Introduction to integrable hierarchies

The example of combinatorial maps

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De Rerum Natura + Équations fonctionnelles et Interactions
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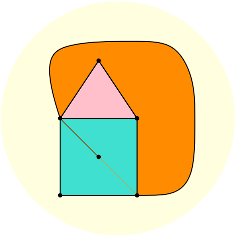
Disclaimer

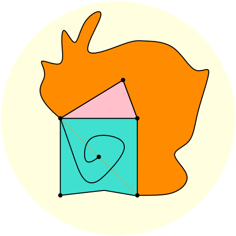
- Every similarity with textbooks and existing articles from other authors is purely intentional.
- This is mostly a mini-course.
- Some work in progress, which could definitely benefit from this community!

Plan

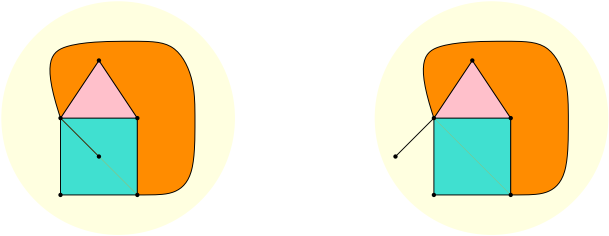
- Combinatorial maps and functional equations
- Integrable systems and hierarchies
- Back to maps

- Combinatorial maps are graphs "properly" embedded in surfaces, up to deformation



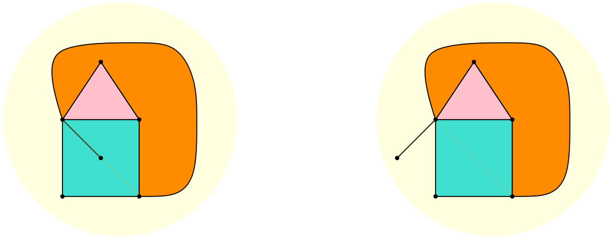


- Combinatorial maps are graphs “properly” embedded in surfaces, up to deformation



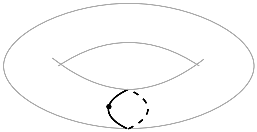
- The graph complement is a disjoint union of disks, called *faces*

- Combinatorial maps are graphs "properly" embedded in surfaces, up to deformation



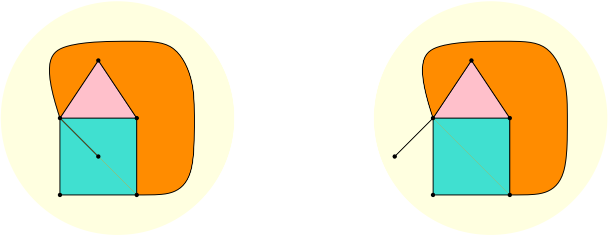
- The graph complement is a disjoint union of disks, called *faces*

Not a map!



Combinatorial maps

- Combinatorial maps are graphs "properly" embedded in surfaces, up to deformation

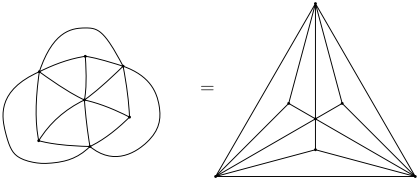


- The graph complement is a disjoint union of disks, called *faces*
- Encoded by cyclic order around vertices $\rightarrow$ permutations and representations of the symmetric group

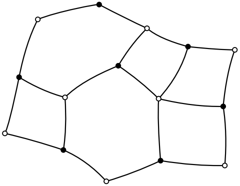
Examples

- They are topological surfaces, nice interplay between combinatorics and topology
- Euler's formula for the genus $g \geq 0$

