A functional equation approach to reflected Brownian motion

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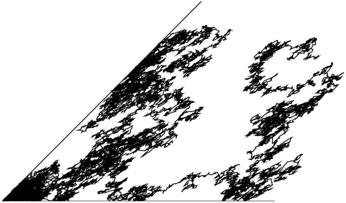
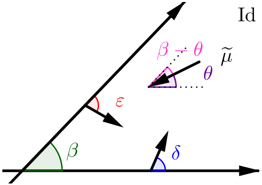
Aim of the talk

Reflected Brownian motion

Introduced in the 80ies by Harrison, Varadhan, Williams





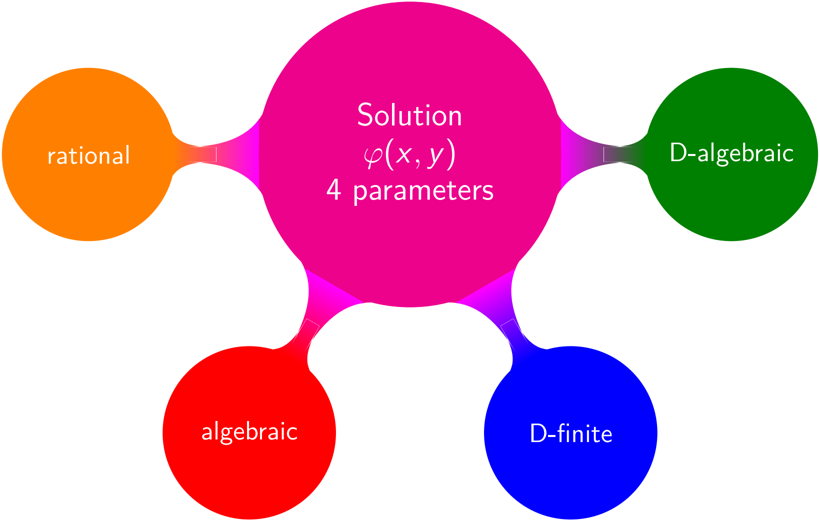
Functional equations

Characterize the distribution, stationary density, etc.

$$-\gamma(x,y)\varphi(x,y) = \gamma_1(x,y)\varphi_1(y)+\gamma_2(x,y)\varphi_2(x)$$

- Complex analysis (boundary value problem)
- Combinatorial tools (Tutte's invariants)
- Galois theory of difference equations
- and more!

A quest of simplicity



A quest of simplicity

Talk based on joint work with...




ON THE STATIONARY DISTRIBUTION OF REFLECTED BROWNIAN MOTION IN A WEDGE: DIFFERENTIAL PROPERTIES

M. BOUSQUET-MÉLOU, A. ELVEY PRICE, S. FRANCESCHI, C. HARDOUIN, AND K. RASCHEL

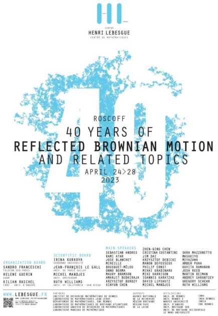
Abstract. We consider the classical problem of determining the stationary distribution of the semimartingale reflected Brownian motion (SRBM) in a two-dimensional wedge. Under standard assumptions on the parameters of the model (opening of the wedge, angles of the reflections, drift), we study the algebraic and differential nature of the Laplace transform of this stationary distribution. We derive necessary and sufficient conditions for this Laplace transform to be rational, algebraic, differentially finite or more generally differentially algebraic. These conditions are explicit linear dependencies between the angles of the model.

A complicated integral expression for this Laplace transform has recently been obtained by two authors of this paper. In the differentially algebraic case, we provide a simple, explicit integral-free expression in terms of a hypergeometric function. It specializes to earlier expressions in several classical cases: the skew-symmetric case, the orthogonal reflections case and the sum-of-exponential densities case (corresponding to the so-called Dieker-Moriarty conditions on the parameters). This paper thus closes, in a sense, the quest of all "simple" cases.

To prove these results, we start from a functional equation that the Laplace transform satisfies, to which we apply tools from diverse horizons. To establish differential algebraicity, a key ingredient is Tutte's *invariant approach*, which originates in enumerative combinatorics. It allows us to express the Laplace transform (or its square) as a rational function of a certain *canonical invariant*, a hypergeometric function in our context. To establish differential transcendence, we turn the functional equation into a difference equation and apply Galoisian results on the nature of the solutions to such equations.

+ other works from Guy Fayolle, Sandro Franceschi, Tomoyuki Ichiba, Ioannis Karatzas

40 years of reflected Brownian motion!



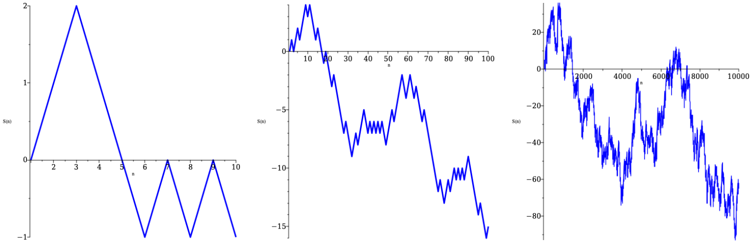
@Mireille: thanks for the inspiring talk!

Reflected Brownian motion: key ideas

Convergence of random walks

Discrete random walks

$S(n) = S(n-1) + X(n)$, with $S(0) = 0$ and $X(1), X(2), \ldots$ iid

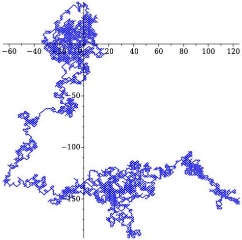


Central limit thm and Donsker ($\mathbb{E}X(n) = 0$ & $\mathbb{E}X(n)^2 = 1$)

$$\frac{S(n)}{\sqrt{n}} \to \mathcal{N}(0,1)$$ Gaussian distribution

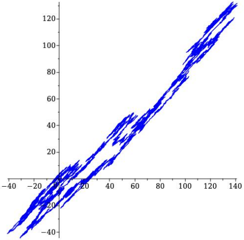
$$\left(\frac{S(\lfloor tn \rfloor)}{\sqrt{n}}\right)_{t \geqslant 0} \to (B(t))_{t \geqslant 0}$$ Gaussian process

