On the solutions of Mahler equations

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- Motivation
- Some properties of the solutions
- 2 Algorithm to recognize regular singular Mahler systems
 - Main result
 - Ideas of the proof of the main result
- 3 Asymptotic behavior of solutions
 - Natural boundary
 - For the order 1 homogeneous equation
 - For the order 2 homogeneous equation

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Definition

• Mahler equation of order n :

$$a_n(z)y(z^{p^n}) + \ldots + a_1(z)y(z^p) + a_0(z)y(z) = 0$$

with $p \in \mathbb{N}_{\geq 2}$, $a_i \in \mathbf{k} := \overline{\mathbb{Q}}(z)$ and $a_0 a_n \neq 0$. We write Ly = 0 with $L := \sum_{i=0}^n a_i \phi_p^i$ and $\phi_p : z \mapsto z^p$.

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• Mahler system : $Y(z^p) = A(z)Y(z)$ denoted by [A].

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$$Ly = 0 \quad \rightsquigarrow \quad [A_L] \text{ where } A_L := \begin{pmatrix} 1 & & \\ & \ddots & \\ & & \ddots & \\ -a_0/a_n & \dots & -a_{n-1}/a_n \end{pmatrix}$$

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Definition

The systems $\phi_p(Y) = AY$ and $\phi_p(Z) = BZ$ are k-equivalents if there exists $T \in GL_n(\mathbf{k})$ such that $B = \phi_p(T)AT^{-1}$. (Z = TY)

Motivation

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• 1929 - 1930 : Mahler was interested in the values of

$$f_p(z) = \sum_{n\geq 0} z^{p^n}$$
 and $g_p(z) = \prod_{n\geq 0} \left(1-z^{p^n}
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Proposition

Let $\alpha \in \overline{\mathbb{Q}}^*$ with $|\alpha| < 1$.

- $f_p(\alpha)$ (resp. $g_p(\alpha)$) is transcendent \rightarrow use $f_p(z^p) = f_p(z) z$;
- algebraic independence of $f_p(\alpha)$ and $g_p(\alpha)$.

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- algebraic independence of $f_p(\alpha)$ and $g_p(\alpha)$.

• 1968, Cobham : the generating series of an automatic sequence is a coordinate of a vector Y such that

$$Y(z) = A(z)Y(z^p)$$
 with $A(z) \in GL_n(\overline{\mathbb{Q}}(z))$.

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What do we need to construct the solutions?

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Definition (the field \mathcal{H} of Hahn series)

Hahn series over $\overline{\mathbb{Q}}$ are the

$$f(z) = \sum_{\gamma \in \mathbb{Q}} f_{\gamma} z^{\gamma} \quad \textit{with} \quad f_{\gamma} \in \overline{\mathbb{Q}}$$

where $supp((f_{\gamma})_{\gamma \in \mathbb{Q}}) = \{\gamma \in \mathbb{Q} \mid f_{\gamma} \neq 0\}$ is a well-ordered set (that is, any nonempty subset has a least element).

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Theorem (Roques, 2021)

Any system [A] with $A \in GL_n(\mathcal{H})$ is \mathcal{H} -equivalent with [C] where $C \in GL_n(\overline{\mathbb{Q}})$ and C is unique up to conjugation by an element of $GL_n(\overline{\mathbb{Q}})$.

Moreover, the solutions of [C] can be constructed with e_c , $c \in \overline{\mathbb{Q}}^*$, and ℓ such that

$$\phi_p(e_c) = ce_c \quad and \quad \phi_p(\ell) = \ell + 1.$$

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$$Y(z^p) = A(z)Y(z), \ A(z) \in \operatorname{GL}_n\left(\overline{\mathbb{Q}}(z)\right)$$
 [A]

Definition

[A] is regular singular at 0 (RS) if there exist $T \in GL_n(\mathcal{P})$, where $\mathcal{P} := \bigcup_{d \ge 1} \overline{\mathbb{Q}}((z^{1/d}))$ are the Puiseux series, $C \in GL_n(\overline{\mathbb{Q}})$ such that $T(z^p) C = A(z)T(z)$. (*)

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Algorithm (Faverjon, P.)

Determine if the Mahler system [A] is RS at 0 by computing the dimension of an explicit $\overline{\mathbb{Q}}$ -vector space V. If it is RS, the algorithm returns C and a series expansion of T at a wanted order.

