Asymptotic expansion of regular graphs

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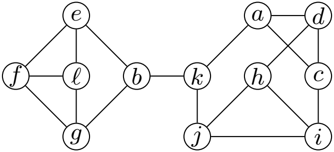
Models

k-regular graphs

- labeled vertices
- unlabeled unoriented edges
- no loop, no multiple edge
- all vertices have degree k

Related models

- degree sequence / set of degrees
- symmetric $(0, 1)$-matrices with constraints on the sum of each row
- bipartite graphs / hypergraphs
- multigraphs



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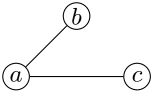

$$\begin{array}{c} \\ a \\ b \\ c \end{array} \begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \end{array}$$

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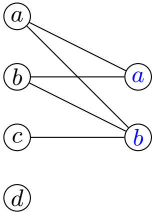

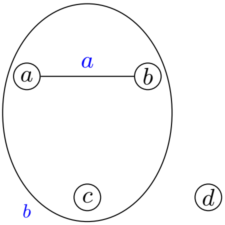

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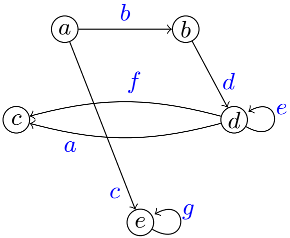

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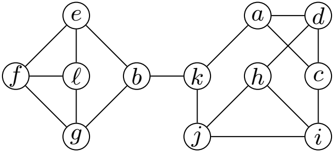

Questions about regular graphs

- exact expression
- asymptotics
- asymptotic expansion
- typical structure

Various approaches, including

- surgery (combinatorics)
- symmetric functions (algebra)
- configuration model (probabilities)
- inversion from multigraphs (analysis)

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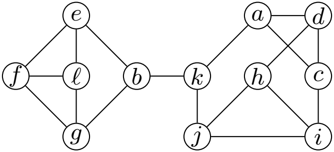

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$$n! = n^n e^{-n} \sqrt{2\pi n} \left(1 + \frac{1}{12} n^{-1} + \frac{1}{288} n^{-2} - \frac{139}{51840} n^{-3} + \cdots + O(n^{-r}) \right)$$

Warm up: k-regular graphs for $k \in \{0, 1, 2\}$

Generating function

$$\mathrm{SG}^{(k)}(z) = \sum_{n \geq 0} \mathrm{SG}_n^{(k)} \frac{z^n}{n!}.$$

0-regular graph: set of isolated vertices $\quad \mathrm{SG}^{(0)}(z) = e^z$

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2-regular graph: set of cycles of length ≥ 3 $\quad \mathrm{SG}^{(2)}(z) = e^{\frac{1}{2}\left(\log\left(\frac{1}{1-z}\right) - z - \frac{z^2}{2}\right)}$

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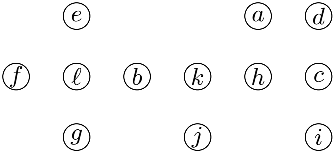

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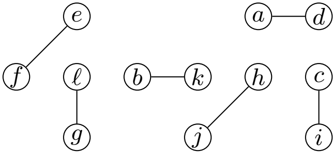

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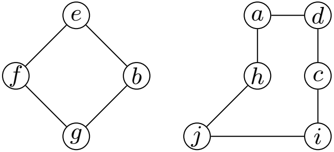

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Symbolic method [Flajolet Sedgewick 2009, Bergeron Labelle Leroux 1998]

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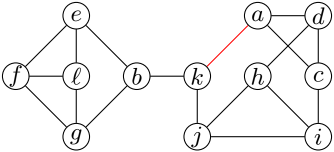
Surgery for 3-regular graphs

Sum of the degrees is twice the number of edges, so n odd implies $\mathrm{SG}_n^{(3)} = 0$.

Surgery: construct a system of equations for the generating functions of 3-regular graphs plus 0, 1 or 2 vertices having degree 2 [Read 1959, Wormald Wright 1979].



