbb{Q}(z)$.

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When does $f(z)$ belong to $\mathcal{L}(\mathcal{S})$?

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$$f_{|p}(z) = A_p(z) f_{|p}(z)^{p^l}.$$

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Definition (Dwork, 1974)

Let L be in $E_p[\delta]$ of order n. We say that L has a strong Frobenius structure (sFs) of period m, if there is $(h_1, \ldots, h_n) \in E_p^n \setminus \{(0, \ldots, 0)\}$ such that, for all solutions f of L in a differential extension of E_p,

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Examples : hypergeometric and Picard–Fuchs

Let $\underline{\alpha} = (\alpha_1, \dots, \alpha_n)$ and $\underline{\beta} = (\beta_1, \dots, \beta_{n-1}, 1)$ be in $(\mathbb{Q} \setminus \mathbb{Z}_{\leq 0})^n$

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$$\mathcal{H}(\underline{\alpha}, \underline{\beta}) = -z \prod_{i=1}^{n} (\delta + \alpha_i) + \prod_{j=1}^{n} (\delta + \beta_j - 1), \ \delta = z \frac{d}{dz}$$

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- If $\mathcal{L} \in \mathbb{Q}(z)[d/dz]$ is a Picard–Fuchs equation then $\mathcal{L}$ has a sFs for almost all p.

MUM at zero

Let K be any field. We say that

$$\mathcal{D} = \delta^n + b_1(z)\delta^{n-1} + \cdots + b_{n-1}(z)\delta + b_n(z) \in K(z)[\delta].$$

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Proposition (I)

If $\mathcal{D}_p \in \overline{\mathbb{F}_p}(z)[\delta]$ is MUM at zero then there exists a polynomial $P(z) \in 1 + z\overline{\mathbb{F}_p}[z]$ such that $\mathcal{D}_p(P) = 0$ and $dim_{\overline{\mathbb{F}_p}((z^p))} Ker(\overline{\mathbb{F}_p}((z)), \mathcal{D}_p) = 1$.

Main result

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Theorem (I, VM)

Let $\mathcal{S}$ be an infinite set of prime numbers. If $f(z) \in \mathcal{MF}(\mathcal{S})$ then there exist a set $\mathcal{S}' \subset \mathcal{S}$ and a constant $C \in \mathbb{R}_{>0}$ such that $\mathcal{S} \setminus \mathcal{S}'$ is finite and, for every $p \in \mathcal{S}'$,

$$f_{|p}(z) = A_p(z) f_{|p}(z)^{p^l},$$

where $A_p(z) \in \mathbb{F}_p(z)$ whose height is bounded by Cp^{2l}.

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which is MUM at zero and, according to Theorem III, has a sFs for every $p > 2$

More generalized p-Lucas

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More generalized p-Lucas

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Remark : Under the assumptions of this theorem, we show that $g_1, \ldots, g_r \in \mathcal{L}(\mathcal{S}')$, where $\mathcal{S}' \subset \mathcal{S}$ and $\mathcal{S} \setminus \mathcal{S}'$ is finite.

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For all integers $k \geq 1$, we construct a differential operator

$$\mathcal{H}_k = \delta^n + e_1\delta^{n-1} + \cdots + e_{n-1}\delta + e_n \in \vartheta_{E_p}[\delta]$$

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where $A_{k,l} = B_k(B_{k+1})^p \cdots B_{k+l-1}(z)^{p^{l-1}}$ and $H(A_{k,l}) \leq nrp^l$.

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But the height of $A_{0,l}\left(\frac{A_{l,l}}{A_{0,l}}\right)^{p^l}$ is less than or equal to nrp^{2l}.

Proof of Theorem VI

We have already seen that

$$f_{|p}(z) = A_{0,l_p}(z)\Lambda_p^l(f(z))_{|p}^{p^l},$$

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Whence, f is a generalized p-Lucas.