A sketch of Dwork's Frobenius structure

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Zeta-functions

Consider the hypersurface

$$f(x_1, \ldots, x_n) = 0$$

where $f \in \mathbb{Z}[x_1, \ldots, x_n]$.
Choose a prime p and $s \geq 1$. Let N_s be the number of solutions of

$$f(x_1, \ldots, x_n) = 0 \text{ in } x_1, \ldots, x_n \in \mathbb{F}_{p^s}.$$

Define

$$\zeta_f(T) = \exp\left(\sum_{s\geq 1} \frac{N_s}{s} T^s\right).$$

Theorem (B.Dwork, 1960)

The function $\zeta_f(T)$ has the form $P(T)/Q(T)$ with $P, Q \in \mathbb{Z}[T]$ and $P(0) = Q(0) = 1$.

This proves the first of the three famous Weil conjectures.

Bernard Dwork, 1923-1998

An example

Consider the elliptic curve E given by $y^2 - x(x-1)(x-t) = 0$ with $t \in \mathbb{Z}$. Then there exists $a_p \in \mathbb{Z}$ such that

$$\zeta_E(T) = \frac{1 - a_p T + pT^2}{(1-T)(1-pT)}.$$

It turns out that always $|a_p| < 2\sqrt{p}$
(Hasse's theorem, special case of third Weil conjecture).

Corollary

Write $1 - a_p T + pT^2 = (1 - \alpha T)(1 - \beta T)$. Note $\beta = \overline{\alpha}$. Then,

$$N_s = 1 - \alpha^s - \beta^s + p^s \text{ for all } s \geq 1$$

In particular, for $y^2 = x(x-1)(x+1)$ and $p \equiv 1 \pmod{4}$ we have $\alpha = a + bi$ with $a, b \in \mathbb{Z}$ and $p = a^2 + b^2$.

Legendre family

In 1962 Dwork expanded his result to the computation of ζ-functions in families of varieties.

We illustrate Dwork's discoveries using the family of elliptic curves

$$y^2 = x(x-1)(x-\tau),$$

with parameter τ. Associate the function

$$f(\tau) = \frac{1}{\pi}\int_1^\infty \frac{dx}{\sqrt{x(x-1)(x-\tau)}}.$$

Expand as powerseries,

$$f(\tau) = \sum_{k\geq 0} \binom{-1/2}{k}^2 \tau^k.$$

Picard-Fuchs equation

The function $f(\tau)$ satisfies the hypergeometric differential equation

$$\tau(\tau-1)f''+(2\tau-1)f'+f/4=0.$$

A second solution is given by $g(\tau):=f(\tau)\log\tau+h(\tau)$ where

$$h(\tau)=\sum_{k>0}\binom{-1/2}{k}^2\left(\sum_{j=k+1}^{2k}\frac{2}{j}\right)\tau^k.$$

This solution correspond to integration over $\int_0^1$.
The matrix

$$Y(\tau)=\begin{pmatrix} f(\tau) & g(\tau) \\ \tau f'(\tau) & \tau g'(\tau) \end{pmatrix}$$

is called a *fundamental solution matrix*.

Frobenius structure I

Choose an odd prime p. There is a relation between $Y(\tau)$ and $Y(\tau^p)$ called *Frobenius structure*. Consider

$$\mathrm{Frob}_p(\tau) := Y(\tau)\begin{pmatrix}1 & \log_p(16^{1-p}) \\ 0 & p\end{pmatrix} Y(\tau^p)^{-1}.$$

Then the entries of Frob_p are powerseries in $\mathbb{Z}_p[\![t]\!]$.
Better yet,

Theorem (B.Dwork, 1962)

Modulo any power p^s the entries of Frob_p are rational functions in τ with a denominator of the form $(1-\tau)^k$.

More formally, the entries lie in the p-adic completion of $\mathbb{Z}[\tau, \frac{1}{1-\tau}]$ (*analytic elements*). Notation $\mathbb{Z}\langle\tau, \frac{1}{1-\tau}\rangle_p$

Dwork's deformation of the ζ-function

Choose $t \in \mathbb{Z}_p$ such that $t^p = t$ (Teichmüller lift) and $t \neq 0, 1$.