In dimension 2



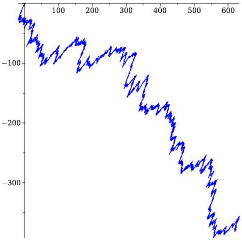
Pair of independent 1d BM

$$(B_1(t), B_2(t))$$



Correlated BM

$$L(B_1(t), B_2(t))$$



Correlated BM with drift

$$(B_1(t), B_2(t)) + \mu t$$

A basic example

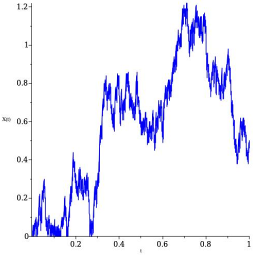
$(|B(t)|)_{t\geqslant 0}$ is a random process with values on $\mathbb{R}_+$

Tanaka formula

$|B(t)| = W(t) + L^0(t)$, with

- $W(t)$ a Brownian motion
- $L^0(t)$ local time at 0

$$L^a(t) = \lim_{\epsilon\to 0} \frac{1}{2\epsilon}\int_0^t \mathbb{1}_{\{a-\epsilon\leqslant W(s)<a+\epsilon\}} ds$$



Reflected Brownian motion in dim 1

$X(t) = X(0) + \sqrt{\sigma}B(t) + \mu t + L^0(t)$, with

- $X(0)$ starting point
- σ variance and μ drift

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A basic example

$(|B(t)|)_{t \geqslant 0}$ is a random process with values on $\mathbb{R}_+$

discrete time

Tanaka formula

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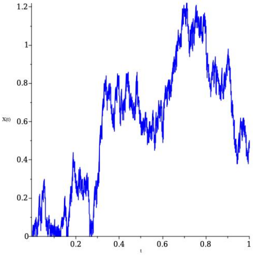


Reflected Brownian motion in dim 1

$X(t) = X(0) + \sqrt{\sigma} B(t) + \mu t + L^0(t)$, with

- $X(0)$ starting point
- σ variance and μ drift

Reflected Brownian motion in dim 2

Reflected Brownian motion in dim 1

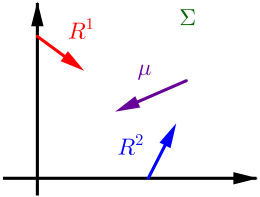
$$X(t) = X(0) + \sqrt{\sigma}B(t) + \mu t + L^0(t)$$

Obliquely reflected Brownian motion in dim 2 [Varadhan Williams '85]

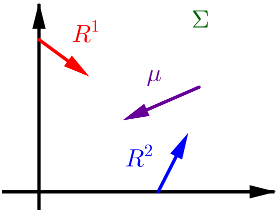
$$X(t) =$$

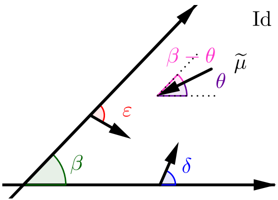
$$X(0) + B(t) + \mu t + (R^1|R^2)\begin{pmatrix} L^{Ox}(t) \\ L^{Oy}(t) \end{pmatrix}$$

- $X(0)$ starting point in $\mathbb{R}_+^2$
- $B(t) = (B_1(t), B_2(t))$ covariance Σ
- μ drift in $\mathbb{R}^2$
- oblique reflections R^1 and R^2
- $L^{Ox}(t)$ local time on the axis Ox



The parameters





Cone normalized	Cone arbitrary
Covariance arbitrary	Covariance normalized
9 parameters?	4 angles

$$\gamma(x,y) = \frac{1}{2}(x,y)\Sigma\begin{pmatrix} x \\ y \end{pmatrix} + \langle \mu, (x,y)\rangle$$ quadratic form

$$\gamma_1(x,y) = \langle R^1, (x,y)\rangle$$
$$\gamma_2(x,y) = \langle R^2, (x,y)\rangle$$ linear forms

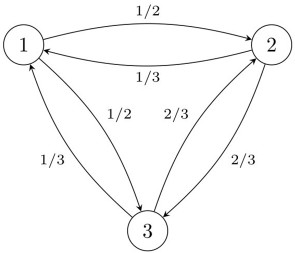
Invariant distribution

Invariant distribution = asymptotic proportion of time

$$\pi(A) = \lim_{t\to\infty} \frac{1}{t}\int_0^t \mathbb{1}_A(X(s))ds$$

In classical Markov chain theory...

$\pi P = \pi$, with P transition matrix



Analogue in continuous time and space

Transition matrix $\Longrightarrow$ Transition semigroup

Invariance $\pi P = \pi \Longrightarrow$ PDE called basic adjoint relationship

Functional equation

The bivariate Laplace transform

Density of the stationary distribution $\pi(u, v)$

$$\varphi(x, y) = \iint_{\mathbb{R}_+^2} e^{xu+yv}\pi(u, v)dudv$$

The functional equation [Dai Miyazawa '11]

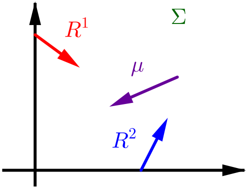
$$-\gamma(x, y)\varphi(x, y) = \gamma_1(x, y)\varphi_1(y) + \gamma_2(x, y)\varphi_2(x)$$

$$\gamma(x, y) = \frac{1}{2}(x, y)\Sigma\begin{pmatrix} x \\ y \end{pmatrix} + \langle\mu, (x, y)\rangle$$

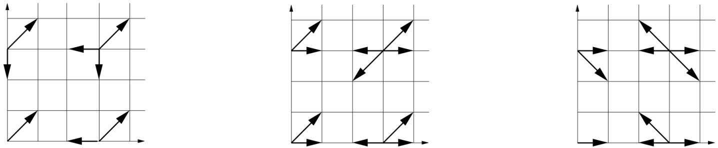
$$\gamma_1(x, y) = \langle R^1, (x, y)\rangle$$

$$\gamma_2(x, y) = \langle R^2, (x, y)\rangle$$

$$\varphi_1(y) = A\varphi(0, y) + B$$



Some history, from discrete random walks to continuous diffusions



- Transition probability in the direction (i, j): $p_{i,j}$
- Characteristic polynomial: $\sum_{(i,j)\in\mathbb{Z}^2} p_{i,j} x^i y^j$
- Reflection (or killing) on the boundary

