PARAMETRIC FACTORIZATION OF ALGEBRO-GEOMETRIC ODOs

Sonia L. Rueda, Universidad Politécnica de Madrid

DRN+EFI

12 June 2024

I will present recent and ongoing joint work
with M.A. Zurro, A. Jiménez-Pastor
R. Hernández-Heredero, R. Delgado
and J.J. Morales-Ruiz

Algorithmic Differential Algebra and Integrability (ADAI)

MINISTERIO DE CIENCIA E INNOVACIÓN
UNIÓN EUROPEA
FONDO EUROPEO DE DESARROLLO REGIONAL
AGENCIA ESTATAL DE INVESTIGACIÓN

Contents

Contents

The theory of commuting ODOs

The theory of commuting ODOs has broad connections with many branches of modern mathematics:

- Non-linear partial differential equations (find new exact solutions).
- Algebra (the Dixmier or Jacobian or Poisson conjectures, representation theory)
- Complex analysis. Deformation quantisation ...

The theory of commuting ODOs

Non-linear differential equations (KdV, Boussinesq, KN...KP)

Korteweg-de Vries equation modeled the solitary waves (solitons) in shallow water.

$$\Updownarrow$$

COMMUTING ODOs $\rightleftarrows$ ALGEBRAIC CURVES

Commutative Ordinary Differential Operators.

By J. L. Burchnall and T. W. Chaundy.

(Communicated by A. L. Dixon, F.R.S.—Received December 22, 1926.—Revised February 1, 1928.)

Schur, Wallemberg, Baker, Krichever, Mumford, Mulase ...

The theory of commuting ODOs

Non-linear differential equations (KdV, Boussinesq, KN...KP)

$\Updownarrow$

COMMUTING ODOs (Lax pairs) $\rightleftarrows$ ALGEBRAIC CURVES

DIRECT PROBLEM $\longrightarrow$

Implicitization

Inverse problem $\longleftarrow$

Parametrization

Beret's conjecture [Guo, Zheglov 2022].

Spectral problem

Schrödinger equation

$$\Psi_{xx} - u(x)\Psi = \lambda\Psi \tag{1}$$

with $u(x)$ satisfying a Korteweg de Vries (KdV) equation of the celebrated KdV hierarchy. For instance, the classical stationary KdV equation

$$u_{xxx} - 6uu_x = 0.$$

λ spectral parameter

(Drach's Ideology, 1919) Brehznev 2008, 2012, 2013.
Integrate (1) as an ODE to obtain a parametric solution $\Psi(x; \lambda)$

Spectral problem

(Σ, ∂) ordinary differential field
field of constants $C = \overline{C}$, characteristic 0.

Given

$$L \text{ in } \Sigma[\partial] \backslash C[\partial]$$

assuming L is

ALGEBRO-GEOMETRIC $\equiv$ NON-TRIVIAL CENTRALIZER $\mathcal{Z}(L)$

Parametric solutions $\Psi(x; \lambda, \mu)$

$$L(\Psi) = \lambda\Psi, \quad B(\Psi) = \mu\Psi$$

for $B \in \mathcal{Z}(L)$, $\partial(\lambda) = 0$, $\partial(\mu) = 0$.

Centralizers and spectral curves

Schur, Flanders, Krichever, Amitsur, Carlson, Ore....
[Goodearl, 1983]

$$\mathcal{Z}(L) = \{A \in \Sigma[\partial] \mid [L, A] = 0\}$$

- Trivial $\mathcal{Z}(L) = C[L]$
- Non-trivial $\mathcal{Z}(L)$ is a free $C[L]$-module, the cardinal of a basis divides $\mathrm{ord}(L)$.

SPECTRAL CURVE $\Gamma := \mathrm{Spec}(\mathcal{Z}(L))$

$\mathcal{Z}(L)$ maximal commutative domain in $\Sigma[\partial]$.

ADAI Goals

`Algorithmic Differential Algebra and Integrability (ADAI)`

Develop Picard-Vessiot (PV) theory for spectral problems.
Use effective differential algebra to develop symbolic algorithms:

1. Compute almost commuting bases and Gelfand-Dikey hierarchies.
2. Compute new algebro-geometric ODOs, order ≥ 3.
3. Parametric factorization of algebro-geometric ODOs.
4. Existence and computation of spectral Picard-Vessiot fields. Generalize

(MRZ 2021) J.J. Morales-Ruiz. S.L. Rueda, and M.A. Zurro. **Spectral Picard-Vessiot fields for algebro-geometric Schrödinger operators.** Annales de l'Institut Fourier, Vol. 71, No. 3, pp. 1287-1324, 2021.

Contents

Centralizers in ring of pseudo-differential operators

Commutative differential ring (R, ∂), whose ring of constants is a field of zero characteristic C

$$R((\partial^{-1})) = \left\{ \sum_{i=-\infty}^{n} a_i \partial^i \mid a_i \in R, n \in \mathbb{Z} \right\}$$

$L \in R[\partial]$, centralizer in the ring of pseudo-differential operators

$$\mathcal{Z}(L) \subset \mathcal{Z}((L)) = \{A \in R((\partial^{-1})) \mid [L, A] = 0\}$$

$\mathrm{ord}(L) = n$, $\exists!$ monic pseudo-differential operator $Q = L^{1/n}$.
Generalized Schur's Theorem [Goodearl, 1983]

$$\mathcal{Z}((L)) = \left\{ \sum_{j=-\infty}^{m} c_j Q^j \mid c_j \in C, m \in \mathbb{Z} \right\}$$

Almost commuting basis

Based on [Wilson 1985]

$$W(L) = \{B \in R[\partial] \mid \mathrm{ord}([L, B]) \leq n - 2\}.$$

Given $A = \sum_{i=-\infty}^{n} a_i \partial^i$ in $R((\partial^{-1}))$ then $A_+ = \sum_{i=0}^{n} a_i \partial^i$.