There are $p - 2$ such choices and they lie in different residue classes modulo p.

Theorem (B.Dwork 1962)

The matrix $\mathrm{Frob}_p(\tau)$ can evaluated at $\tau = t$ and

$$\det(1 - \mathrm{Frob}_p(t)T)$$

is the quadratic part of the ζ-function of $y^2 \equiv x(x-1)(x-t)(\mathrm{mod}\ p)$.

An example

Take $p = 13$ and consider $\mathrm{Frob}_{13}(\tau)$ modulo 13^3.

Its entries are rational functions with numerator of degree 19 and denominator $(1-\tau)^{13}$.

Substitute τ by 239
(Note: $239 \equiv 5 \pmod{13}$ and $239^{13} \equiv 239 \pmod{13^3}$).

We get the matrix

$$M := \mathrm{Frob}_{13}(239) \equiv \begin{pmatrix} 2026 & 1482 \\ 1712 & 169 \end{pmatrix} \pmod{13^3}$$

and $\det(1 - MT) \equiv 1 + 2T + 13T^2 \pmod{13^3}$.

This is the quadratic part of the ζ-function of
$y^2 \equiv x(x-1)(x-5) \pmod{13}$.

WARNING

Let $t \in \mathbb{Z}_p$ with $t^p = t$. Recall that

$$\mathrm{Frob}_p(t) = Y(\tau) \begin{pmatrix} 1 & \log_p(16^{1-p}) \\ 0 & p \end{pmatrix} Y(\tau^p)^{-1} \Big|_{\tau = t}.$$

This should NOT be read as

$$\mathrm{Frob}_p(t) = Y(t) \begin{pmatrix} 1 & \log_p(16^{1-p}) \\ 0 & p \end{pmatrix} Y(t)^{-1}.$$

Deformation method (B.Dwork 1962)

In general consider a family of hypersurfaces $f(\tau, x_1, \ldots, x_n) = 0$ with $f \in \mathbb{Z}[\tau, x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ and parameter τ. Associate a linear differential equation of order D, or a system of D linear first order differential equations (Picard-Fuchs equation, Gauss-Manin system).

Let $Y(\tau)$ be a fundamental solution matrix. Choose a prime p.

Theorem (B.Dwork, 1962)

There exists a constant $D \times D$ matrix C_0 with entries in $\mathbb{Z}_p$ such that the entries of

$$\mathrm{Frob}_p(\tau) := Y(\tau) C_0 Y(\tau^p)^{-1}$$

are analytic elements

AND the specializations $\det(1 - \mathrm{Frob}_p(t)T)$ are the 'interesting' part of the ζ-function of $f(t, x_1, \ldots, x_n) \equiv 0 (\mathrm{mod}\ p)$.

Application (Dwork 1962, Lauder 2002)

Consider family of the form

$$f(\tau, x_1, \ldots, x_n) := x_1^d + \cdots + x_n^d + \tau h(x_1, \ldots, x_n).$$

Let us compute the ζ-function of $f(1, x_1, \ldots, x_n) = 0$.

Construct Picard-Fuchs equation and choose fundamental solution matrix $Y(\tau)$ such that $Y(0)$ is identity matrix.

Clearly $f(0, x_1, \ldots, x_n) = 0$ is a diagonal hypersurface, for which $\mathrm{Frob}_p(0)$ can be computed explicitly.

We deduce that $\mathrm{Frob}_p(0) = Y(0) C_0 Y(0^p)^{-1} = C_0$ and find that

$$\mathrm{Frob}_p(1) = Y(\tau)\mathrm{Frob}_p(0) Y(\tau^p)^{-1}|_{\tau=1}.$$

This idea of Dwork was implemented by Alan Lauder.