Main objectives in the early papers

Computation of the stationary distribution $(\pi_{i,j})_{i,j\geqslant 0}$

Malyshev's work

Доклады Академии наук СССР
1969. Том 187, № 6

УДК 517.948.32 — *МАТЕМАТИКА*

В. А. МАЛЫШЕВ

О РЕШЕНИИ ДИСКРЕТНЫХ УРАВНЕНИЙ ВИНЕРА — ХОПФА В ЧЕТВЕРТЬ-ПЛОСКОСТИ

(Представлено академиком А. Н. Колмогоровым 16 XII 1968)

Дискретные уравнения Винера — Хопфа в четверть-плоскости имеют вид

$$\eta_{ij} = \sum_{k,\, l=0}^{\infty} a_{i-k,\, j-l}\xi_{kl}, \quad ij = 0, 1, 2, \ldots$$

$$\eta(x, y) = \sum_{ij=0}^{\infty} \eta_{ij}\, x^i y^j; \; \xi(x, y) = \sum_{k,\, l=0}^{\infty} \xi_{kl} x^k y^l;$$

$$\xi(y) = \xi(0, y), \quad \xi(x, 0) = \tilde{\xi}(x),$$

$$b(y) = a_{-1,1}y + a_{-1,0} + a_{-1,1}\frac{1}{y}; \quad \tilde{b}(x) = a_{1,-1}x + a_{0,-1} + a_{-1-1}\frac{1}{x}.$$

Несложные выкладки показывают, что система (1) эквивалентна следующему уравнению в производящих функциях (символах):

$$\eta(x, y) = a(x, y)\,\xi(x, y) - \frac{1}{x}\, b(y)\,\xi(y) - \frac{1}{y}\,\tilde{b}(x)\,\tilde{\xi}(x) + a_{-1,-1}\frac{\xi_{00}}{xy}. \quad (4)$$

AN ANALYTICAL METHOD IN THE THEORY OF TWO-DIMENSIONAL POSITIVE RANDOM WALKS

V. A. Malyshev — UDC 519.217

1. Introduction

Consider a homogeneous discrete-time Markov chain whose set of states is a discrete lattice in a quarter-plane, i.e., the set $Z_{++}^2 = \{(i,j) : i,\ j \ge 0 \text{ are integers}\}$. Let $P(k', l'/k, l)$ be the one-step transition probability from state (k, l) to (k', l'). We assume that $P(k', l'/k, l) = 0$ if either $|k' - k| \ge 2$ or $|l' - l| \ge 2$ and if $(k, l) \ne (0, 0)$. Also, we assume that the following homogeneity condition holds:

$$P(k', l'/k, l) = \begin{cases} p_{ij} & \text{for } i = k' - k, \; j = l' - l, \; k, l \geqslant 1, \\ p_{ij}' & \text{for } i = k' - k, \; j = l', \; l = 0, \; k \geqslant 1, \\ p_{ij}'' & \text{for } i = k', \; j = l' - l, \; k = 0, \; l \geqslant 1. \end{cases}$$

The number of nonzero probabilities $P(i, j/0, 0) = p_{ij}^0$ is finite.

$$r(\omega) - \tilde{r}(\omega) = -\frac{\pi_{00}x(\omega)\,q_0(\omega)}{q(\omega)\,\pi_1(\omega)} \overset{\text{def}}{=} \theta(\omega), \qquad (25)$$

$$r(\omega) = r(\xi\omega) = r(\omega + \omega_1), \; \tilde{r}(\omega) = \tilde{r}(\eta\omega) = \tilde{r}(\omega + \omega_1).$$

We now proceed to determine $r(\omega)$ and $\tilde{r}(\omega)$.

Introduction of the key ideas

- Functional equation
- Riemann surface $\sum_{(i,j)\in\mathbb{Z}^2} p_{i,j}x^i y^j = 1$
- Difference equations

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The little yellow book

Unifies Malyshev's approach and Fayolle-Iasnogorodski's '79 paper

Z. Wahrscheinlichkeitstheorie verw. Gebiete
47, 325–351 (1979)

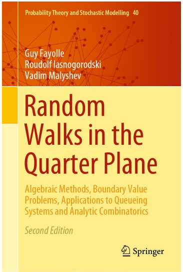
Zeitschrift für
Wahrscheinlichkeitstheorie
und verwandte Gebiete

Two Coupled Processors:
The Reduction to a Riemann-Hilbert Problem

Guy Fayolle[1] and Roudolf Iasnogorodski[2]

[1] Iria-Laboria, Domaine de Voluceau Rocquencourt, F-78150 Le Chesnay
[2] Université d'Orléans, Orléans, France

Résumé. Beaucoup de problèmes liés au couplage de processeurs conduisent à des équations fonctionnelles. En général, les fonctions inconnues représentent les fonctions génératrices d'un processus stationaire. Nous étudions ici un problème particulier, mais la méthode proposée est applicable à des cas très généraux de marches aléatoires à deux dimensions.



- Statement of a difference equation boundary value problem
- Explicit resolution in terms of contour integrals

Now joining together (5.4.9), (5.4.10), (5.4.17) and (5.4.18), we obtain the final *reduced* BVP

$$\boxed{\rho(t)K(t) - \rho(\alpha(t))K(\alpha(t)) = k(t), \quad t \in \mathcal{M},} \qquad (5.4.19)$$

Theorem 5.4.3 *Under the condition (5.4.4), the function π is given by*

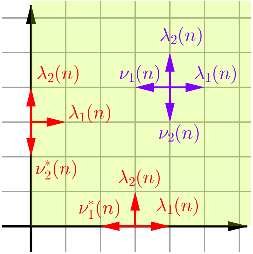
$$\pi(x) = \frac{R(x)H(x)}{2i\pi P(x)S(x)} \int_{\mathcal{M}_d} \frac{k(t)w'(t)dt}{H^+(t)K(\bar{t})(w(t)-w(x))} + T(x), \ \forall x \in G_{\mathcal{M}}, \quad (5.4.21)$$

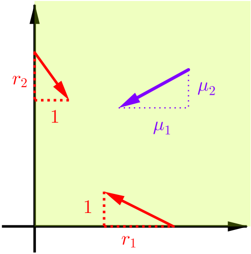
where

(i) $\mathcal{M}_d$ *denotes the portion of the curve* $\mathcal{M}$ *located in the lower half-plane* $\Im z \le 0$;
(ii) k *and* K *have been introduced in (5.4.20);*
(iii) w *is solution of the BVP (5.2.39) on the curve* $\mathcal{M}$ *(see Theorem 5.2.7);*