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To obtain it :

1) (with Laurent series) Do it with the restriction $T \in \operatorname{GL}_n\left(\overline{\mathbb{Q}}((z))\right)$. 2) (from Puiseux series to Laurent series) Find the possible ramifications \mathcal{D}_0 for a solution T of (*) and apply 1) for $A(z^d)$, $d \in \mathcal{D}_0$.

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With Laurent series

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then $v_0(T) \ge \lceil v_0(A)/(p-1) \rceil := \nu$. Write $T(z) := \sum_{m \ge \nu} T_m z^m$.

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2) Recurrence relation to determine the T_m . Let $B := A^{-1}$.

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$$\iff T(z)C^{-1} = B(z)T(z^p)$$

 $\iff \forall m \in \mathbb{Z}, \quad T_mC^{-1} = \sum_{(k,\ell):k+p\ell=m} B_kT_\ell$

If $m > \left\lceil -v_0(B)/(p-1) \right\rceil := \mu$, then $T_m = \sum_{\ell=\nu}^{m-1} B_{m-p\ell} T_\ell C$.

Ideas of the proof of the main result

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Notation.

$$: \quad \overline{\mathbb{Q}}((z))^n \quad \to \quad \overline{\mathbb{Q}}^{n(\mu-\nu+1)}$$
$$g(z) = \sum_{k \in \mathbb{Z}} g_k z^k \quad \mapsto \quad \left(g_{\nu \square \land \land \land \land \land \land } g_{\mu}\right)^\top$$

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With Laurent series

3) Conditions on the column vectors of T.

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3) Conditions on the column vectors of T. Let $\mathbf{t}_{i,j}(z)$ be the columns of \overline{T} and let $C^{-1} = \text{diag}(J_{s_1}(\lambda_1), \ldots, J_{s_r}(\lambda_r))$ be a Jordan matrix.

$$B(z)T(z^{p}) = T(z)C^{-1} \Leftrightarrow \begin{cases} \lambda_{i}\boldsymbol{t}_{i,1}(z) = B(z)\boldsymbol{t}_{i,1}(z^{p}) \\ \lambda_{i}\boldsymbol{t}_{i,j}(z) + \boldsymbol{t}_{i,j-1}(z) = B(z)\boldsymbol{t}_{i,j}(z^{p}), j \geq 2 \end{cases}$$

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Lemma

If [A] is RS with
$$T \in \operatorname{GL}_n\left(\overline{\mathbb{Q}}((z))\right)$$
 then $\dim(V) \ge n$ with

$$V := \left(\bigcap_{k=0}^c \ker(NM^k)\right) \cap \left(\bigcap_{k=0}^c M^k \cdot \ker(N)\right) \text{ and } c := n(\mu - \nu + 1),$$

$$M = (B_{i-pj})_{\nu \le i,j \le \mu} \in \mathcal{M}_c\left(\overline{\mathbb{Q}}\right), \quad N = \begin{cases} (B_{i-pj})_{\nu_0(B)+p\nu \le i < \nu, \nu \le j \le \mu} & \text{if } \nu < \mu \\ 0 & \text{if } \nu = \mu \end{cases}$$

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Idea : $p_{\nu,\mu} : \langle t_{i,j} \rangle \rightarrow V$ is injective.

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• Proof of m = n and $T \in \operatorname{GL}_n(\overline{\mathbb{Q}}((z))) \longrightarrow [A]$ RS.

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• Proof of m = n and $T \in \operatorname{GL}_n(\overline{\mathbb{Q}}((z))) \longrightarrow [A]$ RS.

Theorem (Faverjon, P.)

The system [A] is RS at 0 if and only if $\dim_{\overline{\mathbb{O}}}(V) \ge n$.

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From Puiseux series to Laurent series

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From Puiseux series to Laurent series

Steps :

1) Find the possible ramifications \mathcal{D}_0 . From a work of Chyzak, Dreyfus, Dumas, Mezzarobba (2018) :

 $\mathcal{D}_0 \subset \mathcal{D} := \left\{ d \in \mathbb{N} \mid 1 \leq d \leq p^n - 1, \operatorname{gcd}(d, p) = 1
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2) Apply the previous criterion to $[A(z^d)]$ for a $d \in \mathcal{D}_0$ because :

Theorem (Faverjon, P.)

