On the solutions of Mahler equations

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Outline

Definition

- *Mahler equation of order n :*

$$a_n(z)y(z^{p^n}) + \ldots + a_1(z)y(z^p) + a_0(z)y(z) = 0$$

with $p \in \mathbb{N}_{\geq 2}$, $a_i \in \mathbf{k} := \overline{\mathbb{Q}}(z)$ *and* $a_0 a_n \neq 0$. *We write*

$$Ly = 0 \quad \text{with} \quad L := \sum_{i=0}^{n} a_i \phi_p^i \quad \text{and} \quad \phi_p : z \mapsto z^p .$$

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The systems $\phi_p(Y) = AY$ *and* $\phi_p(Z) = BZ$ *are* **k***-equivalents if there exists* $T \in \mathrm{GL}_n(\mathbf{k})$ *such that* $B = \phi_p(T)AT^{-1}$. *(*$Z = TY$*)*

Outline

- 1929 - 1930 : Mahler was interested in the values of

$$f_p(z) = \sum_{n\geq 0} z^{p^n} \quad \text{and} \quad g_p(z) = \prod_{n\geq 0} \left(1 - z^{p^n}\right).$$

Proposition

Let $\alpha \in \overline{\mathbb{Q}}^\star$ *with* $|\alpha| < 1$.

- $f_p(\alpha)$ *(resp.* $g_p(\alpha)$*) is transcendent* $\rightarrow$ *use* $f_p(z^p) = f_p(z) - z$ *;*
- *algebraic independence of* $f_p(\alpha)$ *and* $g_p(\alpha)$.

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• 1968, Cobham : the generating series of an automatic sequence is a coordinate of a vector Y such that

$$Y(z) = A(z)Y(z^p) \quad \text{with} \quad A(z) \in GL_n(\overline{\mathbb{Q}}(z)).$$

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Definition (the field $\mathcal{H}$ of Hahn series)

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Theorem (Roques, 2021)

Any system $[A]$ with $A \in GL_n(\mathcal{H})$ is $\mathcal{H}$-equivalent with $[C]$ where $C \in GL_n(\overline{\mathbb{Q}})$ and C is unique up to conjugation by an element of $GL_n(\overline{\mathbb{Q}})$.

Moreover, the solutions of $[C]$ can be constructed with e_c, $c \in \overline{\mathbb{Q}}^\star$, and ℓ such that

$$\phi_p(e_c) = ce_c \quad \textit{and} \quad \phi_p(\ell) = \ell + 1\,.$$

Outline

1. Mahler equations
   - Motivation
   - Some properties of the solutions
2. Algorithm to recognize regular singular Mahler systems
   - Main result
   - Ideas of the proof of the main result
3. Asymptotic behavior of solutions
   - Natural boundary
   - For the order 1 homogeneous equation
   - For the order 2 homogeneous equation

$$Y\left(z^{p}\right)=A(z)Y(z),\ \ A(z)\in \mathrm{GL}_n\left(\overline{\mathbb{Q}}(z)\right) \qquad [A]$$

Definition

[A] is regular singular at 0 (RS) if there exist $T\in \mathrm{GL}_n(\mathcal{P})$*, where* $\mathcal{P}:=\bigcup_{d\geq 1}\overline{\mathbb{Q}}((z^{1/d}))$ *are the Puiseux series,* $C\in \mathrm{GL}_n\left(\overline{\mathbb{Q}}\right)$ *such that*

$$T\left(z^{p}\right)C=A(z)T(z)\,. \qquad (*)$$

$$Y(z^p) = A(z)Y(z), \quad A(z) \in \mathrm{GL}_n\left(\overline{\mathbb{Q}}(z)\right) \qquad [A]$$

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Algorithm (Faverjon,P.)

Determine if the Mahler system [A] is RS at 0 by computing the dimension of an explicit $\overline{\mathbb{Q}}$*-vector space* V*. If it is RS, the algorithm returns* C *and a series expansion of* T *at a wanted order.*

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To obtain it :

1) (with Laurent series) Do it with the restriction $T \in \mathrm{GL}_n\left(\overline{\mathbb{Q}}((z))\right)$.

2) (from Puiseux series to Laurent series) Find the possible ramifications $\mathcal{D}_0$ for a solution T of $(*)$ and apply **1)** for $A(z^d)$, $d \in \mathcal{D}_0$.