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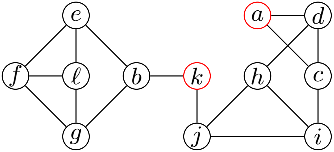

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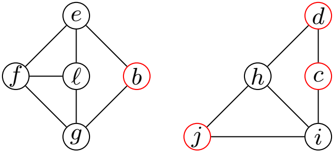

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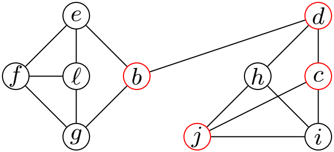


Surgery for 3-regular graphs

The result is a differential equation with polynomial coefficients (D-finite)

$$\left(\frac{3}{2}z^7+3z^5-3z^3\right)\mathrm{SG}^{(3)\prime\prime}(z)+\left(\frac{z^{10}}{2}+3z^8+\frac{3}{2}z^7+3z^6+3z^5-3z^3-16z^2+4\right)\mathrm{SG}^{(3)\prime}(z)-\frac{z^3}{6}(z^4+2z^2-2)^2\,\mathrm{SG}^{(3)}(z)=0$$

Differential equation $\mapsto$ recursion with polynomial coefficients

$$\begin{aligned}3(3n-7)(3n-4)\,\mathrm{SG}^{(3)}_{2n} &= 9(n-1)(2n-1)(3n-7)(3n^2-4n+2)\,\mathrm{SG}^{(3)}_{2n-2}\\ &+(n-1)(2n-3)(2n-1)(108n^3-441n^2+501n-104)\,\mathrm{SG}^{(3)}_{2n-4}\\ &+2(n-2)(n-1)(2n-5)(2n-3)(2n-1)(3n-1)(9n^2-42n+43)\,\mathrm{SG}^{(3)}_{2n-6}\\ &-2(n-3)(n-2)(n-1)(2n-7)(2n-5)(2n-3)(2n-1)(3n-4)(3n-1)\,\mathrm{SG}^{(3)}_{2n-8}\,.\end{aligned}$$

and asymptotic expansion (arbitrary number of error terms).

Same approach works for 4-regular graphs [Read Wormald 1980, Goulden Jackson Reilly 1983]. Becomes too big to be handled by hand for larger k.

Symmetric functions

Generating function of graphs, x_i marks the degree of vertex i

$$G(\boldsymbol{x}) := \sum_{G} \prod_{v \in V(G)} x_v^{\deg(v)} = \prod_{1 \leq i < j} (1 + x_i x_j)$$

Infinitely many variables! But symmetric function.

Number $\mathrm{SG}_n^{(k)}$ of k-regular graphs on n vertices and its generating function

$$\mathrm{SG}_n^{(k)} = [x_1^k \cdots x_n^k] G(\boldsymbol{x}), \qquad \mathrm{SG}^{(k)}(z) = \sum_{n \geq 0} [x_1^k \cdots x_n^k] G(\boldsymbol{x}) \frac{z^n}{n!}.$$

We sketch the proof that $\mathrm{SG}^{(k)}(z)$ is D-finite.

Symmetric functions

Symmetric function families

- Power-sum $p_m(\boldsymbol{x}) = x_1^m + x_2^m + \cdots$
- Homogeneous $h_k(\boldsymbol{x}) = \sum_{\substack{i_1+i_2+\cdots=k \\ \forall j,\ i_j \geq 0}} x_1^{i_1} x_2^{i_2} \cdots$

[Gessel 1990] defines a scalar product on symmetric functions satisfying

$$[x_1^{\lambda_1} \cdots x_n^{\lambda_n}] F(\boldsymbol{x}) = \langle F(\boldsymbol{x}), h_{\lambda_1}(\boldsymbol{x}) \cdots h_{\lambda_n}(\boldsymbol{x}) \rangle$$

so

$$\mathrm{SG}^{(k)}(z) = \sum_{n \geq 0} [x_1^k \cdots x_n^k] G(\boldsymbol{x}) \frac{z^n}{n!} = \sum_{n \geq 0} \langle G(\boldsymbol{x}), h_k(\boldsymbol{x})^n \rangle \frac{z^n}{n!} = \langle G(\boldsymbol{x}), e^{h_k(\boldsymbol{x}) z} \rangle$$