Calabi-Yau threefolds

Consider the family of quintic 3-folds

$$\tau(1 + x_1^5 + x_2^5 + x_3^5 + x_4^5) = x_1 x_2 x_3 x_4.$$

Picard-Fuchs equation: $Ly = 0$ with

$$L = \theta^4 - 5^5\tau(\theta + 1/5)(\theta + 2/5)(\theta + 3/5)(\theta + 4/5), \quad \theta = \tau\frac{d}{d\tau}.$$

There is a basis of solutions of the form

$$y_0 = f_0(\tau), \quad y_1 = f_0(\tau)\log\tau + f_1(\tau),$$
$$y_2 = \frac{1}{2}f_0(\tau)\log^2\tau + f_1(\tau)\log\tau + f_2(\tau),$$
$$y_3 = \frac{1}{6}f_0(\tau)\log^3\tau + \frac{1}{2}f_1(\tau)\log^2\tau + f_2(\tau)\log\tau + f_3(\tau)$$

where $f_i(\tau) \in \tau\mathbb{Q}[\![\tau]\!]$ for $i = 1, 2, 3$ and

$$f_0(\tau) = \sum_{k \geq 0} \frac{(5k)!}{k!^5}\tau^k.$$

Frobenius limit at $\tau = 0$

Let $Y(\tau)$ be the matrix with entries $\theta^i y_j$ and let C_0 be as before. In 2021 Candelas, De la Ossa, Van Straten made a conjecture on families of Calabi-Yau 3-folds which imply that

$$C_0 = \begin{pmatrix} 1 & 0 & 0 & -40p^3\zeta_p(3) \\ 0 & p & 0 & 0 \\ 0 & 0 & p^2 & 0 \\ 0 & 0 & 0 & p^3 \end{pmatrix}$$

This particular case was proven by I.Shapiro (2012) and later by Kedlaya (2021).
In 2023 the speaker and Masha Vlasenko found a way to access such limits via supercongruences.

Baby example of a Frobenius structure

Consider the differential equation

$$(\tau - 1)\frac{dy}{d\tau} = \frac{1}{2}y.$$

It has the solution $y(\tau) = \frac{1}{\sqrt{1-\tau}}$.
Note that for an odd prime p,

$$\frac{y(\tau)}{y(\tau^p)} = \left(\frac{1-\tau^p}{1-\tau}\right)^{1/2}.$$

Also note that

$$\frac{1-\tau^p}{1-\tau} = (1-\tau)^{p-1}\frac{1-\tau^p}{(1-\tau)^p} = (1-\tau)^{p-1}(1+pG(\tau)),$$

where $G(\tau)$ is a rational function in $\mathbb{Z}[\tau]/(1-\tau)^p$.
Hence

$$\frac{y(\tau)}{y(\tau^p)} = (1-\tau)^{(p-1)/2}\left(1 + \frac{p}{2}G(\tau) - \frac{p^2}{8}G(\tau)^2 + \cdots\right)$$

Construction of Frobenius structure

In the following slides we consider a family of algebraic varieties given by an equation $f(\tau, x_1, \ldots, x_n) = 0$ with parameter τ.

We abbreviate $f(\tau, x_1, \ldots, x_n)$ by f and $f(\tau^p, x_1, \ldots, x_n)$ by f^σ

- We construct a module Ω_f of rational differential forms on the complement of $\{f = 0\}$.
- We consider Ω_f modulo the exact forms, denoted by $d\Omega_f$, and assume this yields a free, finitely generated module.
- On $\Omega_f / d\Omega_f$ we have the derivation θ as endomorphism.
- The matrix $M(\tau)$ of θ is the matrix of a system of first order linear differential equations (Gauss-Manin system).
- We take the p-adic completion $\widehat{\Omega}_f$ of Ω_f and define a linear map $\mathscr{C}_p : \widehat{\Omega}_f \to \widehat{\Omega}_{f^\sigma}$ (Cartier map) which descends to a linear map $\mathscr{C}_p : \widehat{\Omega}_f / d\widehat{\Omega}_f \to \widehat{\Omega}_{f^\sigma} / d\widehat{\Omega}_{f^\sigma}$.
- The matrix of this map is $\mathrm{Frob}_p(\tau)$.

Sketch by way of an example

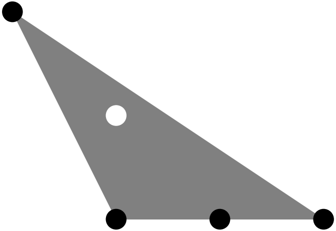
We sketch a version developed in the Dwork-crystal papers (2021-3) by the speaker and Masha Vlasenko.