The three following propositions are equivalent :

- The Mahler system [A] is regular singular at 0,
- 2 dim $V_d \ge n$ for some integer $d \in \mathcal{D}_0$,

• dim $V_d = n$ for every integer $d \in \mathcal{D}_0$.

In that case, [A] is $\overline{\mathbb{Q}}((z^{1/d}))$ -equivalent to a constant system.

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Theorem (Randé, 1992)

If there exists a solution $f \in \overline{\mathbb{Q}}[[z]]$ of Ly = 0 then $f \in \overline{\mathbb{Q}}(z)$ or f is meromorphic in D(0,1) with the unit circle as a natural boundary.

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Theorem (Bell et Coons, 2016)

Let $\alpha_j := a_j(1)$ and $P(X) = \alpha_0 X^n + \ldots + \alpha_n$. If $\alpha_0 \alpha_n \neq 0$ and P(X) has only one non-zero root λ of greatest modulus, then

$$f(z) = rac{\mathcal{C}(z)}{(1-z)^{\log_p(\lambda)}}(1+o(1)) \quad ext{when } z o 1^-$$

with C(z) real analytic, bounded in (0,1) and $C(z^p) = C(z)$.

Natural boundary For the order 1 homogeneous equation For the order 2 homogeneous equation

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Theorem (Randé, 1992)

If there exists a solution $f \in \overline{\mathbb{Q}}[[z]]$ of Ly = 0 then $f \in \overline{\mathbb{Q}}(z)$ or f is meromorphic in D(0,1) with the unit circle as a natural boundary.

Theorem (Bell et Coons, 2016)

Let $\alpha_j := a_j(1)$ and $P(X) = \alpha_0 X^n + \ldots + \alpha_n$. If $\alpha_0 \alpha_n \neq 0$ and P(X) has only one non-zero root λ of greatest modulus, then

$$f(z) = rac{\mathcal{C}(z)}{(1-z)^{\log_{
ho}(\lambda)}}(1+o(1)) \quad ext{when } z o 1^{-1}$$

with C(z) real analytic, bounded in (0,1) and $C(z^p) = C(z)$.

P., Rivoal : Explicit C(z) for the Mahler equations of

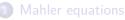
- order 1 homogeneous;
- order 2 homogeneous with mild assumptions on $a_j(z)$;
- order 1 inhomogeous with mild assumptions on $a_i(z_i)$, $a_i = 0$

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Outline



- Motivation
- Some properties of the solutions
- Algorithm to recognize regular singular Mahler systems
 Main result
 - Ideas of the proof of the main result
- 3 Asymptotic behavior of solutions
 - Natural boundary
 - For the order 1 homogeneous equation
 - For the order 2 homogeneous equation

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Example for the order 1 homogeneous equation :

$$y(z^p) = (1 - \alpha z)y(z)$$
 with $\alpha \notin [1, +\infty[\cup \{0\}])$

with solution $f_{\alpha}(z) := \prod_{k=0}^{+\infty} (1 - \alpha z^{p^k})^{-1}$.

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Example for the order 1 homogeneous equation :

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 with $lpha \notin [1, +\infty[\cup \{0\}])$

with solution $f_{\alpha}(z) := \prod_{k=0}^{+\infty} (1 - \alpha z^{p^k})^{-1}$. From [BC16],

$$f_lpha(z) = rac{\mathcal{C}(z)}{(1-z)^{\log_
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Theorem (P.,Rivoal)
Let
$$z = e^{-s}$$
, for all $s > 0$ small enough,
 $C(e^{-s}) = \exp\left(\frac{1}{\ln(p)}\sum_{k\in\mathbb{Z}\setminus\{0\}}\Gamma\left(\frac{2i\pi k}{\ln(p)}\right)\operatorname{Li}_{1+\frac{2i\pi k}{\ln(p)}}(\alpha)s^{-\frac{2i\pi k}{\ln(p)}} + cst\right)$
with $cst := \gamma \log_p(1-\alpha) - \frac{\ln(1-\alpha)}{2} + \frac{\ell(\alpha)}{\ln(p)}$.