Theorem [Gessel 1990]. Let $f(z, p_1(\boldsymbol{x}), p_2(\boldsymbol{x}), \ldots)$ and $g(z, p_1(\boldsymbol{x}), p_2(\boldsymbol{x}), \ldots)$ be D-finite in z and the $p_i(\boldsymbol{x})$'s, with g involving only finitely many $p_i(\boldsymbol{x})$'s and $\langle f, g \rangle$ defining a proper formal power series, then this series is D-finite.

Symmetric functions

Gessel proofs are non-constructive.

[Chyzak Mishna Salvy 2005] gave constructive proofs and [Chyzak Mishna 2024] recently computed the differential equation for $\mathrm{SG}^{(k)}(z)$ for k up to 7.

For any k, if we have enough computational power, we can compute the asymptotic expansion of $\mathrm{SG}_n^{(k)}$.

But what if we don't?

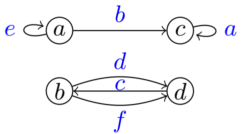
Simplifying the model

Multigraphs

- labeled vertices
- labeled oriented edges
- loops and multiple edges allowed
- degree = number of occurrences in the edges.

Number of k-regular multigraphs on n vertices: $\mathrm{MG}_n^{(k)} = \binom{kn}{k, \ldots, k}$

(c,c)	(a,c)	(d,b)	(b,d)	(a,a)	(b,d)
a	b	c	d	e	f



The only problem for counting k-regular graphs are loops and multiple edges!

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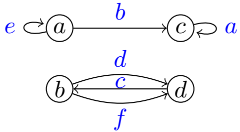

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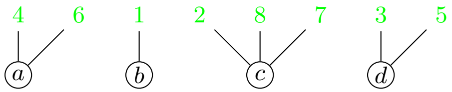
Configuration model

Configuration model [Bollobás 1979] or pairing model [Bender Canfield 1978]

- Draw each degree following a distribution $\sum_d \delta_d = 1$
- Repeat until the sum is even
- Label the half-edges randomly
- Link them to form edges



Generates a uniform k-regular multigraph $\mathcal{MG}_n^{(k)}$ when $\delta_d = \mathbb{1}_{d=k}$.

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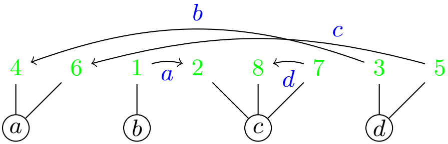

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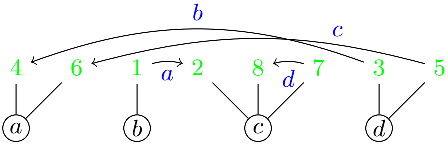

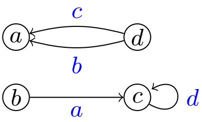

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$\mathcal{MG}_n^{(k)}$ has $m = nk/2$ edges

$$\mathbb{P}\left(\mathcal{MG}_n^{(k)} \text{ is simple}\right) = \frac{\mathrm{SG}_n^{(k)}\, 2^m m!}{\mathrm{MG}_n^{(k)}}$$

Counting the occurrences of a subgraph in $\mathcal{MG}_n^{(k)}$ is easy

$$\mathbb{E}(\#\text{ loops in } \mathcal{MG}_n^{(k)}) = \frac{k\text{-regular multigraphs with a distinguished loop}}{\mathrm{MG}_n^{(k)}}$$

Almost surely no double loop, triple edge, or loop touching a double edge.

Number of loops and double edges $\sim$ Pois $\left(\frac{k^2-1}{4}\right)$

$$\mathbb{P}\left(\mathcal{MG}_n^{(k)} \text{ is simple}\right) \underset{n\to+\infty}{\sim} \mathbb{P}(\text{no loop and no double edge}) \underset{n\to+\infty}{\sim} e^{-(k^2-1)/4}$$

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Inversion from multigraphs

First, we extracted more error terms by removing other loops and double edges configurations using inclusion-exclusion.