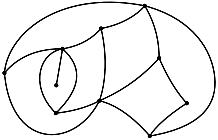
$$2g = 2 - F + E - V \geq 0$$



Planar triangulation

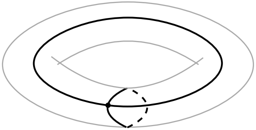


Planar bipartite map

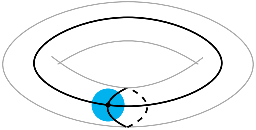


Planar quadrangulation

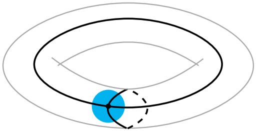
Non-zero genus

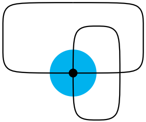


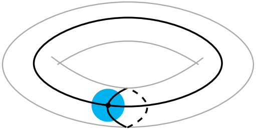
Non-zero genus

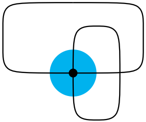


Non-zero genus

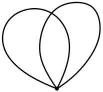




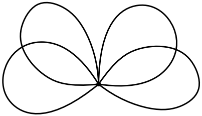




Drawing in the plane of map with genus: Crossings



Map of genus 1



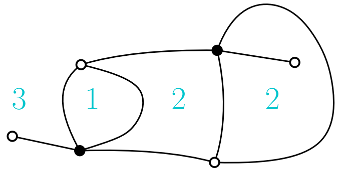
Map of genus 2

Enumeration problems

- Count maps by genus and size
- E.g. planar maps by number of edges [Tutte 60s]

$$27t^2 M(t)^2 + (1 - 18t)M(t) + 16t - 1 = 0$$

  implies $[t^n]M(t) = \frac{2 \cdot 3^n}{(n+1)(n+2)} \binom{2n}{n}$
- What kind of equations do the GF satisfy?
- Consider bipartite maps. The degree of a face is half its number of sides.
- A partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ with $\lambda_1 \geq \cdots \geq \lambda_l \geq 0$ like $(3, 2, 2, 1)$
- Encode the degrees of white vertices in a partition



$\lambda = (3, 2, 2, 1)$

- Notice that the size, i.e. number of edges, is $n = \sum_{i=1}^{l} \lambda_i$.

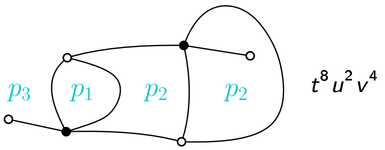
Generating functions

- Consider an infinite set of indeterminates $x_1, x_2, x_3, \ldots$ where x_i is associated to each face of degree i
- For $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$

$$x_\lambda = x_1 \cdots x_l \qquad\qquad x_{(3,2,2,1)} = x_3 x_2^2 x_1$$

- Denote $\mathcal{M}_\lambda$ the set of bip. maps, connected or not with face partition λ, having $n = \sum_i \lambda_i$ edges.
- $\tau \equiv \tau(t, u, v, x_1, x_2, \ldots)$ the GF of labeled bip. maps, connected or not

$$\tau = \sum_{n \geq 0} \frac{t^n}{n!} \sum_{|\lambda|=n} \sum_{\mathcal{M}_\lambda} u^{V_\circ} v^{V_\bullet} x_\lambda \in \mathbb{Q}[u, v, x_1, x_2, \ldots][[t]]$$



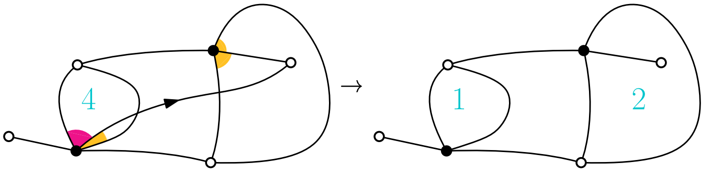
- Connected: $F = \log \tau$
- Control genus $F^{(g)} = [w^{2-2g}] F_{|t \to t/w, x_i \to w x_i, u, v \to u/w, v/w}$

Virasoro constraints

- “Rooting” operation: mark a corner in a face $\to x_i^* \equiv \frac{i\partial}{\partial x_i}$
- Thm τ is uniquely determined by the equations $L_i\tau = 0$ for $i \geq 0$

$$L_i = -x_{i+1}^* + t\sum_{j+k=i} x_j^* x_k^* + t\sum_{j\geq 1} x_j x_{i+j}^* + t(u+v)x_i^* + tuv\delta_{i,0}$$