We use the family of elliptic curves $E_\tau : f(\tau, x, y) = 0$ with $f(\tau, x, y) = y^2 - x(x-1)(x-\tau)$ as leading example.

Generalization to arbitrary f is more or less straightforward.

Let Δ be the Newton polytope of f, i.e convex hull of the exponent vectors $(0,2), (3,0), (2,0), (1,0)$.

Let Δ° be its interior.



with $\Delta^\circ_{\mathbb{Z}} = \{(1,1)\}$.

Regular functions

Consider the $\mathbb{Z}[\tau]$-module Ω_f generated by the functions

$$(k-1)!\frac{x^r y^s}{f^k} \text{ with } k \geq 1 \text{ and } (r,s) \in k\Delta^\circ, \quad s \text{ odd.}$$

They are actually differential forms $(k-1)!\frac{x^r y^s}{f^k}\frac{dx}{x} \wedge \frac{dy}{y}$ on the complement of $E_\tau : f = 0$, with $\frac{dx}{x} \wedge \frac{dy}{y}$ removed.
Define the derivatives by

$$d\Omega_f = x\frac{\partial}{\partial x}\Omega_f + y\frac{\partial}{\partial y}\Omega_f.$$

Side remark: when we work over $\mathbb{C}$,

$$\Omega_f / d\Omega_f \cong H^2_{\mathrm{DR}}(\mathbb{T}^2 \setminus E_\tau) \cong H^1_{\mathrm{DR}}(E_\tau) \cong \mathbb{C}\frac{dx}{y} + \mathbb{C}\frac{xdx}{y}.$$

Finiteness

In our example one can show:

Proposition

The quotient $\Omega_f/d\Omega_f$ is a free $\mathbb{Z}[\tau, \frac{1}{2\tau(1-\tau)}]$-module with generators

$$\frac{xy}{f}, \quad \frac{x^2y}{f^2}.$$

A derivation

Apply the derivation $\theta = \tau \frac{d}{d\tau}$ to Ω_f in the naive way. For example,

$$\theta\left(\frac{xy}{y^2 - x(x-1)(x-\tau)}\right) = \frac{-\tau x^2(x-1)y}{(y^2 - x(x-1)(x-\tau))^2}.$$

Trivially $\theta \circ x\frac{\partial}{\partial x} = x\frac{\partial}{\partial x} \circ \theta$ and similarly for $y\frac{\partial}{\partial y}$.

Hence θ maps $d\Omega_f$ to itself as well as $\Omega_f / d\Omega_f$.

Let $M(\tau)$ be the matrix of θ with respect to the basis $\omega_1 = \frac{xy}{f}$, $\omega_2 = \theta(\omega_1)$.

In our example we get

$$M(\tau) = \begin{pmatrix} 0 & 1 \\ \frac{\tau}{1-\tau} & \frac{\tau}{4(1-\tau)} \end{pmatrix}.$$

The equation $\frac{d\mathbf{y}}{d\tau} = M(\tau)\mathbf{y}$ is called the corresponding Gauss-Manin system.

Cartier operator

Let p be an odd prime.
We define the Cartier operator $\mathscr{C}_p$ on a Laurent series by

$$\mathscr{C}_p : \sum_{m,n} a_{m,n} x^m y^n \mapsto \sum_{m,n} a_{pm,pn} x^m y^n.$$

Lemma

We have

- $\mathscr{C}_p \circ x\frac{\partial}{\partial x} = px\frac{\partial}{\partial x} \circ \mathscr{C}_p$ and similar for $y\frac{\partial}{\partial y}$.
- $\mathscr{C}_p(g(x^p, y^p)h(x,y)) = g(x,y)\mathscr{C}_p(h(x,y))$.

Laurent series expansion

We rewrite

$$\frac{xy}{f} = \frac{xy}{y^2 - x(x-1)(x-\tau)}$$

as

$$\frac{x}{y} \times \frac{1}{1 - \frac{x(x-1)(x-\tau)}{y^2}}$$

and then as geometric expansion

$$\sum_{k \geq 0} \frac{x}{y} \times \frac{x^k(x-1)^k(x-\tau)^k}{y^{2k}}.$$

This is a Laurent series of the form $\sum_{m,n} a_{mn} x^m y^n$ with support in $-n/2 \leq m \leq -3n/2$. The functions

$$\frac{x^r y^s}{f^k}$$

can be expanded similarly.