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Notation.

$$\begin{aligned} p_{\nu,\mu} : \quad \overline{\mathbb{Q}}((z))^n \quad &\to \quad \overline{\mathbb{Q}}^{n(\mu-\nu+1)} \\ \boldsymbol{g}(z) = \textstyle\sum_{k\in\mathbb{Z}} \boldsymbol{g_k} z^k \quad &\mapsto \quad \left(\boldsymbol{g_\nu} \;\cdots\; \boldsymbol{g_\mu}\right)^\top \end{aligned}$$

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$$B(z)T\left(z^p\right) = T(z)C^{-1} \iff \begin{cases} \lambda_i \boldsymbol{t}_{i,1}(z) = B(z)\boldsymbol{t}_{i,1}\left(z^p\right) \\ \lambda_i \boldsymbol{t}_{i,j}(z) + \boldsymbol{t}_{i,j-1}(z) = B(z)\boldsymbol{t}_{i,j}\left(z^p\right), j \geq 2 \end{cases}$$

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If $[A]$ *is RS with* $T \in \mathrm{GL}_n\left(\overline{\mathbb{Q}}((z))\right)$ *then* $\dim(V) \geq n$ *with*
$V := \left(\bigcap_{k=0}^{c} \ker(NM^k)\right) \cap \left(\bigcap_{k=0}^{c} M^k \cdot \ker(N)\right)$ *and* $c := n(\mu - \nu + 1)$,

$$M = (B_{i-pj})_{\nu \leq i,j \leq \mu} \in \mathcal{M}_c\left(\overline{\mathbb{Q}}\right), \quad N = \begin{cases} (B_{i-pj})_{v_0(B)+p\nu \leq i < \nu,\, \nu \leq j \leq \mu} & \text{if } \nu < \mu \\ 0 & \text{if } \nu = \mu \end{cases}$$

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Idea : $p_{\nu,\mu} : < \boldsymbol{t}_{i,j} > \to V$ is injective.

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- Denote by $(\mathbf{e_i})_{1\leq i\leq m}$ a basis of V. Let

$$E = (\mathbf{e_1} \mid \cdots \mid \mathbf{e_m}) := \begin{pmatrix} E_\nu \\ \vdots \\ E_\mu \end{pmatrix}.$$

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- Proof of $m = n$ and $T \in \mathrm{GL}_n(\overline{\mathbb{Q}}((z)))$ $\quad \rightarrow \quad$ [A] RS.

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- Proof of $m = n$ and $T \in \mathrm{GL}_n(\overline{\mathbb{Q}}((z)))$ $\rightarrow$ $[A]$ RS.

Theorem (Faverjon,P.)

The system $[A]$ is RS at 0 if and only if $\dim_{\overline{\mathbb{Q}}}(V) \geq n$.

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Steps :

1) Find the possible ramifications $\mathcal{D}_0$. From a work of Chyzak, Dreyfus, Dumas, Mezzarobba (2018) :

$$\mathcal{D}_0 \subset \mathcal{D} := \{d \in \mathbb{N} \mid 1 \leq d \leq p^n - 1\,, \gcd(d, p) = 1\}\,.$$

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2) Apply the previous criterion to $[A(z^d)]$ for a $d \in \mathcal{D}_0$ because :

Theorem (Faverjon,P.)

The three following propositions are equivalent :

1. *The Mahler system $[A]$ is regular singular at 0,*
2. $\dim V_d \geq n$ *for some integer* $d \in \mathcal{D}_0$,
3. $\dim V_d = n$ *for every integer* $d \in \mathcal{D}_0$.

In that case, $[A]$ is $\overline{\mathbb{Q}}((z^{1/d}))$-equivalent to a constant system.

Outline

Theorem (Randé, 1992)

If there exists a solution $f \in \overline{\mathbb{Q}}[[z]]$ of $Ly = 0$ then $f \in \overline{\mathbb{Q}}(z)$ or f is meromorphic in $D(0,1)$ with the unit circle as a natural boundary.

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Theorem (Bell et Coons, 2016)

Let $\alpha_j := a_j(1)$ *and* $P(X) = \alpha_0 X^n + \ldots + \alpha_n$. *If* $\alpha_0 \alpha_n \neq 0$ *and* $P(X)$ *has only one non-zero root* λ *of greatest modulus, then*

$$f(z) = \frac{C(z)}{(1-z)^{\log_p(\lambda)}}(1 + o(1)) \quad \text{when } z \to 1^-$$

with $C(z)$ *real analytic, bounded in* $(0,1)$ *and* $C(z^p) = C(z)$.