$$\mathcal{Z}((L))_+ := \{A_+ \mid A \in \mathcal{Z}((L))\}$$

(DJHRZ 2024) $L \in R[\partial]$, $\mathrm{ord}(L) = n$,

$$W(L) = \mathcal{Z}((L))_+.$$

C-vector space of almost commuting operators with basis

$$\mathcal{B}(L) := \{P_m := (Q^m)_+ \ \mid m \in \mathbb{N}, Q = L^{1/n}\}$$

Formal differential operators

$U = \{u_0, \ldots, u_{n-2}\}$ differential variables over $\mathbb{Q}$.

$$L = \partial^n + u_{n-2}\partial^{n-2} \cdots + u_1\partial + u_0 \in \mathbb{Q}\{U\}[\partial]$$

(DHJRZ 2024) Symbolic algorithms in $\mathbb{Q}\{U\}[\partial]$:

- Almost commuting basis of $W_M(L)$

$$\{P_m := (Q^m)_+ \ | \ 0 \leq m \leq M, Q = L^{1/n}\}.$$

- Gelfand-Dikey (GD) hierarchy $H_{m,j} \in \mathbb{Q}\{U\}$

$$[L, P_m] = H_{m,0} + H_{m,1}\partial + \ldots + H_{m,n-2}\partial^{n-2}.$$

Wilson's Theorem

$U = \{u_2, \ldots, u_n\}$ differential variables over $\mathbb{Q}$.

Weight function w on $\mathbb{Q}\{U\}$ by $w(u_\ell) = \ell$, and

$$w(u_\ell^{(k)}) = \ell + q, \ k = 1, 2, \ldots.$$

extend to $\mathbb{Q}\{U\}[\partial]$ by $w(\partial) = 1$, homogeneity of $\partial p = p\partial + \partial(p)$, for every $p \in \mathbb{Q}\{U\}$ then

$$L = \partial^n + u_2\partial^{n-2} \cdots + u_{n-1}\partial + u_n \in \mathbb{Q}\{U\}[\partial]$$

homogeneous of weight n

$$\mathcal{H}(L) = \{P_j = (L^{j/n})_+ \mid j \in \mathbb{N}\}$$

is the unique C-basis of $W(L)$ under *homogeneity condition*: each P_j is homogeneous of weight j. (monic and in normal form).

Triangular System

Fix $2 \leq m \leq M$, differential variables $Y = \{y_2, \ldots, y_M\}$ over $\mathbb{Q}\{U\}$. Extend weight function $w\left(y_\ell^{(k)}\right) = \ell + k$.

$$\tilde{P}_m := \partial^m + y_2\partial^{m-2} + \cdots + y_{m-1}\partial + y_m$$

homogeneous of weight m,

$$[L, \tilde{P}_m] = E_{m,0} + E_{m,1}\partial + \ldots + E_{m,n+m-3}\partial^{n+m-3},$$

homogeneous of weight $n + m$ in $\mathbb{Q}\{U, Y\}[\partial]$

(DHJRZ 2024) $n, m \geq 2$, the system

$$\mathbb{E}_{m,n} := \{E_{m,n+m-i}(U, Y_m) = 0 \mid i = 3, \ldots, m+1\}$$

has a unique solution $Z = \{q_2, \ldots, q_m\} \in \mathbb{Q}\{U\}$ in the variables $Y_m = \{y_2, \ldots, y_m\}$, with weights $w(q_\ell) = \ell$.

Triangular System

$Z = \{q_2, \dots, q_m\} \subset \mathbb{Q}\{U\}$ the *weighted solution of* $\mathbb{E}_{m,n}$

(DHJRZ 2024) $i = 3, \dots, m+1$, the differential polynomial

$$E_{m,m+n-i}(U, Y) = n y'_{i-1} + e_{m,i-2}(U, Y_{i-2}),$$

with $e_{m,i-2} \in \mathbb{Q}\{U\}\{Y_{i-2}\}$ homogeneous of weight i for $i \geq 3$ and they are total derivatives of polynomials in $\mathbb{Q}\{U\}$.

We have recursive formulas

$$q_2 := \frac{m}{n} u_2, \;\; q_i := \frac{1}{n} \partial^{-1} e_{m,i}(q_2, \dots, q_{i-1}), \;\; i = 3, \dots, m.$$

$\mathtt{INTEG}(e_{m,i})$ with [Bilge 1992], [Boulier et al. 2016] for partial derivations and general rankings.

$$F(z) \in \mathbb{Q}\{z\}, \text{ write as } F(z) = \partial(A(z)) + B(z)$$

in $\mathbb{Q}\{U\}$ iterate over $u_2 > u_3 > \cdots > u_n$.

Gelfand-Dickey hierarchy for fix n

$$L = \partial^n + u_2\partial^{n-2} + \ldots + u_{n-1}\partial + u_n.$$

Differential field of constants (C, ∂) zero characteristic, $\mathbb{Q} \subset C$.
The Lax equations in this case are

$$L_t = [A_m, L] \ , \quad \text{with } t = t_m \ ,$$

where

$$A_m = \partial^m + a_2\partial^{m-2} + \ldots + a_{m-1}\partial + a_m \in W(L) \subseteq C\{U\}[\partial].$$

in terms of the almost commuting basis

$$A_m = P_m + c_{m,m-1}P_{m-1} + \cdots + c_{m,0}P_0, \ \ c_{m,j} \in C.$$

$$u_{i,t} = H_{m,n-i}(U) + \sum_{j=1}^{m-1} c_{m,j}H_{j,n-i}(U) \ , \text{for} \ \ i = 2, \ldots, n, \ \ m \geq 2.$$

$n = 2$, Korteweg-de Vries (KdV) hierarchy $L_2 = \partial^2 + u_2$

$$u_{2,t} = H_{m,0}(U) + \sum_{j=1}^{m-1} c_{m,j} H_{j,0}(U)\ , \quad m \geq 2.$$

$$u_{2,t} = KdV_m(u_2) + \sum_{j=1}^{m-1} c_{m,j} KdV_j(u_2)\ , \quad m \geq 2.$$