Then we noticed magical cancellations and realized how rich the model describing the interlacing of loops and double edges was.

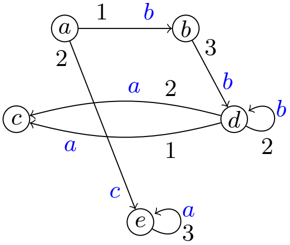
In fact [Caizergues, P. 2023], a related model is

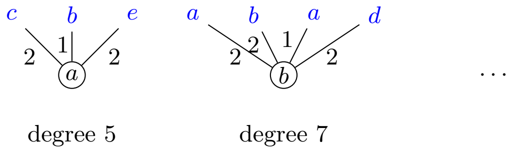
- rich enough to be reducible to simple graphs by the symbolic method
- simple enough to have a reasonnable generating function.

Inversion from multigraphs

Multigraphs with

- z marking the vertices
- w_j marking the edges of weight j
- δ_d marking the vertices of degree d (sum of the weights of the incident edges)
- degrees bounded by k





$$\mathrm{WMG}(z, \boldsymbol{w}, \boldsymbol{\delta}) = \sum_{\boldsymbol{m} \in \mathbb{N}^k} (2\boldsymbol{m})! [\boldsymbol{x}^{2\boldsymbol{m}}] e^{z \sum_{d=1}^{k} \delta_d [y^d] \exp(\sum_j x_j y^j)} \frac{(\boldsymbol{w}/2)^{\boldsymbol{m}}}{\boldsymbol{m}!}$$

Inversion from multigraphs

Exponential Hadamard product

$$\sum_n a_n \frac{z^n}{n!} \odot_z \sum_n b_n \frac{z^n}{n!} = \sum_n a_n b_n \frac{z^n}{n!}$$

Double factorial transform

$$\sum_m \frac{(2m)!}{2^m m!} [z^{2m}] F(z) x^m = e^{x^2/2} \odot_x F(x)$$

Application to multigraphs with weighted edges

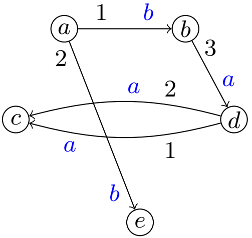
$$\Delta(y, \boldsymbol{\delta}) = \sum_{d=1}^{k} \delta_d \frac{y^d}{d!} \qquad P(\boldsymbol{x}, \boldsymbol{\delta}) = \Delta(y, \boldsymbol{\delta}) \odot_{y=1} e^{\sum_{j=1}^{k} x_j y^j}$$

$$\mathrm{WMG}(z, \boldsymbol{w}, \boldsymbol{\delta}) = e^{w_1 x_j^2/2 + \cdots + w_k x_k^2/2} \odot_{x_1=1} \cdots \odot_{x_k=1} e^{zP(\boldsymbol{x}, \boldsymbol{\delta})}$$

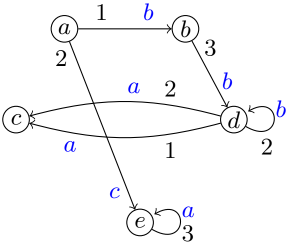
Finitely many variables and D-finite.

Inversion from multigraphs

Construct each (multi)graph family from a simpler one,
then invert the relation between their generating functions.



Loopless multigraph
with weighted edges



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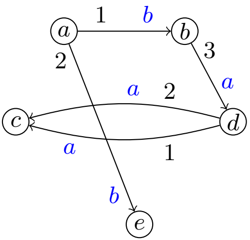

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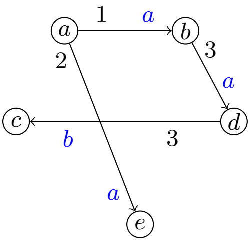

Loopless multigraph
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without multiple edge