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P.,Rivoal : Explicit $C(z)$ for the Mahler equations of

- order 1 homogeneous ;
- order 2 homogeneous with mild assumptions on $a_j(z)$;
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Example for the order 1 homogeneous equation :

$$y(z^p) = (1 - \alpha z)y(z) \quad \text{with } \alpha \notin [1, +\infty[\cup \{0\}$$

with solution $f_\alpha(z) := \prod_{k=0}^{+\infty} \left(1 - \alpha z^{p^k}\right)^{-1}$.

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Theorem (P.,Rivoal)

Let $z = e^{-s}$, for all $s > 0$ small enough,

$$C(e^{-s}) = \exp\left(\frac{1}{\ln(p)} \sum_{k \in \mathbb{Z}\setminus\{0\}} \Gamma\left(\frac{2i\pi k}{\ln(p)}\right) \mathrm{Li}_{1+\frac{2i\pi k}{\ln(p)}}(\alpha) s^{-\frac{2i\pi k}{\ln(p)}} + cst\right)$$

with $cst := \gamma \log_p(1-\alpha) - \frac{\ln(1-\alpha)}{2} + \frac{\ell(\alpha)}{\ln(p)}$.

Ideas.

$$f_\alpha(z) := \prod_{k=0}^{+\infty} \left(1-\alpha z^{p^k}\right)^{-1} \quad \rightarrow \quad G_\alpha(s) := \ln(f_\alpha(e^{-s})) .$$

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- Cahen-Mellin formula : $e^{-\omega} = \frac{1}{2i\pi}\int_{a-i\infty}^{a+i\infty}\Gamma(z)\omega^{-z}\mathrm{d}z$ for $\omega > 0$, $a > 0$.

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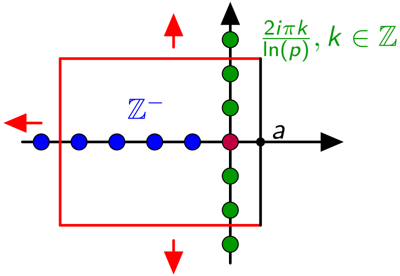

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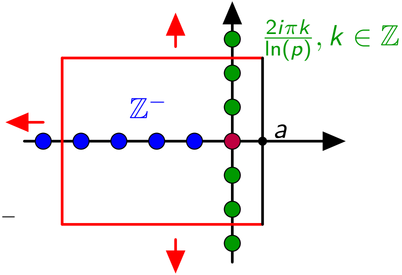

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Outline

$$y(z) = a(z)y(z^p) + b(z)y(z^{p^2})$$

Brent, Coons, Zudilin (2015) : $p = 4$, $a(z) = 1 + z + z^2$ and $b(z) = -z^4$.

Extend it to $a(z), b(z) \in \mathbb{R}(z)$ such that

(H1) $a(z), b(z) \in \mathbb{R}^+[[z]]$;

(H2) $a(0) + b(0) = 1$;

(H3) $a(z)$ and $b(z)$ are defined at $z = 1$;

(H4) $a(z)$ and $b(z)$ have no pole in $D(0, 1)$;

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- Consider the Mellin transforms :

$$\mathcal{F}(s) := \int_0^{+\infty} \ln(f(e^{-t}))t^{s-1}\mathrm{d}t \quad \text{and} \quad \mathcal{M}(s) := \int_0^{+\infty} \ln(\mu(e^{-t}))t^{s-1}\mathrm{d}t.$$

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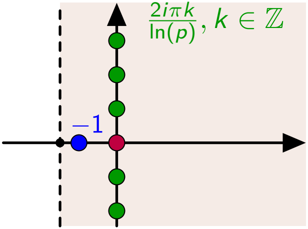

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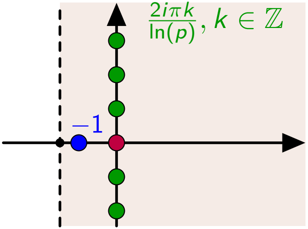

- Mellin dictionary gives :

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with $C(e^{-s}) = \exp\left(\frac{1}{\ln(p)} \sum_{k\in\mathbb{Z}\setminus\{0\}} \mathcal{M}\left(\frac{2\pi ki}{\ln(p)}\right) s^{-\frac{2\pi ki}{\ln(p)}} + c_0\right)$.

Thanks !