Natural boundary For the order 1 homogeneous equation For the order 2 homogeneous equation

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Ideas.

$$f_{\alpha}(z) := \prod_{k=0}^{+\infty} \left(1 - lpha z^{p^k}
ight)^{-1} \quad o \quad G_{\alpha}(s) := \ln(f_{\alpha}(e^{-s})) \,.$$

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Ideas.

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• Cahen-Mellin formula : $e^{-\omega} = \frac{1}{2i\pi} \int_{a-i\infty}^{a+i\infty} \Gamma(z) \omega^{-z} dz$ for $\omega > 0$, a > 0.

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$$G_{\alpha}(s) = \sum_{k \ge 0} \sum_{m \ge 1} m^{-1} \alpha^m e^{-sp^k m} = \frac{1}{2i\pi} \int_{a-i\infty}^{a+i\infty} s^{-z} \frac{\Gamma(z) \operatorname{Li}_{1+z}(\alpha)}{1-p^{-z}} \mathrm{d}z \,.$$

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• Apply the residue theorem.

Natural boundary For the order 1 homogeneous equation For the order 2 homogeneous equation

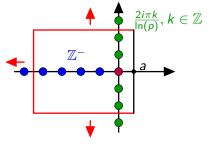
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 $\begin{array}{c} \bullet \\ \bullet \\ \bullet \end{array} \xrightarrow{2i\pi k}, k \in \mathbb{Z}$

Ideas.

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• Cahen-Mellin formula : $e^{-\omega} = \frac{1}{2i\pi} \int_{a-i\infty}^{a+i\infty} \Gamma(z) \omega^{-z} dz$ for $\omega > 0$, a > 0.

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• Apply the residue theorem.

$$f_{\alpha}(z) = \frac{C(z)}{(1-z)^{\log_{p}(\frac{1}{1-\alpha})}} (1+o(1)) \quad \text{when } z \to 1^{-}$$

$$C(e^{-s}) = \exp\left(\frac{1}{\ln(p)} \sum_{k \in \mathbb{Z}^{*}} \Gamma\left(\frac{2i\pi k}{\ln(p)}\right) \operatorname{Li}_{1+\frac{2i\pi k}{\ln(p)}}(\alpha) s^{-\frac{2i\pi k}{\ln(p)}} + \frac{\operatorname{cst}}{\operatorname{cst}}\right) \xrightarrow{\mathbb{Z}^{*}} \mathbb{Z} \xrightarrow{\mathbb{Z}^{*}} \mathbb{Z}$$

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- Motivation
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- For the order 1 homogeneous equation
- For the order 2 homogeneous equation

Natural boundary For the order 1 homogeneous equation For the order 2 homogeneous equation

$$y(z) = a(z)y(z^p) + b(z)y(z^{p^2})$$

Brent, Coons, Zudilin (2015) : p = 4, $a(z) = 1 + z + z^2$ and $b(z) = -z^4$.

Extend it to $a(z), b(z) \in \mathbb{R}(z)$ such that (H1) $a(z), b(z) \in \mathbb{R}^+[[z]]$;

- (H2) a(0) + b(0) = 1;
- (H3) a(z) and b(z) are defined at z = 1;
- (H4) a(z) and b(z) have no pole in D(0,1);
- (H5) a(z) and b(z) are not both constant;
- (H6) For all $z \in [0,1]$, $|rz^{r-1}b(z)| < a(z^r)^2$.

Natural boundary For the order 1 homogeneous equation For the order 2 homogeneous equation

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$$\begin{split} y(z) &= a(z)y(z^p) + b(z)y(z^{p^2}) \\ \text{Brent, Coons, Zudilin (2015)} : p = 4, \ a(z) = 1 + z + z^2 \text{ and } b(z) = -z^4. \\ \text{Extend it to } a(z), b(z) \in \mathbb{R}(z) \text{ such that (H1) } a(z), b(z) \in \mathbb{R}^+[[z]]; \\ & (\text{H2}) \ a(0) + b(0) = 1; \\ & (\text{H3}) \ a(z) \text{ and } b(z) \text{ are defined at } z = 1; \\ & (\text{H4}) \ a(z) \text{ and } b(z) \text{ are not both constant}; \\ & (\text{H6}) \text{ For all } z \in [0, 1], \ |rz^{r-1}b(z)| < a(z^r)^2. \\ & \rightarrow \text{ a solution } f \text{ holomorphic in } D(0, 1) \text{ such that } f(z) > 0 \text{ for all } z \in (0, 1). \end{split}$$