Early study of reflected BM

[Harrison Reiman Varadhan Williams 80ies]





$$\lambda_i(n), \nu_i(n) \to \frac{1}{2},\ \sqrt{n}(\lambda_i - \nu_i) \to \mu_i \text{ and } \nu_i^*(n) \to \frac{r_i+1}{2}$$

[Reiman '84]

Analytic approach

In the 80ies

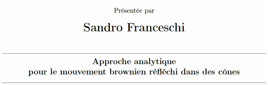
Independent works of [Foschini '82, Foddy '84, Baccelli Fayolle '87]

In some very particular cases

- realized functional equation approach inspired by F-I-M
- first few results

Sandro Franceschi & collaborators

Optimal hypotheses + other models (transient)



Using different tools

Combinatorics

[Bousquet-Mélou Elvey Price etc.]

Galois theory

[Dreyfus Hardouin Roques Singer etc.]

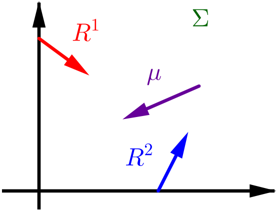
Complex analysis

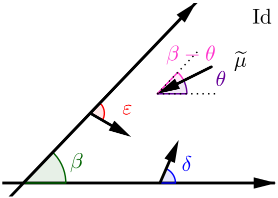
[Fayolle Franceschi R. etc.]

Probability

Main results: characterization and computation of the solutions

The parameters



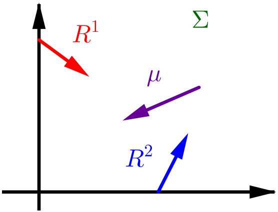


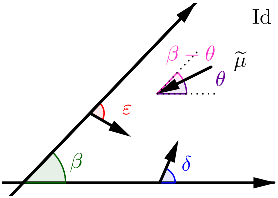
Cone normalized
Covariance arbitrary
9 parameters?

Cone arbitrary
Covariance normalized
4 angles

$$\gamma(x,y) = \frac{1}{2}(x,y)\Sigma\begin{pmatrix} x \\ y \end{pmatrix} + \langle \mu, (x,y)\rangle$$

$$\gamma_1(x,y) = \langle R^1, (x,y)\rangle$$
$$\gamma_2(x,y) = \langle R^2, (x,y)\rangle$$





Cone normalized
Covariance arbitrary
9 parameters?

Cone arbitrary
Covariance normalized
4 angles

$$\gamma(x,y) = \frac{1}{2}(\sigma_{11}x^2 + 2\sigma_{12}xy + \sigma_{22}y^2) + \mu_1 x + \mu_2 y$$

$$\gamma_1(x,y) = r_{11}x + r_{21}y$$
$$\gamma_2(x,y) = r_{12}x + r_{22}y$$

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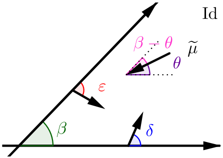
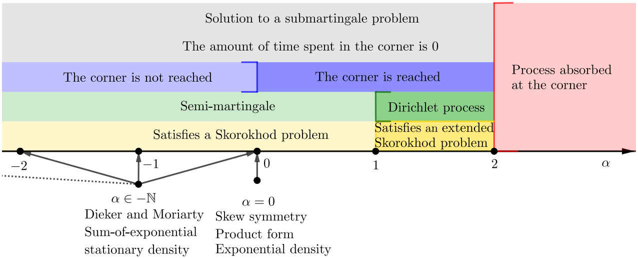
A geometric quantity

$$\alpha = \frac{\delta + \varepsilon - \pi}{\beta}$$





Observation

Two new parameters govern the evolution of reflected BM

$\alpha_1 + \alpha_2 = 2\alpha - 1$

$$\alpha_1 = \frac{2\varepsilon + \theta - \beta - \pi}{\beta}$$

$$\alpha_2 = \frac{2\delta - \theta - \pi}{\beta}$$

Main results (2/3)

$$\alpha = \frac{\delta + \varepsilon - \pi}{\beta}, \quad \alpha_1 = \frac{2\varepsilon + \theta - \beta - \pi}{\beta}, \quad \alpha_2 = \frac{2\delta - \theta - \pi}{\beta}$$

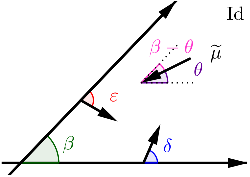
Theorem

Necessary and sufficient condition for the Laplace transform $\varphi(x, y)$ to be rational/algebraic/D-finite/D-algebraic

	D-algebraic	D-finite	Algebraic	Rational
$\frac{\beta}{\pi} \notin \mathbb{Q}$	Condition $(\mathcal{C})$	Condition $(\mathcal{C}')$	$\alpha \in \mathbb{Z}$, or $\alpha_1, \alpha_2 \in \mathbb{Z}$	$\alpha \in \mathbb{Z}$
$\frac{\beta}{\pi} \in \mathbb{Q}$	always	Condition $(\mathcal{C})$		$\alpha \in \mathbb{Z}$

$(\mathcal{C})$ $\alpha \in \mathbb{Z} + \frac{\pi}{\beta}\mathbb{Z}$ or $\alpha_1, \alpha_2 \in \mathbb{Z} + \frac{\pi}{\beta}\mathbb{Z}$

$(\mathcal{C}')$ $\alpha \in -\mathbb{N}_0 + \frac{\pi}{\beta}\mathbb{Z}$ or $\alpha_1, \alpha_2 \in \mathbb{Z} \cup (-\mathbb{N} + \frac{\pi}{\beta}\mathbb{Z})$



Main results (2/3)

$$\alpha = \frac{\delta + \varepsilon - \pi}{\beta}, \quad \alpha_1 = \frac{2\varepsilon + \theta - \beta - \pi}{\beta}, \quad \alpha_2 = \frac{2\delta - \theta - \pi}{\beta}$$