- Proof $x_{i+1}^*\tau$ counts maps with a rooted face of degree $i+1$
- Remove the root edge and consider the different cases
- The root edge “joins” two different faces



$$t\sum_{j+k=i} x_j^* x_k^*$$

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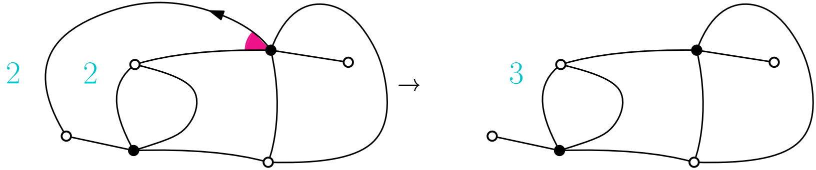
- Proof $x_{i+1}^*\tau$ counts maps with a rooted face of degree $i+1$
- Remove the root edge and consider the different cases
- The root edge "cuts" a face



$$t \sum_{j\geq 1} x_j x_{i+j}^*$$

Virasoro constraints

- "Rooting" operation: mark a corner in a face $\to x_i^* \equiv \frac{i\partial}{\partial x_i}$
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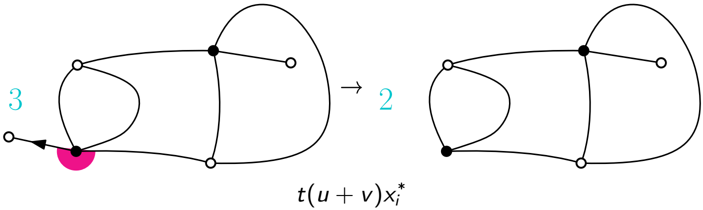
- Proof $x_{i+1}^*\tau$ counts maps with a rooted face of degree $i+1$
- Remove the root edge and consider the different cases
- The root edge connects to a "new" vertex

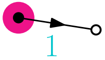


Virasoro constraints

- "Rooting" operation: mark a corner in a face $\to x_i^* \equiv \frac{i\partial}{\partial x_i}$
- Thm τ is uniquely determined by the equations $L_i\tau = 0$ for $i \geq 0$

$$L_i = -x_{i+1}^* + t \sum_{j+k=i} x_j^* x_k^* + t \sum_{j\geq 1} x_j x_{i+j}^* + t(u+v)x_i^* + tuv\delta_{i,0}$$

- Proof $x_{i+1}^*\tau$ counts maps with a rooted face of degree $i+1$
- Remove the root edge and consider the different cases
- The root face has degree 1



tuv

- $[L_i, L_j] = t(i-j)L_{i+j-1}$
- Important remark: if we set degrees as follows:

$$\deg(x_i) = -i, \qquad \deg(x_i^*) = i$$

- then $[t^n]\tau$ is homogeneous of degree $-n$
- But L_i is not homogeneous, because it is inductive on the size

Applications

Cases

1. $x_i = 1$ for all $i \geq 1 \to$ all maps
2. $d \geq 2$ and keep $x_1, \dots, x_d$ formal and set $x_{d+1} = x_{d+2} = \cdots = 0$
   allow only for a finite number of face degrees

- Extract an equation in the planar sector

$$\sum_i z^i [w^2] e^{-F} L_i e^F_{|t \to t/w, x_i \to w x_i, u, v \to u/w, v/w} = 0$$

Applications

Cases

1. $x_i = 1$ for all $i \geq 1 \to$ all maps
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   allow only for a finite number of face degrees

- Gives a equation on $W(z) = \sum_{i\geq 1} z^i x_i^* F^{(0)}$ involving unknown series, said to have one catalytic variable
- Case 1

$$tzW(z)^2 + (tz(u+v) - 1)W(z) + tz\frac{W(z) - W(1)}{z-1} + tuv = 0$$

- Case 2

$$tzW(z)^2 + \left(t\sum_{i=1}^{d} x_i z^{-i+1} + tz(u+v) - 1\right)W(z)$$

$$- t\sum_{i=2}^{d}\sum_{j=1}^{i-1} x_i z^{-(i-1-j)} x_j^* F^{(0)} + tuv = 0$$

- Quadratic equations, but with unknown/mysterious series in t!