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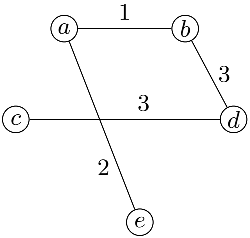
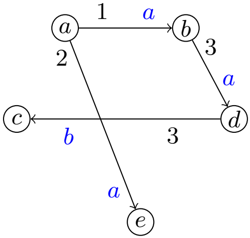


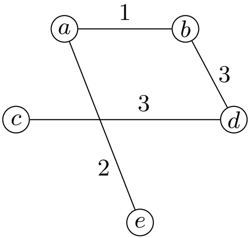
Graph with weighted edges



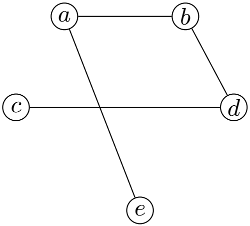
Loopless multigraph
with weighted edges
without multiple edge

Inversion from multigraphs

Construct each (multi)graph family from a simpler one,
then invert the relation between their generating functions.



Graph with weighted edges



Graph

Inversion from multigraphs

Degrees bounded by k. Define the polynomials

$$\Delta(y,\boldsymbol{\delta}) = \sum_{d=1}^{k} \delta_d \frac{y^d}{d!} \qquad P(\boldsymbol{x},w,\boldsymbol{\delta}) = \Delta(y,\boldsymbol{\delta}) \odot_{y=1} \frac{e^{\sum_{j=1}^{k} x_j y^j}}{\sqrt{1+wy^2}}$$

The generating function of graphs with marked degrees is

$$G(z,w,\boldsymbol{\delta}) = e^{-\sum_{d=1}^{k} \frac{(-w)^j}{j}\frac{x_j^2}{2}} \odot_{x_1=1} \cdots \odot_{x_k=1} e^{zP(\boldsymbol{x},w,\boldsymbol{\delta})}.$$

Finitely many variables. D-finite by stability of D-finite series.

In particular, for $\delta_0 = \cdots = \delta_{k-1} = 0$ and $\delta_k = w = 1$, extracting $[z^n]$ gives

$$\mathrm{SG}_n^{(k)} = (-1)^{kn/2} e^{x_1^2/2+\cdots+x_k^2/(2k)} \odot_{x_1=1} \cdots \odot_{x_k=1} \left([y^k] \frac{e^{-i\sum_{j=1}^{k} x_j y^j}}{\sqrt{1-y^2}} \right)^n$$

Asymptotic expansion

Integral representation

$$\frac{(2m)!}{2^m m!} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} t^{2m} e^{-t^2/2} dt$$

implies

$$e^{x^2/2} \odot_x P(x) = \sum_n \frac{(2n)!}{2^n n!} [z^{2n}] P(z) x^{2n} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} P(xt) e^{-t^2/2} dt$$

so for k-regular graphs

$$\mathrm{SG}_n^{(k)} = \frac{(-1)^{kn/2}\sqrt{k!}}{(2\pi)^{k/2}} \int_{\mathbb{R}^k} \left([y^k] \frac{e^{-i\sum_{j=1}^k t_j y^j}}{\sqrt{1-y^2}} \right)^n e^{-\sum_{j=1}^k j t_j^2/2} d\boldsymbol{t}$$

Laplace method

Among many references [de Bruijn 1958, Pemantle Wilson 2013]. Assume

- I is a compact interval neighborhood of 0
- $A(t)$ and $\phi(t)$ are analytic
- $\mathrm{Re}(\phi(t))$ reaches its minimum only at $t = 0$
- $\phi'(0) = 0$, $\phi''(0) \neq 0$

then for any $r \geq 0$

$$\int_I A(t) e^{-n\phi(t)} dt = e^{-n\phi(0)} \sqrt{\frac{2\pi}{\phi''(0)n}} \left(f_0 + f_1 n^{-1} + \cdots + f_{r-1} n^{-(r-1)} + O(n^{-r}) \right)$$

where $\Psi(t) = \left(\dfrac{\phi(t) - \phi(0)}{\phi''(0)t^2/2} \right)^{-1/2}$ and $f_j = \dfrac{(2j-1)!!}{\phi''(0)^j} [t^{2j}] A(t) \Psi(t)^{2j+1}$.