$n = 3$, Bousinesq systems $L_3 = \partial^3 + u_2\partial + u_3$

$$u_{i,t} = H_{m,3-i}(U) + \sum_{j=1}^{m-1} c_{m,j} H_{j,3-i}(U)\ , \text{for} \quad i = 2, 3, \quad m \geq 2.$$

Boussinesq system $m = 5$

$$[L_3, \tilde{P}_5] = E_{5,5}\partial^5 + E_{5,4}\partial^4 + E_{5,3}\partial^3 + E_{5,2}\partial^2 + E_{5,1}\partial + E_{5,0},$$

where

$$\begin{aligned}
E_{5,5} &= \mathbf{3y_2' - 5u_2'},\\
E_{5,4} &= \mathbf{3y_3' + 3y_2'' - 10u_2'' - 5u_3'},\\
E_{5,3} &= \mathbf{3y_4' + 3y_3'' + y_2''' + u_2y_2' - 3u_2'y_2 - 10u_2''' - 10u_3''},\\
E_{5,2} &= \mathbf{3y_5' + 3y_4'' + y_3''' + u_2y_3' - 3u_2''y_2 - 2u_2'y_3 - 3u_3'y_2 - 5u_2^{(4)} - 10u_3'''},\\
E_{5,1} &= 3y_5'' + y_4''' + u_2y_4' - u_2'''y_2 - u_2''y_3 - u_2'y_4 - 3u_3''y_2 - 2u_3'y_3 - u_2^{(5)} - 5u_3^{(}\\
E_{5,0} &= u_2y_5' - u_3'''y_2 - u_3''y_3 - u_3'y_4 - u_3^{(5)} + y_5'''.
\end{aligned}$$

Solving the triangular system w.r.t. $\{y_2, y_3, y_4, y_5\}$ we obtain $Z = \{q_2, q_3, q_4, q_5\}$ and

$$P_5 = \mathcal{E}_Z(\tilde{P}_5) = \partial^5 + \frac{5}{3}u_2\partial^3 + \left(\frac{5}{3}u_2' + \frac{5}{3}u_3\right)\partial^2 + \left(\frac{10}{9}u_2'' + \frac{5}{3}u_3' + \frac{5}{9}u_2^2\right)\partial + \left(\frac{10}{9}u_3'' + \frac{10}{9}u_2u_3\right).$$

The system of the GD hierarchy at level $m = 5$ is,

$$u_{2,t} = H_{5,1} + c_{5,4}H_{4,1} + c_{5,2}H_{2,1} + c_{5,1}H_{1,1},$$
$$u_{3,t} = H_{5,0} + c_{5,4}H_{4,0} + c_{5,2}H_{2,0} + c_{5,1}H_{1,0}$$

where

$$\begin{cases} -H_{5,0}(u_2,u_3) & = & \frac{10}{9}u_2'u_2u_3 + \frac{5}{9}u_2^2u_3' + \frac{10}{9}u_2'''u_3 + \frac{20}{9}u_2''u_3' \\ \quad = \mathcal{E}_Z(E_{5,0}) & & +\frac{5}{3}u_2'u_3'' + \frac{5}{9}u_2u_3''' - \frac{5}{3}u_3''u_3 - \frac{5}{3}u_3'^2 + \frac{1}{9}u_3^{(5)} \\ -H_{5,1}(u_2,u_3) & = & \frac{5}{9}u_2'u_2^2 + \frac{5}{9}u_2'''u_2 + \frac{5}{9}u_2''u_2' + \frac{5}{3}u_2''u_3 + \frac{5}{3}u_2'u_3' \\ \quad = \mathcal{E}_Z(E_{5,1}) & & -\frac{10}{3}u_3'u_3 + \frac{1}{9}u_2^{(5)} \end{cases}$$

SageMath module `almost_commuting.py`

`https://github.com/Antonio-JP/da_wilson`

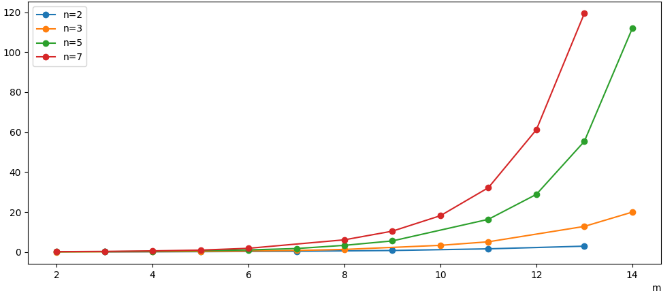

Figure: Time (in seconds) spent when calling the method `almost_commuting_wilson` for inputs (n, m) ranging $n = 2, 3, 5, 7$ and $m = 2, \ldots, 14$. Each line represents the different times for a specific value of n while values for $m \equiv 0$ $(mod\ n)$ are skipped since they are not representative for time considerations.

Contents

Spectral curve of L

(RZ 2024)

Generalized Schur's Theorem

$$\mathcal{Z}((L)) = \left\{ \sum_{j=-\infty}^{m} c_j P^{j/n} \mid c_j \in C, m \in \mathbb{Z} \right\}$$

$n = \mathrm{ord}(L)$, in the ring of pseudo-differential operators $\Sigma((\partial^{-1}))$.