Cartier on rational functions

What is $\mathscr{C}_p(\Omega_f)$? For example:

$$\begin{aligned}\mathscr{C}_p\left(\frac{xy}{f(\tau,x,y)}\right) &= \mathscr{C}_p\left(\frac{xyf(\tau,x,y)^{p-1}}{f(\tau,x,y)^p}\right)\\ &= \mathscr{C}_p\left(\frac{xyf(\tau,x,y)^{p-1}}{f(\tau^p,x^p,y^p)-pG(\tau,x,y)}\right)\end{aligned}$$

where $pG(\tau,x,y)=f(\tau^p,x^p,y^p)-f(\tau,x,y)^p$.
Expand in geometric series

$$\mathscr{C}_p\left(\sum_{r=0}^{\infty} xyf(\tau,x,y)^{p-1}\frac{p^r G(\tau,x,y)^r}{f(\tau^p,x^p,y^p)^{r+1}}\right)$$

$$= \sum_{r=0}^{\infty}\frac{p^r}{r!}\frac{r!}{f(\tau^p,x,y)^{r+1}}\mathscr{C}_p\left(xyf(\tau,x,y)^{p-1}G(\tau,x,y)^r\right)$$

The latter sum is in $\widehat{\Omega}_{f^\sigma}=\varprojlim \Omega_{f^\sigma}/p^s\Omega_{f^\sigma}$, the p-adic completion of Ω_{f^σ}. Here $f^\sigma=f(\tau^p,x,y)$.

Cartier matrix

So $\mathscr{C}_p : \widehat{\Omega}_f \to \widehat{\Omega}_{f^\sigma}$.

Because of $\mathscr{C}_p \circ x\frac{\partial}{\partial x} = px\frac{\partial}{\partial x} \circ \mathscr{C}_p$ we have $\mathscr{C}_p : d\widehat{\Omega}_f \to d\widehat{\Omega}_{f^\sigma}$.

Hence $\mathscr{C}_p : \widehat{\Omega}_f / d\widehat{\Omega}_f \to \widehat{\Omega}_{f^\sigma} / d\widehat{\Omega}_f$.

In our standard example $\widehat{\Omega}_f$ and $\widehat{\Omega}_{f^\sigma}$ are modules over $\mathbb{Z}_p\langle\tau, \frac{1}{\tau(\tau-1)}\rangle_p$, the p-adic completion of $\mathbb{Z}[t, \frac{1}{2\tau(\tau-1)}]$.

Let $\mathrm{Frob}_p(\tau)$ be the matrix of $\mathscr{C}_p$ with respect to suitable bases.

In our example we could use the bases

$$\omega_1 = \frac{xy}{f},\ \omega_2 = \theta(\omega_1) \quad \text{and} \quad \omega_1^\sigma = \frac{xy}{f^\sigma},\ \omega_2^\sigma = \theta(\omega_1)^\sigma.$$

Formula for Frobenius structure

Remark that $\mathscr{C}_p \circ \theta = \theta \circ \mathscr{C}_p$.
We can write this relation in terms of matrices and get

$$\theta \mathrm{Frob}_p(\tau) = M(\tau)\mathrm{Frob}_p(\tau) - p\mathrm{Frob}_p(\tau)M(\tau^p). \qquad \text{(A)}$$

Let $Y(\tau)$ be a fundamental solution matrix of the Gauss-Manin system.
Then (A) implies that there exists a constant matrix C_0 such that

$$\mathrm{Frob}_p(\tau) = Y(\tau)C_0Y(\tau^p)^{-1}$$

In our example the entries of $\mathrm{Frob}_p(\tau)$ are a priori in $\mathbb{Z}_p\langle\tau, \frac{1}{\tau(\tau-1)}\rangle_p$.
From the formula we also see that the entries are in $\mathbb{Q}_p[\![\tau]\!]$. Hence the entries lie in $\mathbb{Z}_p\langle\tau, \frac{1}{\tau-1}\rangle_p$.