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where $\Psi(t) = \left(\frac{\phi(t) - \phi(0)}{\phi''(0) t^2 / 2} \right)^{-1/2}$ and $f_j = \frac{(2j-1)!!}{\phi''(0)^j} [t^{2j}] A(t) \Psi(t)^{2j+1}$.

Or $T(x) = x \Psi(T(x))$ and $\sum_{j \geq 0} f_j z^j = e^{zx^2/(2\phi''(0))} \odot_{x=1} A(T(x)) T'(x)$

Asymptotic expansion

Assume kn is even. After a few changes of variables

$$B_0(u, y, \boldsymbol{t}) = \sum_{\ell=1}^{k} [z^\ell] \frac{\left(1 + \frac{u}{1+t_1}\left(\frac{k-1}{2}\frac{yz}{(1+t_1)^2} + \sum_{j=2}^{k} t_j z^{j-1}\right)\right)^{k-\ell}}{\sqrt{1-z^2}} \frac{k!}{(k-\ell)!}\left(\frac{uy}{1+t_1}\right)^{\ell}$$

$$B_1(u, y, \boldsymbol{t}) = \exp\left(-\frac{\log(1 + B_0(u, y, \boldsymbol{t})) - k(k-1)\frac{u^2 t_2 y}{(1+t_1)^2}}{y^2} + \frac{(k-1)^2}{4(1+t_1)^4} + \frac{(k+1)k(k-1)}{4}u^2\right)$$

$$B_2(y, \boldsymbol{t}) = B_1\left(-\frac{1}{\sqrt{k}}, y, \boldsymbol{t}\right) + B_1\left(\frac{1}{\sqrt{k}}, y, \boldsymbol{t}\right)$$

$$\phi(t) = \frac{t^2}{2} + t - \log(1+t)$$

$$\mathrm{SG}_n^{(k)} = \frac{(kn/e)^{kn/2}\sqrt{k}}{k!^{n-1/2}} e^{-(k^2-1)/4}\left(\frac{n}{2\pi}\right)^{k/2} \int_{\mathbb{R}_{>-1}\times\mathbb{R}^{k-1}} B_2(in^{-1/2}, \boldsymbol{t}) e^{-kn\phi(t_1) - \sum_{j=2}^{k} njt_j^2/2} d\boldsymbol{t}.$$

is amenable to the Laplace method.

Asymptotic expansion

For any $k \geq 3$, the number of k-regular graphs on n vertices has asymptotic expansion

$$\mathrm{SG}_n^{(k)} \approx \frac{(nke^{-1})^{nk/2}}{k!^n} \frac{e^{-(k^2-1)/4}}{\sqrt{2}} \left(c_0^{(k)} + c_1^{(k)} n^{-1} + c_2^{(k)} n^{-2} \cdots \right)$$

Each $c_j^{(k)}$ is in $\mathbb{Q}[k, k^{-1}, \mathbb{1}_{3 \leq k}, \ldots, \mathbb{1}_{2j+2 \leq k}]$.

The fomulae for $c_j^{(k)}$ and $\mathrm{SG}_j^{(k)}$ are similar.

$$\begin{aligned}
c_0^{(k)} &= 2, \\
c_1^{(k)} &= -\frac{1}{4}k^4 + k^3 - \frac{3}{4}k^2 - \frac{5}{2}k + \frac{7}{2} - \frac{7}{6}k^{-1} + \left(\frac{1}{3}k^3 - 2k^2 + \frac{13}{3}k - 4\frac{4}{3}k^{-1} \right) \mathbb{1}_{3 \leq k} \\
&\quad + \left(\frac{1}{4}k^4 - \frac{3}{2}k^3 + \frac{11}{4}k^2 - \frac{3}{2}k \right) \mathbb{1}_{4 \leq k} \\
c_2^{(k)} &= \cdots
\end{aligned}$$

Connected k-regular graphs (work in progress)

A graph is a set of connected components, so the generating function $\mathrm{CSG}^{(k)}(z)$ of connected k-regular graphs satisfies

$$\mathrm{SG}^{(k)}(z) = e^{\mathrm{CSG}^{(k)}(z)} \qquad \text{so} \qquad \mathrm{CSG}^{(k)}(z) = \log(\mathrm{SG}^{(k)}(z)).$$

Factorially divergent series [Borinsky 2017, Bender 1975].