Commutative differential domain

$$\mathcal{Z}(L) = \mathcal{Z}((L)) \cap \Sigma[\partial]$$

$Spec(\mathcal{Z}(L))$ is an abstract algebraic curve Γ

Compute the defining ideal of Γ

BC Ideal of a pair

Commuting P and Q in $\Sigma[\partial]$

$$e_{P,Q} : C[\lambda, \mu] \to \Sigma[\partial]$$

homomorphism of C-algebras defined by

$$g(P, Q) := e_{P,Q}(g) = e_{P,Q}(\sigma_{i,j}\lambda^i\mu^j) = \sigma_{i,j}P^iQ^j.$$

Define the Burchnall-Chaundy BC ideal of the pair P and Q as

$$\mathrm{BC}(P, Q) := \mathrm{Ker}(e_{P,Q}) = \{g \in C[\lambda, \mu] \mid g(P, Q) = 0\}.$$

Its elements are BC polynomials

Spectral curve of a pair

Commuting P and Q in $\Sigma[\partial]\backslash C[\partial]$

$\mathcal{Z}(P)$ finitely generated $C[P]$-module $\Rightarrow \mathtt{BC}(P, Q)$ non zero ideal.

$\Sigma[\partial]$ domain $\Rightarrow \mathtt{BC}(P, Q)$ prime ideal.

Spectral curve $\Gamma_{P,Q} := V(\mathtt{BC}(P, Q))$

Coordinate ring of $\Gamma_{P,Q}$

$$\frac{C[\lambda, \mu]}{\mathtt{BC}(P, Q)} \simeq C[P, Q].$$

Spectral curve of a pair

There exists an irreducible polynomial $f \in C[\lambda, \mu]$ such that

$$\mathtt{BC}(P, Q) = (f)$$

$$\Gamma_{P,Q} = \{\ (\lambda_0, \mu_0) \in C^2 \mid f(\lambda_0, \mu_0) = 0 \}\,.$$

How do we compute f?

Computing BC ideals

Given (monic) $P,\ Q \in \Sigma[\partial]$, then $P-\lambda,\ Q-\mu$ in $\mathbb{D}=\Sigma[\lambda,\mu]$.

$\mathrm{ord}(P)=n$, $\mathrm{ord}(Q)=m$

$$h(\lambda,\mu) = \partial\mathrm{Res}(P-\lambda, Q-\mu) = \mu^n - \lambda^m + \dots$$

a non trivial polynomial in $\Sigma[\lambda,\mu]$

Generalize [Wilson, 1985], [Previato, 1991].

(RZ 2023) Arbitrary (Σ,∂), $Const(\Sigma)=C=\overline{C}$.

If $[P,Q]=0$ then $h(\lambda,\mu)\in \mathtt{BC}(P,Q)$.

1. Proof by Poisson's Formula $h(\lambda,\mu)\in C[\lambda,\mu]$.
2. Proof by elimination ideals $h(P,Q)=0$.

Rosen-Morse potential $u_1 = \frac{-2}{\cosh^2(x)}$

$$L_1 = -\partial^2 + u_1,\ [L_1, A_3] = \mathrm{KdV}_0(\mathrm{u}_1) + \mathrm{KdV}_1(\mathrm{u}_1) = 0$$

$$f_1(\lambda,\mu) = -\mu^2 - \lambda(\lambda-1)^2 =$$
$$= \partial\mathrm{Res}(L_1-\lambda, A_3-\mu) =$$

$$\begin{vmatrix} -1 & 0 & \frac{-2}{(\cosh(x))^2} - \lambda & 8\,\frac{\sinh(x)}{(\cosh(x))^3} & \frac{4}{(\cosh(x))^2} - 12\,\frac{(\sinh(x))^2}{(\cosh(x))^4} \\ 0 & -1 & 0 & \frac{-2}{(\cosh(x))^2} - \lambda & 4\,\frac{\sinh(x)}{(\cosh(x))^3} \\ 0 & 0 & -1 & 0 & \frac{-2}{(\cosh(x))^2} - \lambda \\ -1 & 0 & \frac{-3}{(\cosh(x))^2} + 1 & 9\,\frac{\sinh(x)}{(\cosh(x))^3} - \mu & \frac{3}{(\cosh(x))^2} - 9\,\frac{(\sinh(x))^2}{(\cosh(x))^4} \\ 0 & -1 & 0 & \frac{-3}{(\cosh(x))^2} + 1 & 3\,\frac{\sinh(x)}{(\cosh(x))^3} - \mu \end{vmatrix}$$

Elimination ideals

Left ideal

$$(P-\lambda, Q-\mu) = \{C(P-\lambda) + D(Q-\mu) \mid C, D \in \Sigma[\lambda,\mu][\partial]\}$$

Two sided ideals

$$\mathcal{E}(P-\lambda, Q-\mu) := (P-\lambda, Q-\mu) \cap \Sigma[\lambda,\mu].$$

and

$$\mathcal{E}_C(P-\lambda, Q-\mu) := (P-\lambda, Q-\mu) \cap C[\lambda,\mu].$$

By definition of the differential resultant

$$h(\lambda,\mu) = \partial\mathrm{Res}(P-\lambda, Q-\mu) \in \mathcal{E}_C(P-\lambda, Q-\mu).$$

Thus both elimination ideals are nonzero.

Elimination ideals

Commuting P and Q in $\Sigma[\partial] \setminus C[\partial]$, both of positive order,

$$f = \sqrt{h}, \text{ with } h = \partial\mathrm{Res}(P - \lambda, Q - \mu).$$

(RZ 2023)

1. The radical of the elimination ideal $\mathcal{E}_C(P - \lambda, Q - \mu)$ equals
$$\mathtt{BC}(P, Q) = (f).$$
2. The radical of the elimination ideal $\mathcal{E}(P - \lambda, Q - \mu)$ equals $[f]$.