More on this

- Thm [Bousquet-Mélou–Jehanne] The mysterious series are algebraic (and there is an algorithm to produce the system).
- "Trivial" cases: one mysterious series in case 1 and case 2 with $x_k = x\delta_{k,2}$

$$tzW(z)^2 + (tz(u+v)-1)W(z) + tz\frac{W(z)-W(1)}{z-1} + tuv = 0$$

$$tzW(z)^2 + \left(txz^{-1} + tz(u+v) - 1\right)W(z) - tx(x_1^* F^{(0)}) + tuv = 0$$

all bip. maps and bip. quadrangulations

- "Non-trivial" cases, e.g. $x_k = x\delta_{k,3}$ i.e. bipartite hexangulations

$$tzW(z)^2 + \left(txz^{-2} + tz(u+v) - 1\right)W(z) - tx(z^{-1}x_1^* F^{(0)} + x_2^* F^{(0)}) + tuv = 0$$

- In general in case 2, the mysterious series are $x_1^* F^{(0)}, \dots, x_{d-1}^* F^{(0)}$.

Questions

- How would this look in higher genus?
- Do we need to solve for mysterious series and how?
- What would be an equivalent of the BMJ thm?
- Typically polynomial equations become differential equations

$$W(z)^k \to \frac{d^k}{dz^k}\psi(z)$$

- "Quantization"

In higher genus: Rationality

- Same method as above involves $x_i^* x_j^* F^{(0)}$ at genus 1, $x_i^* x_j^* x_k^* F^{(1)}$ at genus 2, and so on.

$$\sum_{j+k=i} x_j^* x_k^* e^F \rightarrow \sum_{j+k=i} x_j^* F \, x_k^* F + (x_j^* x_k^* F)$$

- Much less explicit results
- Thm [Bender-Canfield-Richmond] In case 1, $x_1^* F^{(g)}$ is rational w.r.t. algebraic series (of trees)
- Proof of BCR relies on writing equations $L_i \tau = 0$ as inductive system on genus and number of marked faces
- Bijective proof of BCR at fixed genus by Albenque-Lepoutre/Lepoutre, bypasses the constraints! (How dare they!)

In higher genus: Topological recursion

- Same method as above involves $x_i^* x_j^* F^{(0)}$ at genus 1, $x_i^* x_j^* x_k^* F^{(1)}$ at genus 2, and so on.
- Nowadays, in case 2, there is an algorithm to calculate the series with n marked face and genus g w.r.t. $2g + n$ (Eynard & co.) called topological recursion (TR)
- This procedure applies well beyond the world of maps, in *enumerative geometry*
- Proof of TR relies on writing equations $L_i \tau = 0$ as inductive system on genus and number of marked faces
- TR does everything it can to eliminate the mysterious series, but here we want the opposite!

Some recurrence formula for maps

- $t_g^n = |\{\#$ triangulations of genus g with n triangles$\}|$
  [Kazakov-Kostov-Nekrasov99 in appendix, Goulden-Jackson08]

$$(n+1)t_g^n = 4n(3n-2)(3n-4)t_{g-1}^{n-1} + 4 \sum_{\substack{i+j=n-2 \\ h+k=g}} (3i+2)(3j+2)t_h^i t_k^j$$

- $m_g^n = |\{\#$ maps of genus g with n edges & weight u per vertex$\}|$
  [Carrell-Chapuy14, Kazarian-Zograf15]

$$(n+1)m_g^n = 2(1+u)(2n-1)m_g^{n-1} + \frac{1}{2}(2n-3)(2n-2)(2n-1)m_{g-1}^{n-2} + 3 \sum_{\substack{i+j=n-2 \\ h+k=g}} (2i+1)(2j+1)m_h^i m_k^j$$

- $b_g^n = \{\#$ bip. maps, weight u per white vertex & v per black vertex$\}|$
  [Kazarian-Zograf15]

$$(n+1)b_g^n = \alpha^n(u,v)b_{g-1}^n + \beta^n(u,v)b_{g-2}^n + \gamma^n b_{g-2}^{n-1} + \sum_{\substack{i+j=n-2 \\ h+k=g}} \mu_i \mu_j b_h^i b_k^j$$

What is remarkable?