Natural boundary For the order 1 homogeneous equation For the order 2 homogeneous equation

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 $y(z) = a(z)y(z^{p}) + b(z)y(z^{p^{2}})$ Brent, Coons, Zudilin (2015) : p = 4, $a(z) = 1 + z + z^{2}$ and $b(z) = -z^{4}$. Extend it to $a(z), b(z) \in \mathbb{R}(z)$ such that (H1) $a(z), b(z) \in \mathbb{R}^{+}[[z]]$; (H2) a(0) + b(0) = 1; (H3) a(z) and b(z) are defined at z = 1; (H4) a(z) and b(z) have no pole in D(0,1); (H5) a(z) and b(z) are not both constant; (H6) For all $z \in [0,1]$, $|rz^{r-1}b(z)| < a(z^{r})^{2}$.

 \rightarrow a solution f holomorphic in D(0,1) such that f(z) > 0 for all $z \in (0,1)$.

Introduce

$$\mu(z):=\frac{f(z)}{f(z^p)}\,.$$

Natural boundary For the order 1 homogeneous equation For the order 2 homogeneous equation

$$\begin{split} y(z) &= a(z)y(z^p) + b(z)y(z^{p^2}) \\ \text{Brent, Coons, Zudilin (2015)} : p = 4, \ a(z) = 1 + z + z^2 \text{ and } b(z) = -z^4. \\ \text{Extend it to } a(z), b(z) \in \mathbb{R}(z) \text{ such that (H1) } a(z), b(z) \in \mathbb{R}^+[[z]]: \\ & (\text{H2}) \ a(0) + b(0) = 1; \\ & (\text{H3}) \ a(z) \text{ and } b(z) \text{ are defined at } z = 1; \\ & (\text{H4}) \ a(z) \text{ and } b(z) \text{ are not both constant}; \\ & (\text{H6}) \text{ For all } z \in [0, 1], \ |rz^{r-1}b(z)| < a(z^r)^2. \end{split}$$

ightarrow a solution f holomorphic in D(0,1) such that f(z) > 0 for all $z \in (0,1)$.

Introduce

$$\mu(z):=\frac{f(z)}{f(z^p)}\,.$$

• Consider the Mellin transforms :

$$\mathcal{F}(s) := \int_{0}^{+\infty} \ln(f(e^{-t})) t^{s-1} \mathrm{d}t \quad \text{and} \quad \mathcal{M}(s) := \int_{0}^{+\infty} \ln(\mu(e^{-t})) t^{s-1} \mathrm{d}t + \sum_{12/14} \ln(\mu(e^{-t}))$$

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$$\text{Recall}: \mathcal{F}(s) := \int_0^{+\infty} \ln(f(e^{-t}))t^{s-1} dt \quad \text{and} \quad \mathcal{M}(s) := \int_0^{+\infty} \ln(\mu(e^{-t}))t^{s-1} dt.$$

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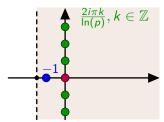
Recall :
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 and $\mathcal{M}(s) := \int_0^{+\infty} \ln(\mu(e^{-t}))t^{s-1} dt$.
• $\mathcal{F}(s) = \frac{\mathcal{M}(s)}{1-p^{-s}}$ with $\mathcal{M}(s) = \underbrace{\widetilde{\mathcal{M}}(s)}_{1-p^{-s}} + \ln(\mu_1)c_1^{-s}\Gamma(s).$

analytic in :

Natural boundary For the order 1 homogeneous equation For the order 2 homogeneous equation

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• Mellin dictionary gives :

$$f(z) = rac{C(z)}{(1-z)^{\log_p(\mu_1)}}(1+o(1)) \quad ext{when } z o 1^-$$

with $C(e^{-s}) = \exp\left(\frac{1}{\ln(p)}\sum_{k\in\mathbb{Z}\setminus\{0\}}\mathcal{M}\left(\frac{2\pi ki}{\ln(p)}\right)s^{-\frac{2\pi ki}{\ln(p)}} + c_0\right)$

Marina Poulet

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Thanks!

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