Recall $f \in C[\lambda, \mu]$,

$$(f) = C[\lambda, \mu]f \text{ and } [f] = \Sigma[\lambda, \mu]f \text{ differential ideal.}$$

New coefficient field

$P, Q \in \Sigma[\partial]$

$$[P, Q] = 0 \Rightarrow \partial\mathrm{Res}(P - \lambda, Q - \mu) = f(\lambda, \mu)^r \in C[\lambda, \mu].$$

As differential operators in $\Sigma[\lambda, \mu][\partial]$,

$$\partial\mathrm{Res}(P - \lambda, Q - \mu) \neq 0 \Rightarrow \mathrm{gcrd}(P - \lambda, Q - \mu) = 1.$$

$$\Sigma(\Gamma_{P,Q}) = Fr\left(\frac{\Sigma[\lambda, \mu]}{[f]}\right)$$

As differential operators in $\Sigma(\Gamma_{P,Q})[\partial]$,

$$\partial\mathrm{Res}(P - \lambda, Q - \mu) = 0 \Rightarrow \mathrm{gcrd}(P - \lambda, Q - \mu) \neq 1.$$

Centralizer $\mathrm{ord}(L) = 3$

Given $L \in \Sigma[\partial] \backslash C[\partial]$, with $\mathcal{Z}(L) \neq C[L]$.

$\mathcal{Z}(L)$ is a free $C[L]$-module of rank 3.

$\{1, A_1, A_2\}$ basis of $\mathcal{Z}(L)$ as a $C[L]$-module.
A_i is a monic operator in $\mathcal{Z}(L) \backslash C[L]$ of minimal order

$$o_i := \mathrm{ord}(A_i) \equiv i \ (mod\ 3).$$

$$\mathcal{Z}(L) = C[L] \oplus C[L]A_1 \oplus C[L]A_2 = C[L, A_1, A_2]$$

BC ideal $\mathrm{ord}(L) = 3$

$$e_L : C[\lambda, \mu_1, \mu_2] \to \Sigma[\partial]$$
$$e_{P,Q}(\lambda) = L, \ \ e_{P,Q}(\mu_1) = A_1, \ \ e_{P,Q}(\mu_2) = A_2.$$

Image of e_L,

$$\mathcal{Z}(L) = C[L, A_1, A_2]$$

Given $g \in C[\lambda, \mu_1, \mu_2]$ denote

$$g(L, A_1, A_2) := e_L(g).$$

$$\mathtt{BC}(L) := \mathrm{Ker}(e_L) = \{g \in C[\lambda, \mu_1, \mu_2] \mid g(L, A_1, A_2) = 0\}.$$

Spectral curve $\operatorname{ord}(L) = 3$

$$\operatorname{ord}(L) = 3 \text{ in } \Sigma[\partial], \quad \mathcal{Z}(L) = C[L, A_1, A_2], \quad \operatorname{ord}(A_i) \equiv_3 i$$

$$\begin{cases} f_i = \partial\mathrm{Res}(L - \lambda, A_i - \mu_i), & i = 1, 2 \\ f_3^r = \partial\mathrm{Res}(A_1 - \mu_1, A_2 - \mu_2) \end{cases}$$

are irreducible in $C[\lambda, \mu_1, \mu_2]$ since

$$\mathtt{BC}(L, A_i) = (f_i) \text{ and } BC(A_1, A_2) = (f_3)$$

$$(0) \subset (f_i) \subset (f_1, f_2) \subseteq (f_1, f_2, f_3) \subseteq \mathtt{BC}(L), \quad i = 1, 2.$$

$$\Gamma := V(\mathtt{BC}(L)) \subseteq \gamma := V(f_1, f_2, f_3) \subseteq \beta := V(f_1, f_2).$$

Space algebraic curve $\beta = V(f_1) \cap V(f_2)$ is the intersection of the irreducible surfaces defined by $f_1(\lambda, \mu_1) = 0$ and $f_2(\lambda, \mu_2) = 0$.

Spectral curve $\operatorname{ord}(L) = 3$

Theorem: (RZ 2024) $\mathtt{BC}(L)$ is a prime ideal

$$\mathtt{BC}(L) = (f_1, f_2, f_3)$$

Irreducible affine algebraic curve in C^3

$$\Gamma = V(\mathtt{BC}(L))$$

$$\mathcal{Z}(L) \simeq C[\Gamma] = \frac{C[\lambda, \mu_1, \mu_2]}{\mathtt{BC}(L)}$$

If $\operatorname{ord}(A_2) = 2$ then $A_1 = A_2^2$ implying that $f_3 = (\mu - \gamma^2)^2$.

$$\mathcal{Z}(L) = \mathcal{C}(A_2) = C[L, A_2] \simeq \frac{C[\lambda, \mu]}{(f_2)}$$

coordinate ring of a plane algebraic curve.

Planar spectral curves

[Dickson, Gesztesy, Unterkofler, 1999] $\Sigma = \mathbb{C}(x), \partial = d/dx$

$$L = \partial^3 - \frac{15}{x^2}\partial + \frac{15}{x^3} + h \ .$$

$$\mathcal{Z}(L) = C[L, A_1, A_2] \ , \ \mathrm{ord}(A_1) = 4, \mathrm{ord}(A_2) = 8.$$

We compute the generators of the ideal $\mathtt{BC}(L) = (f_1, f_2, f_3)$ using differential resultants

$$f_1 = -\mu_1^3 + (\lambda - h)^4 \ , f_2 = -\mu_2^3 + (\lambda - h)^8, f_3^4 = (\mu_2 - \mu_1^2)^4.$$

Since f_3 is the BC polynomial of A_1 and A_2 we have $A_2 = A_1^2$, implying that

$$\mathcal{Z}(L) = C[L, A_1] \simeq \frac{C[\lambda, \mu_1]}{(f_1)}$$

Non-planar spectral curves

(RZ 2022) $\Sigma = \mathbb{C}(x), \partial = d/dx$

$$L = \partial^3 - \frac{6}{x^2}\partial + \frac{12}{x^3} + h \,, \;\; h \in \mathbb{C}.$$

$\mathcal{Z}(L) = \mathbb{C}[L, A_1, A_2]$ with $\mathrm{ord}(A_1) = 4$, $\mathrm{ord}(A_2) = 5$.
Using differential resultants we compute

$$f_1 = -\mu^3 + (\lambda - h)^4, \;\; f_2 = -\gamma^3 + (\lambda - h)^5, \;\; f_3 = \gamma^4 - \mu^5.$$

$\mathtt{BC}(L) = (f_1, f_2, f_3)$ is a prime ideal.