Theorem

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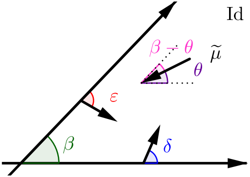
$\frac{\beta}{\pi} \notin \mathbb{Q}$	Condition $(\mathcal{C})$	Condition $(\mathcal{C}')$	$\alpha \in \mathbb{Z}$, or $\alpha_1, \alpha_2 \in \mathbb{Z}$	$\alpha \in \mathbb{Z}$
$\frac{\beta}{\pi} \in \mathbb{Q}$	always linear relations	Condition $(\mathcal{C})$		$\alpha \in \mathbb{Z}$

$(\mathcal{C})$ $\alpha \in \mathbb{Z} + \frac{\pi}{\beta}\mathbb{Z}$ or $\alpha_1, \alpha_2 \in \mathbb{Z} + \frac{\pi}{\beta}\mathbb{Z}$

$(\mathcal{C}')$ $\alpha \in -\mathbb{N}_0 + \frac{\pi}{\beta}\mathbb{Z}$ or $\alpha_1, \alpha_2 \in \mathbb{Z} \cup (-\mathbb{N} + \frac{\pi}{\beta}\mathbb{Z})$



Main results (3/3)

Theorem

Condition $(\mathcal{C}) \implies$ elementary expression of $\varphi(x,y)$ in terms of (possibly irrational) powers

- $\varphi_1(y)$ is a rational function in y and $T_{\pi/\beta}(ay+b)$, where a and b are explicit and $T_c(z)$ is the generalized Chebychev polynomial

$$T_c(z) = \frac{1}{2}\left(\left(z+\sqrt{z^2-1}\right)^c + \left(z-\sqrt{z^2-1}\right)^c\right)$$

- Similar statement for $\varphi_2(x)$
- Functional equation gives $\varphi(x,y)$

$$-\gamma(x,y)\varphi(x,y) = \gamma_1(x,y)\varphi_1(y) + \gamma_2(x,y)\varphi_2(x)$$

Corollary

Condition $(\mathcal{C}) \implies \varphi(x,y)$, $\varphi_1(y)$ and $\varphi_2(x)$ all D-algebraic

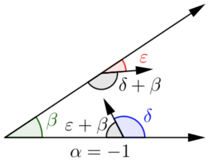
Three examples

Rational cases: $\alpha \in \mathbb{Z}$

Theorem

If $\alpha \in \mathbb{Z}$, then

- $\varphi_1(y) = \frac{1}{\text{polynomial}(y)}$
- Corresponding density: $\sum \kappa v^i e^{-av}$



boundary invariant measure

- If all poles of $P(y)$ are distinct, then sum-of-exponential density [Dieker Moriarty '09]
- Multiple poles may occur. For instance if $\delta+\varepsilon+\beta=\pi$ and $\theta-2\delta=2\beta+\pi$, then

$$\varphi_1(y) = \frac{\kappa}{(a-y)^2}$$

Erlang distribution, with density $a^2 v e^{-av}$

A D-finite case: $\delta + \varepsilon + \beta = 2\pi$

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Theorem

Under the above assumption

$$\varphi_1(y) = \kappa \frac{T_{\pi/\beta}(ay+b) - A}{(B-y)(C-y)}$$

with all constants explicit and

$$T_c(z) = \frac{1}{2}\left(\left(z+\sqrt{z^2-1}\right)^c + \left(z-\sqrt{z^2-1}\right)^c\right)$$

- $\varphi_1(y)$ is D-finite, even algebraic if $\pi/\beta \in \mathbb{Q}$
- Linear differential equation satisfied by T_c yields an explicit order-4 recurrence relation for the moments

probabilistic application

A double algebraic case: $\alpha_1 = \alpha_2 = 0$

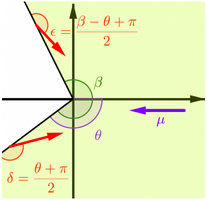
$\alpha_1 = \frac{2\varepsilon + \theta - \beta - \pi}{\beta}$

$\alpha_2 = \frac{2\delta - \theta - \pi}{\beta}$

Theorem

If $\alpha_1 = \alpha_2 = 0$, we have

- $\varphi_1(y) = \frac{\kappa}{\sqrt{A-y}}$
- density $\frac{e^{-v/A}}{\sqrt{v}}$



- Explicit density in polar coordinate

$$\pi(r\cos a, r\sin a) = \frac{\cos(\frac{\theta - a}{2})}{\sqrt{r}} \exp\left(-c \cdot r \cdot \cos^2(\frac{\theta - a}{2})\right)$$

- Extends special case of [Harrison '78]

Some ideas of proofs

A functional equation for $\varphi_1(y)$

Algebraic manipulations starting from the functional equation

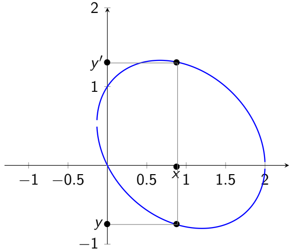
$$-\gamma(x,y)\varphi(x,y) = \gamma_1(x,y)\varphi_1(y) + \gamma_2(x,y)\varphi_2(x)$$

Quadratic kernel

$\gamma(x,y) = \frac{1}{2}(\sigma_{11}x^2+2\sigma_{12}xy+\sigma_{22}y^2)+\mu_1 x+\mu_2 y$

An involution

$y \leftrightarrow y'$, with $\gamma(x,y) = \gamma(x,y') = 0$



- If $\gamma(x,y) = \gamma(x,y')$, by elimination of $\varphi_2(x)$

$$\frac{\gamma_1(x,y)}{\gamma_2(x,y)}\varphi_1(y) = \frac{\gamma_1(x,y')}{\gamma_2(x,y')}\varphi_1(y')$$

- Equivalently, for some algebraic functions $A(y)$ and $B(y)$

$$\varphi_1(B(y)) = A(y)\varphi_1(y)$$

Rational uniformisation of $\gamma(x, y) = 0$

The Riemann surface

$$\gamma(x, y) = \tfrac{1}{2}(\sigma_{11}x^2 + 2\sigma_{12}xy + \sigma_{22}y^2) + \mu_1 x + \mu_2 y = 0$$

can be parametrized by
$$\begin{cases} X(s) &= a_1 + b_1\left(s + \frac{1}{s}\right) \\ Y(s) &= a_2 + b_2\left(\frac{s}{e^{i\beta}} + \frac{e^{i\beta}}{s}\right) \end{cases}$$