Specialization of τ

On this page we write $f(\tau)$ for $f(\tau, x_1, \ldots, x_n)$.
Let $t \in \mathbb{Z}_p$ be such that $t^p = t$. We specialize τ to t.
Then

$$\mathscr{C}_p : \widehat{\Omega}_{f(\tau)} \to \widehat{\Omega}_{f(\tau^p)}$$

specializes to

$$\mathscr{C}_p : \widehat{\Omega}_{f(t)} \to \widehat{\Omega}_{f(t)},$$

i.e $\mathscr{C}_p$ becomes an endomorphism of the $\mathbb{Z}_p$-module $\widehat{\Omega}_{f(t)}$
and also of the $\mathbb{Z}_p$-module $\widehat{\Omega}_{f(t)}/d\widehat{\Omega}_{f(t)}$.

Its characteristic polynomial gives us the 'interesting' part of the ζ-function.
(This requires a long non-trivial calculation originating from Dwork's 1960 paper)

Applications of Frobenius structure

- 'p-adic cycles'
  (Dwork (1969))
- p-integrality of mirror maps
  (Vologodsky (2008), FB-Vlasenko (2022)).
- p-integrality of instanton numbers
  (Stienstra (2003), Vologodsky (2008), FB-Vlasenko(2022))

Powerseries solutions

Recall the differential equation $\tau(\tau-1)y'' + (2\tau-1)y' + y/4 = 0$ having solution

$$f(\tau) = \sum_{k\geq 0} \binom{-1/2}{k}^2 \tau^k$$

as solution near $\tau = 0$. A second solution contains $\log\tau$.
Near $\tau = -1$ we have a basis of two powerseries solutions

$$1 - \frac{u^2}{16} - \frac{u^3}{16} - \frac{83u^4}{1536} - \frac{117u^5}{2560} - \frac{1593u^6}{40960} - \frac{687u^7}{20480} - \frac{107739u^8}{3670016} + O\left(u^9\right)$$

and

$$u + \frac{3u^2}{4} + \frac{9u^3}{16} + \frac{7u^4}{16} + \frac{903u^5}{2560} + \frac{3003u^6}{10240} + \frac{10241u^7}{40960} + \frac{891u^8}{4096} + O\left(u^9\right)$$

with $u = \tau + 1$. Every power of every prime occurs in the denominator of some coefficient.

p-adic cycles

Add $1/4$ times the second solution to the first,

$$1+\frac{u}{4}+\frac{u^2}{8}+\frac{5u^3}{64}+\frac{85u^4}{1536}+\frac{87u^5}{2048}+\frac{141u^6}{4096}+\frac{949u^7}{32768}+\frac{91845u^8}{3670016}+O\left(u^9\right).$$

Experimentally we see that 5 does NOT occur in the denominators.

Theorem (Dwork, 1969)

Let $a \in \mathbb{Z}_p$ be such that $a(a-1)$ is a p-adic unit and $y^2 \equiv x(x-1)(x-a) \pmod{p}$ is not super-singular. Then, up to a constant factor, there is a unique solution in $\mathbb{Z}_p[\![\tau - a]\!]$.

Dwork called these unique solutions 'p-adic cycles'.
In general their coefficients do not lie in $\mathbb{Q}$, but in $\mathbb{Q}_p$
That the solution at $\tau = -1$ is an exception may have to do with the fact that $y^2 = x(x-1)(x+1)$ is a CM-elliptic curve.
This may also explain why the above solution seems to have no primes $p \equiv 1 \pmod{4}$ in its coefficient denominators.

A famous example

P. Candelas, X. de la Ossa, P. Green, L. Parkes, *An exactly soluble superconformal theory from a mirror pair of Calabi–Yau manifolds*, Phys. Lett. B 258 (1991), no. 1-2, 118 - 126.