First explicit example of a non-planar spectral curve.

The curve defined by $\mathtt{BC}(L)$ is a non-planar curve Γ parametrized by

$$\aleph(\tau) = (h - \tau^3, \tau^4, -\tau^5), \tau \in \mathbb{C}.$$

Contents

New coefficient field

$\mathrm{ord}(L) = 3$

Theorem: (RZ 2024)

$[\mathtt{BC}(L)]$ is a prime differential ideal of $\Sigma[\lambda, \mu_1, \mu_2]$

Differential domain

$$\Sigma[\Gamma] = \frac{\Sigma[\lambda, \mu_1, \mu_2]}{[\mathtt{BC}(L)]}$$

Its fraction field

$$\Sigma(\Gamma)$$

is a differential field with the extended derivation.

Intrinsic right factor

$\mathrm{ord}(L) = 3$ in $\Sigma[\partial]$, $\mathcal{Z}(L) = C[L, A_1, A_2]$

Theorem: (RZ 2024) The greatest common right divisor in $\Sigma(\Gamma)[\partial]$

$$\partial + \phi = \mathrm{gcrd}(L - \lambda, A_1 - \mu_1, A_2 - \mu_2)$$

equals $\mathrm{gcrd}(L - \lambda, A_1 - \mu_1) = \mathrm{gcrd}(L - \lambda, A_2 - \mu_2)$
and divides $\mathrm{gcrd}(A_1 - \mu_1, A_2 - \mu_2))$.

Assume $L = \partial^3 + u_1\partial + u_0$

$$L - \lambda = \left(\partial^2 - \phi\partial + u_1 - 2\phi' + \phi^2\right) \cdot (\partial + \phi),$$

in $\Sigma(\Gamma)[\partial]$, under the condition

$$\phi^3 + u_1\phi - 3\,\phi\phi' - u_0 + \phi'' + \lambda = 0.$$

Non-planar spectral curve

$$\aleph(\tau) = (-\tau^3 + 1, \tau^4, -\tau^5),\ \tau \in \mathbb{C}.$$

The first differential subresultants of $L - \lambda$, $A_1 - \mu_1$ and $A_2 - \mu_2$ pairwise are equal to

$$\phi_{i,0} + \phi_{i,1}\partial,\ \ i = 1, 2, 3,\ \ j = 0, 1,$$

with

$$\begin{aligned}
\phi_{1,0} &= (1-\lambda)\mu_1 - \tfrac{4\mu_1}{x^3} + \tfrac{8(\lambda-1)}{x^4}, & \phi_{1,1} &= (\lambda-1)^2 - \tfrac{2\mu_1}{x^2} + 4\tfrac{(\lambda-1)}{x^3},\\
\phi_{2,0} &= (1-\lambda)^3 - \tfrac{4(1-\lambda)^2}{x^2} + \tfrac{8\mu_2}{x^4}, & \phi_{2,1} &= (\lambda-1)^3 - \tfrac{4(1-\lambda)^2}{x^2} + \tfrac{8\mu_2}{x^3},\\
\phi_{3,0} &= -\mu_2\left(\mu_1^2 + \tfrac{4\mu_2}{x^3} - \tfrac{8\mu_1}{x^4}\right), & \phi_{3,1} &= \mu_1^3 - \tfrac{2\mu_2^2}{x^2} + \tfrac{4\mu_2\mu_1}{x^3}.
\end{aligned}$$

We have $\mathrm{ord}(A_1) = 4$ and $\mathrm{ord}(A_2) = 5$ thus

$$\phi = \overline{\phi}_i = \frac{\phi_{0,i}}{\phi_{1,i}} + [\mathrm{BC}(L)], \quad i = 1, 2, 3.$$

$$\phi(\tau) := \phi_i(\aleph(\tau)) = \frac{-\tau^3 x^3 + 2\tau^2 x^2 - 4\tau x + 4}{(\tau^2 x^2 - 2\tau x + 2)x}.$$

Thus

$$L + \tau^3 - 1 = \left(\partial^2 + \phi(\tau)\partial + \phi(\tau)^2 + 2\phi(\tau)' - \frac{6}{x^2}\right) \cdot (\partial + \phi(\tau))$$

At every point $P_0 = \aleph(\tau_0)$ of the spectral curve Γ of L we recover a right factor $\partial + \phi(\tau_0)$, for $\tau_0 \neq 0$.

τ free parameter. Full factorization? $\rightarrow$ Parametric Picard-Vessiot theory [Cassidy, Singer, 2006], [Arreche 2016]

Algebro-geometric Schrödinger operators

$$L_s = -\partial^2 + u_s$$

where u_s are KdV-solitons, solutions of $\mathrm{KdV}_s(u, \bar{c}^s) = 0$.

Rational	Rosen-Morse	Elliptic
$u_s = \dfrac{s(s+1)}{x^2}$	$u_s = \dfrac{-s(s+1)}{\cosh^2(x)}$	$u_s = s(s+1)\wp(x; g_2, g_3)$

[Veselov, A.P., 2011. **On Darboux-Treibich-Verdier Potentials**. Letters in Mathematical Physics, 96(1), 209-216.]