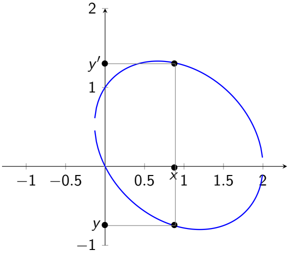
$q = e^{2i\beta}$

Involutions

$X(\frac{1}{s}) = X(s)$ and $Y(\frac{q}{s}) = Y(s)$

The involution $y \leftrightarrow y'$

$\gamma(x, y) = \gamma(x, y') = 0$ with $x = X(s)$
then $y = Y(s)$ and $y' = Y(\frac{1}{s})$



A q-difference equation

Former equation

$\varphi_1(y') = A(y)\varphi_1(y)$

We evaluate at $y = Y(s)$ and set $\begin{cases} A(Y(s)) &= \widetilde{A}(s) \\ \varphi_1(Y(s)) &= \widetilde{\varphi}_1(s) \end{cases}$

A q-difference equation...

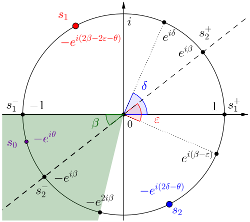
$\widetilde{\varphi}_1(qs) = \widetilde{A}(s)\widetilde{\varphi}_1(s)$

...with $|q| = 1$

Recall that $q = e^{2i\beta}$

$s_1 = -e^{i(2\beta - 2\varepsilon - \theta)}$ and $s_2 = -e^{i(2\delta - \theta)}$

Then $\widetilde{A}(s) = \dfrac{(s - s_1)(s_2 s - 1)}{(s - s_2)(s_1 s - 1)}$



Factorisation

Key observation

If there exists a rational function $R(s)$ such that

$$\widetilde{A}(s) = \frac{R(s)}{R(qs)} \qquad \text{(decoupling)}$$

then the q-diff equation $\widetilde{\varphi}_1(qs) = \widetilde{A}(s)\widetilde{\varphi}_1(s)$ becomes

$$(R\widetilde{\varphi}_1)(qs) = (R\widetilde{\varphi}_1)(s)$$

- Then $(R\widetilde{\varphi}_1)(e^{i\omega})$ is 2β periodic

explicit trigonometric solution

- It may happen that $\widetilde{A}(s)^2$ decouples

Computations under Condition ($\mathcal{C}$)

Theorem

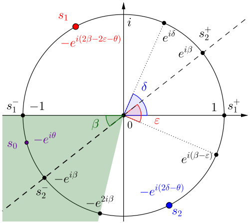
Decoupling $\iff$ ($\mathcal{C}$)

Existence of a decoupling

$$\widetilde{A}(s) = \frac{(s - s_1)(s_2 s - 1)}{(s - s_2)(s_1 s - 1)} = \frac{R(s)}{R(qs)}$$

$$\Updownarrow$$

($\mathcal{C}$) $\alpha \in \mathbb{Z} + \frac{\pi}{\beta}\mathbb{Z}$ or $\alpha_1, \alpha_2 \in \mathbb{Z} + \frac{\pi}{\beta}\mathbb{Z}$



Corollary

Condition ($\mathcal{C}$) $\Longrightarrow$ $\varphi(x, y)$, $\varphi_1(y)$ and $\varphi_2(x)$ all D-algebraic

Theorem

D-algebraicity $\Longrightarrow$ Condition $(\mathcal{C})$

Galois theory of q-difference equations gives necessary condition on the function $\widetilde{A}(s)$ for $\widetilde{\varphi}_1$ to be algebraic

If $\widetilde{\varphi}_1(qs) = \widetilde{A}(s)\widetilde{\varphi}_1(s)$ is D-algebraic, then with $\partial = is\frac{d}{ds}$

$$c_0 \frac{\partial \widetilde{A}}{\widetilde{A}} + c_1 \partial \left(\frac{\partial \widetilde{A}}{\widetilde{A}} \right) + \cdots + c_N \partial^N \left(\frac{\partial \widetilde{A}}{\widetilde{A}} \right) = h(qs) - h(s)$$

Reasoning on the poles

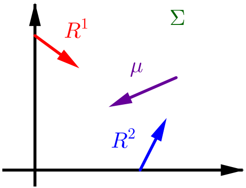
yields Condition $(\mathcal{C})$

Degenerate reflected Brownian motion

Degenerate obliquely reflected BM in dim 2 [Ichiba Karatzas]

$$X(t) = X(0) + B(t) + \mu t + (R^1|R^2)\begin{pmatrix} L^{Ox}(t) \\ L^{Oy}(t) \end{pmatrix}$$

- $X(0)$ starting point in $\mathbb{R}_+^2$
- μ drift in $\mathbb{R}^2$
- oblique reflections R^1 and R^2
- $L^{Ox}(t)$ local time on the axis Ox
- $B(t) = (B_1(t), B_2(t))$ covariance Σ

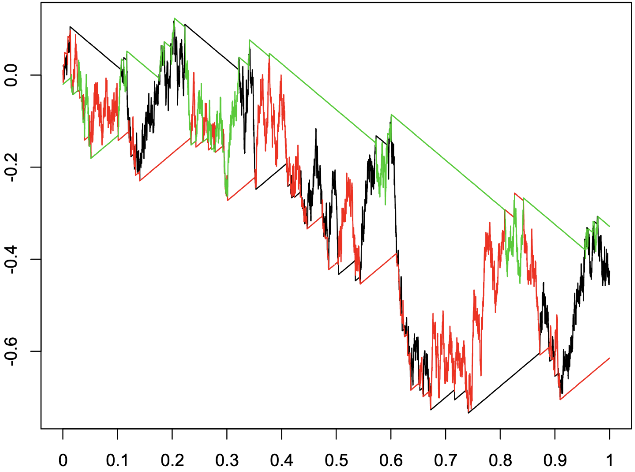


rank 1

Two options: $\Sigma = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ and $\Sigma = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$

Motivations

Competing three particle systems



[Ichiba Karatzas]