Consider $\quad A(z) = \sum_{n \geq 0} a_n z^n \quad$ with $\quad a_n \approx \sum_{j \geq 0} c_j \alpha^{n+\beta-j} \Gamma(n+\beta-j).$

Define the map $\quad \mathcal{M}(A)(y) = \sum_{j \geq 0} c_j y^j.$

If $B(z)$ has positive radius of convergence, then

$$\mathcal{M}(B \circ A)(y) = B'(A(y)) \mathcal{M}(A)(y).$$

Consequence. Asympt expansion of $\mathrm{SG}_n^{(k)} \mapsto$ asympt expansion of $\mathrm{CSG}_n^{(k)}$.

Conclusion

Bipartite graphs (already known by Michael Borinsky, but unpublished!)

Combinatorial interpretation of the coefficients of the asymptotic expansion (see also [Borinsky 2017], [Dovgal Nurligareev 2023])

Link between symmetric functions and inversion from multigraphs.

Thank you!

Laplace method

Assume

- I is a compact interval neighborhood of 0
- $A(t)$ and $\phi(t)$ are analytic
- $\mathrm{Re}(\phi(t))$ reaches its minimum only at $t = 0$
- $\phi'(0) = 0$, $\phi''(0) \neq 0$

then for any $r \geq 0$

$$\int_I A(t) e^{-n\phi(t)} dt = e^{-n\phi(0)} \sqrt{\frac{2\pi}{\phi''(0)n}} \left(f_0 + f_1 n^{-1} + \cdots + f_{r-1} n^{-(r-1)} + O(n^{-r}) \right)$$

where $F(z) = \sum_m f_m z^m$ is the formal power series

$$\Psi(t) = \left(\frac{\phi(t) - \phi(0)}{\phi''(0) t^2 / 2} \right)^{-1/2} \qquad T(x) = x \Psi(T(x))$$

$$F(z) = e^{zx^2/(2\phi''(0))} \odot_{x=1} A(T(x)) T'(x)$$

$$[z^k] F(z) = \frac{(2k-1)!!}{\phi''(0)^k} [t^{2k}] A(t) \Psi(t)^{2k+1}.$$

Proof

$$\int_I A(t)e^{-n\phi(t)}dt = e^{-n\phi(0)} \int_I A(t)e^{-n(\phi(t)-\phi(0))}dt$$

$e^{-(\phi(t)-\phi(0)}$ reaches its maximum 1 only at $t = 0$.
$\forall \epsilon > 0$, the contribution outside $[-\epsilon, \epsilon]$ is exponentially small.

$$\begin{aligned}
\int_{-\epsilon}^{\epsilon} A(t)e^{-nt^2/2}dt &= \int_{-\epsilon\sqrt{n}}^{\epsilon\sqrt{n}} A(tn^{-1/2})e^{-t^2/2}\frac{dt}{\sqrt{n}} \\
&= \int_{-\epsilon\sqrt{n}}^{\epsilon\sqrt{n}} \left(a_0 + a_1 tn^{-1/2} + \cdots + O(t^{2r}n^{-r})\right) e^{-t^2/2}\frac{dt}{\sqrt{n}} \\
&\approx \sqrt{\frac{2\pi}{n}} \sum_{m=0}^{r-1} a_{2m}\frac{(2m)!}{2^m m!}n^{-m} + O(n^{-r})
\end{aligned}$$

Change of variable to reduce the general case

$$\phi''(0)\frac{x^2}{2} = \phi(T(x)) - \phi(0), \qquad T(x) = x\Psi(T(x)), \qquad \Psi(t) = \frac{\phi(t) - \phi(0)}{\phi''(0)t^2/2}$$