Algebro-geometric Schrödinger operator

Algebro-geometric Schrödinger operator $L_s = -\partial^2 + u_s$

$$\mathcal{Z}(L_s) = C[L_s, A_{2s+1}].$$

$$f_s(\lambda,\mu) = \partial\mathrm{Res}(L_s - \lambda, A_{2s+1} - \mu) = -\mu^2 + R_{2s+1}(\lambda)$$

f_s is irreducible in $\Sigma[\lambda,\mu]$

$$C(\Gamma_s) = Fr\left(\frac{C[\lambda,\mu]}{(f_s)}\right) \text{ and } \Sigma(\Gamma_s) = Fr\left(\frac{\Sigma[\lambda,\mu]}{[f_s]}\right)$$

Factorization of $L_s - \lambda = (-\partial - \phi_s)(\partial - \phi_s)$ in $\Sigma(\Gamma_s)[\partial]$.

$$\partial - \phi_s = \mathrm{gcrd}(L_s - \lambda, A_{2s+1} - \mu)$$

Contents

Spectral Picard-Vessiot extension

Theorem: (MRZ 2021) The field of constants of $(\Sigma(\Gamma_s), \tilde{\partial})$ is $C(\Gamma_s)$.

$C(\Gamma_s)$ is not algebraically closed!

Definition:(MRZ 2021) A differential field extension $\mathcal{L}$ of $\Sigma(\Gamma_s)$ is called a **Spectral Picard-Vessiot extension over the curve** Γ_s of the equation $(L_s - \lambda)(\Psi) = 0$ if:

1. $\mathcal{L} = \Sigma(\Gamma_s)\langle\Psi_1, \Psi_2\rangle$, the differential field extension of $\Sigma(\Gamma_s)$ generated by $\{\Psi_1, \Psi_2\}$, a fundamental set of solutions of $(L_s - \lambda)(\Psi) = 0$.
2. $\mathcal{L}$ and $\Sigma(\Gamma_s)$ have the same field of constants $C(\Gamma_s)$.

[Kaplansky, 1976], [Crespo, Hajto, 2011]

From the right factor to the solutions

$L_s - \lambda = (-\partial - \phi_s)(\partial - \phi_s)$ in $\Sigma(\Gamma_s)[\partial]$

$$\phi_+ := \phi_s = \frac{\mu + \alpha(\lambda)}{\varphi_2(\lambda)} \text{ and } \phi_- := \frac{-\mu + \alpha(\lambda)}{\varphi_2(\lambda)}.$$

Nonzero solutions Ψ_+ and Ψ_- of $(L_s - \lambda)(\Psi) = 0$ are defined by the differential relations

$$\partial(\Psi_+) = \phi_+\Psi_+ \text{ and } \partial(\Psi_-) = \phi_-\Psi_-$$

Ψ_+ and Ψ_- belong to the differential closure of the field $\Sigma(\Gamma_s)$.

$$\Psi_+\Psi_- = \frac{\varphi_2 w(\Psi_+, \Psi_-)}{2\mu} \in \Sigma(\Gamma_s).$$

Existence of spectral Picard-Vessiot extension

We have

$$\Sigma(\Gamma_s)\langle \Psi_+, \Psi_- \rangle = \Sigma(\Gamma_s)\langle \Psi_+ \rangle = \Sigma(\Gamma_s)\langle \Psi_- \rangle$$

a Liouvillian extension of $\Sigma(\Gamma_s)$ since

$$\frac{\partial(\Psi_+)}{\Psi_+} = \phi_+ \text{ and } \frac{\partial(\Psi_-)}{\Psi_-} = \phi_- \text{ belong to } \Sigma(\Gamma_s).$$

$\Psi_s := \Psi_+$

$$\begin{array}{ccc} \Sigma(\Gamma_s) & \hookrightarrow & \Sigma(\Gamma_s)\langle \Psi_s \rangle \\ \uparrow & & \uparrow \\ C(\Gamma_s) & = & C(\Gamma_s) \end{array}$$

Existence of spectral Picard-Vessiot extension

L_s algebro-geometric Schrödinger operator

$$L_s - \lambda = -(\partial - \phi_s)(\partial - \phi_s) \text{ in } \Sigma(\Gamma_s)[\partial].$$

Solution Ψ_s of $(L_s - \lambda)\Psi = 0$ defined by $\partial(\Psi_s) = \phi_s \Psi_s$.

Using [Bronstein, 2013]

Theorem:(MRZ 2021)

1. Ψ_s is transcendental over $\Sigma(\Gamma_s)$.
2. The field of constants of $\Sigma(\Gamma_s)\langle\Psi_s\rangle$ is $C(\Gamma_s)$.

Theorem:(MRZ 2021) $\Sigma(\Gamma_s)\langle\Psi_s\rangle$ is a Spectral Picard-Vessiot extension over the curve Γ_s of the equation $(L_s - \lambda)(\Psi) = 0$.

Recovering classical Picard–Vessiot

$L_s = -\partial^2 + u_s$ algebro-geometric,
$K = C\langle u_s\rangle$, field of constants $C = \overline{C}$

$$
\begin{array}{ccccc}
C(\Gamma_s) & \longrightarrow & K(\Gamma_s) & \xrightarrow{(-\partial+\phi_s)(\Psi_s)=0} & K(\Gamma_s)\langle\Psi_s\rangle \\
\uparrow & & \uparrow & & \uparrow \\
C & \xrightarrow{\text{at } P_0\in\Gamma_s} & K & \xrightarrow{(-\partial+\phi_0)(y_0)=0} & K\langle y_0\rangle
\end{array}
$$

$P_0 = (\lambda_0, \mu_0)$ in Γ_s (singular or not),
$P_0 \notin Z_s = \{(\lambda, 0) \,|\, R_{2s+1}(\lambda) = 0\}$

$K\langle y_0\rangle$ is the PV extension of $K = C\langle u_s\rangle$ for $L_s y = \lambda_0 y$.

Rational spectral curves

τ is an algebraic indeterminate over $K = \mathbb{C}\langle u_s \rangle$

$$\mathbb{C}(\Gamma_s) \overset{\rho_1}{\simeq} \mathbb{C}(\tau) \quad \text{with } \rho_1(\lambda) = \chi_1(\tau) \text{ and } \rho_1(\mu) = \chi_2(\tau)$$

$$\begin{array}{ccc} K(\Gamma_s) & \xrightarrow{\rho_1} & K(\tau) \\ \uparrow & & \uparrow \\ \mathbb{C}(\Gamma_s) & \xrightarrow{\rho_1} & \mathbb{C}(\tau) \end{array}$$

ρ_1 extends to an isomorphism ϱ: $K(\Gamma_s)[\partial] \simeq K(\tau)[\partial]$.