Other motivations

Degeneracy of RBM

Reflected BM $\longrightarrow$ Analytic description

$\downarrow$ $\downarrow$

Degenerate RBM $\longrightarrow$ **?**

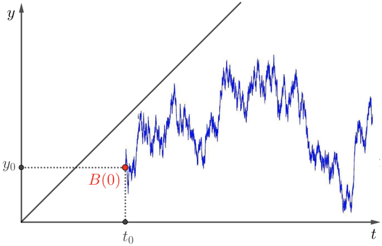
Confluence

$q = e^{2i\beta} \to 1$

Space-time Brownian motion (t, B_t)

[Bougerol Defosseux '22]

[Franceschi '24]

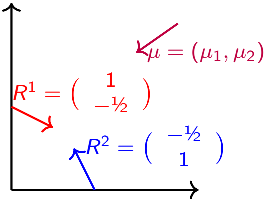


Main results

Main results (1/2)

A particular model

- Symmetric reflections (rank-based diffusions)
- Normalisation $\mu_1 + \mu_2 = 1$
- One parameter μ_1



Theorem [Franceschi Ichiba Karatzas R.]

$$\pi(u,0) = \sum_{n\in\mathbb{Z}} (n-1)n(n+1)(n-1+\mu_1)(n+\mu_1)(n+1+\mu_1)(n+\tfrac{\mu_1}{2})e^{-n(n+\mu_1)u}$$

$$\pi(u,0) = \frac{d}{du}\left(\frac{d}{du} + 1 - \mu_1\right)\left(\frac{d}{du} + 1 + \mu_1\right)\theta_{\mu_1}(e^{-u})$$

with $\theta_{\mu_1}(q) = \sum_{n\in\mathbb{Z}} (n + \frac{\mu_1}{2}) q^{n(n+\mu_1)}$

- First proof: functional equation
- Second proof: Ansatz for the bivariate density $\pi(u,v)$

First proof: functional equation

Theorem [Franceschi Ichiba Karatzas R.]

$\pi(u,0) =$
$\sum_{n\in\mathbb{Z}}(n-1)n(n+1)(n-1+\mu_1)(n+\mu_1)(n+1+\mu_1)(n+\frac{\mu_1}{2})e^{-n(n+\mu_1)u}$

The Laplace transform

$$\int_0^\infty \pi(u,0)e^{-ux}du = \frac{x(x+1-\mu_1)(x+1+\mu_1)}{\cos(\pi\sqrt{\mu_1^2-4x})-\cos(\pi\mu_1)}$$

globally meromorphic

Proof: infinite sum

$$\frac{1}{\tan x} = \frac{1}{x}+2x\sum_{k\geqslant 1}\frac{1}{x^2-(k\pi)^2}$$

Mittag-Leffler expansion

Infinite product

$$\sin x = x\prod_{k\geqslant 1}\Big(1-\frac{x^2}{(k\pi)^2}\Big)$$

$\pi(u,0) = \sum_{k\neq -1,0,1}\mathcal{E}(k(k+\mu_1))$

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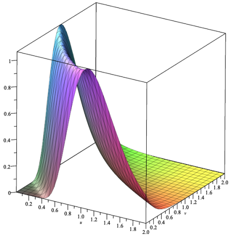
Theorem

$$\pi(u,v) = c\sum_{n\geqslant 0} c_n e^{-a_n u - b_n v} + c' \sum_{n\geqslant 0} c'_n e^{-a'_n u - b'_n v}$$

- Recursive computation of the constants
- a_n, b_n quadratic
- $c_n = P_8(n) + (-1)^n Q_8(n)$

same for a'_n, b'_n



Corollary

$$\pi(u,0) = \sum_{n\in\mathbb{Z}} (n-1)n(n+1)(n-1+\mu_1)(n+\mu_1)(n+1+\mu_1)(n+\tfrac{\mu_1}{2})e^{-n(n+\mu_1)u}$$

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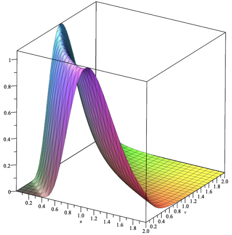
Theorem

$$\pi(u,v) = c\sum_{n\geqslant 0} c_n e^{-a_n u - b_n v} + c'\sum_{n\geqslant 0} c'_n e^{-a'_n u - b'_n v}$$

- Recursive computation of the constants
- a_n, b_n quadratic
- $c_n = P_8(n) + (-1)^n Q_8(n)$

same for a'_n, b'_n



Asymmetric reflections

More general reflections [Dreyfus Flin Franceschi ’24+]

Basic Adjoint Relationship

PDE called basic adjoint relationship [Harrison Williams]

$$\begin{cases} \mathcal{G}^*\pi(u,v) = 0 & \text{(I)} \\ \partial_{R_1^*}\pi(0,v) + 2\mu_1\pi(0,v) = 0 & \text{(V)} \\ \partial_{R_2^*}\pi(u,0) + 2\mu_2\pi(u,0) = 0 & \text{(H)} \end{cases}$$

with

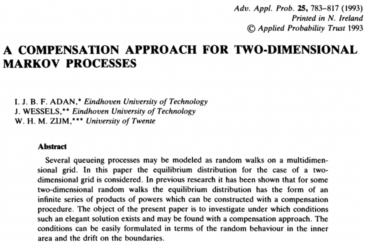
$$\begin{cases} \mathcal{G}^* = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)^2 + \mu_1 \frac{\partial}{\partial x} + \mu_2 \frac{\partial}{\partial y} \\ R = \begin{pmatrix} 1 & -1/2 \\ -1/2 & 1 \end{pmatrix} \\ R^* = 4\Sigma - 2R\,\mathrm{diag}(R)^{-1} = \begin{pmatrix} 2 & -3 \\ -3 & 2 \end{pmatrix} \\ \Sigma = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \end{cases}$$

Compensation approach

Main objective (discrete setting)

Use the **invariance property $\pi P = \pi$** to compute directly the stationary distribution $\pi_{i,j}$ without the generating function

$$\sum_{i,j \geqslant 0} \pi_{i,j} x^i y^j$$


Adv. Appl. Prob. **25**, 783–817 (1993)
Printed in N. Ireland

A COMPENSATION APPROACH FOR TWO-DIMENSIONAL MARKOV PROCESSES

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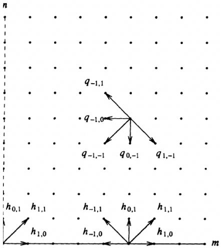
Abstract

Several queueing processes may be modeled as random walks on a multidimensional grid. In this paper the equilibrium distribution for the case of a two-dimensional grid is considered. In previous research it has been shown that for some two-dimensional random walks the equilibrium distribution has the form of an infinite series of products of powers which can be constructed with a compensation procedure. The object of the present paper is to investigate under which conditions such an elegant solution exists and may be found with a compensation approach. The conditions can be easily formulated in terms of the random behaviour in the inner area and the drift on the boundaries.