Recall the Picard-Fuchs equation associated to the quintic 3-fold.

$$\theta^4 y - 5^5\tau(\theta + 1/5)(\theta + 2/5)(\theta + 3/5)(\theta + 4/5)y = 0, \quad \theta = \tau\frac{d}{d\tau}.$$

Powerseries solution

$$y_0(\tau) = \sum_{k\geq 0} \frac{(5k)!}{k!^5}\tau^k.$$

Other solutions

Basis of solutions given by $y_i(\tau),\ i = 0, 1, 2, 3$, where $y_0(\tau)$ is already given. Next, $y_1(\tau) = y_0(\tau) \log \tau + f_1(\tau)$ where

$$f_1(\tau) = \sum_{n \geq 1} \frac{(5n)!}{(n!)^5} \tau^n \sum_{j=n+1}^{5n} \frac{1}{j}.$$

Similarly $y_2(\tau) = \frac{1}{2} y_0(\tau) \log^2 \tau + f_1(\tau) \log \tau + f_2(\tau)$.

Let us define
$q(\tau) = \exp(y_1(\tau)/y_0(\tau)) = \tau \exp(f_1(\tau)/y_0(\tau) \in \tau\mathbb{Q}[\![\tau]\!]$. This is called the *canonical coordinate*.

Theorem, Lian-Yau (1996)

We have $q(\tau) = \tau + 770\tau^2 + 1014275\tau^3 + \cdots \in \tau\mathbb{Z}[\![\tau]\!]$.

p-integrality of the mirror map

Let $Y(\tau)$ be the 4×4-matrix with entries $\theta^i y_j$ with $i, j = 0, 1 \ldots, 4$.
Suppose our equation has a Frobenius structure, i.e. there is a constant 4×4-matrix C_0 such that $\mathrm{Frob}_p(\tau) = Y(\tau) C_0 Y(\tau^p)^{-1}$ has entries in $\mathbb{Z}_p[\![\tau]\!]$.

Theorem (FB-Vlasenko 2022)

Let $\phi_0(\tau), \ldots, \phi_3(\tau)$ be the top row of $\mathrm{Frob}_p(\tau)$. Suppose that the top left entry of C_0 is 1 and p divides $\phi_i(\tau)$ for $i = 1, 2, 3$. Then

$$q(\tau) \in \mathbb{Z}_p[\![\tau]\!].$$

Yukawa coupling

Express τ as power series in q (mirror map),

$$\tau = q - 770q + 171525q^3 - 81623000q^4 + \cdots$$

Trivially $y_0(\tau)/y_0(\tau) = 1$ and by construction, $y_1(\tau)/y_0(\tau) = \log q$. But

$$y_2(\tau)/y_0(\tau) = \frac{1}{2}\log^2 q + 575q + \frac{975375}{4}q^2 + \frac{1712915000}{9}q^3 + \cdots$$

Apply θ_q^2, where $\theta_q = q\frac{d}{dq}$.
Define $K(q) = 5\theta_q^2(y_2/y_0)$, the *Yukawa coupling*. We get

$$K(q) = 5 + 2875q + 4876875q^2 + 8564575000q^3 + \cdots$$

Rewrite in Lambert expansion,

$$K(q) = 5 + \frac{k_1 q}{1-q} + \frac{k_2 q^2}{1-q^2} + \frac{k_3 q^3}{1-q^3} + \cdots$$

The numbers $a_n := k_n/n^3$ are called the *instanton numbers*.

Instanton numbers

We have

$$\begin{aligned} a_1 &= 2875 \\ a_2 &= 609250 \\ a_3 &= 317206375 \\ a_4 &= 242467530000 \\ &\vdots \end{aligned}$$

Physicist's prediction

For every d the number a_d is an integer that counts the number of degree d rational curves that lie on a generic hypersurface (3-fold) of degree 5 in $\mathbb{P}^4$.

Known for $d = 1$ (H.Schubert, 1886), for $d = 2$ (S.Katz, 1986) and $d = 3$ (G.Ellingsrud, S.Strømme, 1993). PHYSICS WINS!

Integrality

In 1995 Givental provided a link between Gromov-Witten invariants of a family of Calabi-Yau threefolds. But these invariants are a priori rational numbers.

Conjecture

The instanton numbers are in $\mathbb{Z}$.

Proof idea: use Dwork's p-adic cohomology theory (Jan Stienstra, 2003). Kontsevich, Schwarz and Vologodsky took this up and developed ideas to solve this problem around 2007.

Theorem (FB - Masha Vlasenko, 2022)

The denominators of the instanton numbers can only contain prime divisors $2, 3, 5$.

END

Thank you!