[Makar-Limanov, 2021]. Centralizers of rank one in the first Weyl algebra provide rational curves.

One-parameter form factorization

$$L_s - \chi_1(\tau) = \varrho(L_s - \lambda)$$

$$A_{2s+1} - \chi_2(\tau) = \varrho(A_{2s+1} - \mu)$$

in $K(\tau)[\partial]$, with $\tilde{\phi}_s = \rho_1(\phi_s)$

$$L_s - \chi_1(\tau) = (-\partial - \tilde{\phi}_s)(\partial - \tilde{\phi}_s), \text{ in } K(\tau)[\partial].$$

Theorem:(MRZ 2021) The Liouvillian extension $K(\tau)\langle\Upsilon_s\rangle$ of $K(\tau)$, by a nonzero solution $\Upsilon_s \in \widehat{K(\tau)}$ of $(\partial - \tilde{\phi}_s)\Upsilon = 0$, is a transcendental extension with field of constants $\mathbb{C}(\tau)$.

One-parameter form Spectral Picard-Vessiot

Spectral PV extension $K(\Gamma_s)\langle\Psi_s\rangle$ of $K(\Gamma_s)$ for $L_s - \lambda$

$$\begin{array}{ccc} K(\Gamma_s)\langle\Psi_s\rangle & \xrightarrow{\hat{\rho}_1} & K(\tau)\langle\Upsilon_s\rangle \\ \uparrow & & \uparrow \\ \mathbb{C}(\Gamma_s) & \xrightarrow{\rho_1} & \mathbb{C}(\tau) \end{array}$$

Parametric Picard-Vessiot theory τ free parameter.

Advanteges:

- Effective computation of Liouvillian solution Υ_s.

$$K(\tau) = \mathbb{C}\langle u_s \rangle(\tau) = \mathbb{C}(\tau)\langle u_s \rangle \subset K(\tau)(\Upsilon_s).$$

  If u_s is a monomial over the differential field $\mathbb{C}(\tau)$ one can address the integration of $(\partial - \tilde{\phi}_s)\Upsilon = 0$ in the differential algebraic setting of (Bronstein, 2013).

- We established the appropriate algebraic setting to solve the spectral problem analytically for rational curves, possibly with singularities. Whenever $u = u(x)$ is an analytic potential in some complex domain, we describe the analytic character of the Liouvillian solution.

Rosen-Morse $u_s = \frac{-s(s+1)}{\cosh^2(x)}$, $s \geq 1$

$K = \mathbb{C}(z = e^x)$, $\partial = d/dx$, field of constants $\mathbb{C}$.
Rational curves Γ_s, $(\chi_1(\tau), \chi_2(\tau)) = \left(-\tau^2, -\tau \prod_{\kappa=1}^{s}(\tau^2 - \kappa^2)\right)$.

$$\phi_1 = \frac{\mu + \frac{1}{2}\partial(\varphi)}{\varphi}$$

$$\tilde{\phi}_1 = \frac{(z^2+1)^3\chi_2(\tau) + z^4 - z^2}{(z^2+1)((z^2+1)^2\chi_1(\tau) + z^4 + z^2 + 1)} \in \mathbb{C}(\tau)(z)$$

Since $z = e^x$ is transcendental over $\mathbb{C}(\tau)$, using the symbolic integration package of Maple to obtain Υ_s as $\mathrm{int}(\tilde{\phi}_s, x)$.

$$\Upsilon_1 = \frac{(\tau - 1)\, z^2 + \tau + 1}{z^2 + 1} e^{x\tau} \in \mathbb{C}(\tau, e^x, e^{x\tau}).$$

All functions are analytic outside the analytic set

$$E = \{\ z^4 + z^2 + 1 - (z^2+1)^2\tau^2 = 0\ \} \subset \mathbb{C}^2.$$

Spectral problem

(Σ, ∂) ordinary differential field
with field of constants $C = \overline{C}$, characteristic 0.

Given a normalized ODO and p prime or not

$$L = \partial^p + u_{p-2}\partial^{p-2} + \cdots + u_1\partial + u_0 \text{ in } \Sigma[\partial],$$

assuming L algebro-geometric

NON-TRIVIAL CENTRALIZER $\mathcal{Z}(L) \simeq C[\Gamma]$

$\Sigma(\Gamma)$

Spectral PV extension of $\Sigma(\Gamma)$ for $L(\Psi) = \lambda\Psi$

- (DHJRZ 2024) Delgado, R., Hernandez-Heredero, R.J., Jimenez-Pastor, A., Rueda, S. L., Zurro, M-A. **Computing almost commuting bases of ODOs and Gelfand-Dikey hierarchies**. arXiv prerprint
- (RZ 2024) Rueda, S. L., Zurro, M.A. **Spectral Curves for Third-Order ODOs**. Axioms. 2024; 13(4):274.
- (MRZ 2021) J.J. Morales-Ruiz. S.L. Rueda, and M.A. Zurro. **Spectral Picard-Vessiot fields for algebro-geometric Schrödinger operators .** Annales de l'Institut Fourier, Vol. 71, No. 3, pp. 1287-1324, 2021.
- (MRZ 2020) J.J. Morales-Ruiz. S.L. Rueda, and M.A. Zurro. **Factorization of KdV Schrödinger operators using differential subresultants.** Adv. Appl. Math., 120:102065, 2020.
- (PRZ 2019) E. Previato, S.L. Rueda, and M.A. Zurro. **Commuting Ordinary Differential Operators and the Dixmier Test**. SIGMA Symmetry Integrability Geom. Methods Appl., 15(101):23 pp., 2019.