[Adan Wessels Zijm '93]

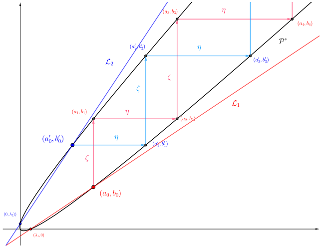
Compensation approach (1/2)

$$\begin{cases} \mathcal{G}^*\pi(u,v) = 0 & \text{(I)} \\ \partial_{R_1^*}\pi(0,v) + 2\mu_1\pi(0,v) = 0 & \text{(V)} \\ \partial_{R_2^*}\pi(u,0) + 2\mu_2\pi(u,0) = 0 & \text{(H)} \end{cases}$$

Ansatz

$$\pi(u,v) = \sum_n c_n e^{-a_n u - b_n v}$$

- e^{-au-bv} solution to (I) $\Leftrightarrow$ $(a,b) \in \mathcal{P}^*$
- e^{-au-bv} solution to (I) and (V) $\Leftrightarrow$ $(a,b) = (a_0, b_0)$
- $c_0 e^{-a_0 u - b_0 v} + c_1 e^{-a_1 u - b_1 v}$ solution to (I) and (H) $\Leftrightarrow$ condition c_0, c_1



$$\overbrace{c_0 e^{-a_0 u - b_0 v}}^{\in(I)\cap(V)} + c_1 e^{-a_1 u - b_1 v} + \ldots$$

$$\underbrace{\overbrace{c_0 e^{-a_0 u - b_0 v}}^{\in (I)\cap(V)} + c_1 e^{-a_1 u - b_1 v}}_{\in (I)\cap(H)}$$

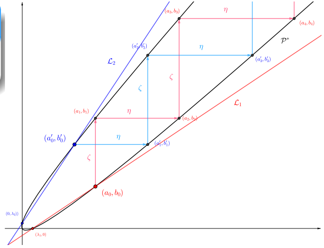
$$\overbrace{c_1 e^{-a_1 u - b_1 v} + c_2 e^{-a_2 u - b_2 v}}^{\in (I)\cap(V)} \quad \underbrace{c_2 e^{-a_2 u - b_2 v} + c_3 e^{-a_3 u - b_3 v}}_{\in (I)\cap(H)} \quad \overbrace{c_3 e^{-a_3 u - b_3 v} + c_4 e^{-a_4 u - b_4 v}}^{\in (I)\cap(V)} + \cdots$$

Compensation approach (2/2)

Two orbits

possible starting points

- $c_0 e^{-a_0 u - b_0 v} + c_1 e^{-a_1 u - b_1 v}$ solution to (I) and (H) $\Leftrightarrow$ condition c_0, c_1
- $c'_0 e^{-a'_0 u - b'_0 v} + c'_1 e^{-a'_1 u - b'_1 v}$ solution to (I) and (V) $\Leftrightarrow$ condition c'_0, c'_1



Theorem

$$\pi(u, v) = c \sum_{n \geqslant 0} c_n e^{-a_n u - b_n v} + c' \sum_{n \geqslant 0} c'_n e^{-a'_n u - b'_n v}$$

Thank you for your attention

Stochastic reflection
5–9 August 2024
Isaac Newton Institute, Cambridge

Appendix

Classes of functions

- Rational

$$\psi(x) = \frac{1-x}{1-x-x^2}$$

- Algebraic

$$1 - \psi(x) + x\psi(x)^2 = 0$$

- D-finite

$$x(1-16x)\psi''(x) + (1-32x)\psi'(x) - 4\psi(x) = 0$$

- D-algebraic

$$(2x + 5\psi(x) - 3x\psi'(x))\psi''(x) = 48x$$

- D-transcendantal

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$$\theta_{\mu_1}(q) = \sum_{n\in\mathbb{Z}}(n + \frac{\mu_1}{2})q^{n(n+\mu_1)}$$

Not surprisingly, the Jacobi theta-like function θ_μ in (16) admits a direct probabilistic interpretation (see (82) below) in terms of Brownian motion conditioned to stay forever in the interval $[0,1]$. More specifically, for $t > 0$ and $x, y \in (0,1)$, let $q_t(x,y)$ be the associated transition probability density. Using the recent results by Bougerol and Defosseux [5, Eq. (2.1)], one has

$$q_t(x,y) = \frac{\sin(\pi y)}{\sin(\pi x)} e^{\pi^2 t/2} p_t(x,y),$$

where $p_t(x,y)$ is the transition probability density function of the killed Brownian motion in $[0,1]$, namely,

$$p_t(x,y) = \frac{1}{2\sqrt{2\pi t}} \sum_{n\in\mathbb{Z}} \left(\exp\left(-\frac{(x-y+2n)^2}{2t}\right) - \exp\left(-\frac{(x+y-2+2n)^2}{2t}\right)\right), \tag{80}$$

see Section 6 in Appendix A.1 of [4]. As explained in [5, Sec. 2.1], it is actually possible to start the process at $x = 0$ (using the idea of entrance density measure), and obtain the density function

$$q_t(0,y) = \lim_{x\to 0} q_t(x,y) = \sin(\pi y) \sum_{n\in\mathbb{Z}} n \sin(n\pi y) \exp\left(-\pi^2(n^2-1)\frac{t}{2}\right), \tag{81}$$

see [5, Eq. (2.5)]. The Jacobi transformation of our Lemma 15 leads directly to

$$\theta_\mu(e^{-2/t}) = \frac{1}{\sin(\pi\mu)} \left(\frac{\pi t}{2}\right)^{3/2} \exp\left(\frac{\mu^2}{2t} - \frac{\pi^2 t}{2}\right) q_t(0,\mu). \tag{82}$$

As a conclusion, up to a simple prefactor function, the theta function θ_μ exactly describes the entrance density measure of the killed Brownian motion in $[0,1]$ starting from 0.