- Remarkably simple! Bypass the TR/marked faces thing
- Coefficients are independent of genus $\rightarrow$ ODEs!
- ODE on $\frac{dF}{dt}$ for the "trivial" cases (w.r.t. the planar case):
  - For bipartite maps: case 1 and case 2 with $x_k = x\delta_{k,2}$ i.e. bip. quadrangulations
  - For general maps: case $x_k = 1$ and case $x_k = x\delta_{k,3}$ i.e. triangulations
- "Trivial" cases again for orientable and non-orientable maps [VB-Chapuy-Dołęga]
- ODE in case 2, but with shifts on u, v [Louf19]
- Question: ODE for general case 2?
  Looks like yes
  Evidence of principle, explicit algorithms still buffering

A table

Rooted maps of genus g with n edges, orientable or not

$$h_n^g = \frac{2}{(n+1)(n-2)} \Bigg(n(2n-1)(2h_{n-1}^g + h_{n-1}^{g-1/2}) + \frac{(2n-3)(2n-2)(2n-1)(2n)}{2} h_{n-2}^{g-1}$$

$$+12 \sum_{\substack{g_1=0..g \\ g_1+g_2=g}} \sum_{\substack{n_1=0..n \\ n_1+n_2=n}} \frac{(2n_2-1)(2n_1-1)n_1}{2} h_{n_2-1}^{g_2} h_{n_1-1}^{g_1} - \sum_{\substack{g_1=0..g \\ g_1+g_2=g}} \sum_{\substack{n_1=0..n-1 \\ n_1+n_2=n}} \sum_{\substack{g_0=0..g_1 \\ g_1-g_0 \in \mathbb{N}}} \binom{n_1+2-2g_0}{n_1-2g_1} 2^{2(1+g_1-g_0)} h_{n_1}^{g_0}$$

$$\Bigg(\frac{(2n_2-1)(2n_2-2)(2n_2-3)}{2} h_{n_2-2}^{g_2-1} - \delta_{(n_2,g_2)\neq(n,g)} \frac{n_2+1}{4} h_{n_2}^{g_2} + \frac{2n_2-1}{2} (2h_{n_2-1}^{g_2} + h_{n_2-1}^{g_2-1/2})$$

$$+ 6 \sum_{\substack{g_3=0..g_2 \\ g_3+g_4=g_2}} \sum_{\substack{n_3=0..n_2 \\ n_3+n_4=n_2}} \frac{(2n_3-1)(2n_4-1)}{4} h_{n_3-1}^{g_3} h_{n_4-1}^{g_4} \Bigg) \Bigg)$$

$n \backslash g$	5/2	3	7/2	4
5	8229	0	0	0
6	516958	166377	0	0
7	19381145	13093972	4016613	0
8	562395292	595145086	382630152	113044185
9	13929564070	20431929240	20549348578	12704958810
10	309411522140	587509756150	818177659640	790343495467
11	6344707786945	14923379377192	26881028060634	35918779737610
12	122357481545872	345651571125768	770725841809552	1330964564940140
13	2247532739398856	7452363840633244	19946409152977346	42611002435124552
14	39681114425793904	151717486205709730	476412224477845444	1220973091185233106
15	677939355268197412	2946794762696249280	10665684328125155376	32054128913697072040
16	11265765391845733784	55029552840385680100	226357454725004343024	783804517126931727890

How do we get those recurrence formulas?

- The constraints $L_i\tau = 0$ determine τ, so... Yet, not able with $L_i\tau = 0$ only!!
- Use KP equation as a black box instead

$$-F_{3,1} + F_{2,2} + \frac{1}{2}F_{1,1}^2 + \frac{1}{12}F_{1^4} = 0$$

with $f_i \equiv \frac{\partial f}{\partial x_i}$ Details to come, be patient!

- Recall degrees such that $[t^n]\tau$ is homogeneous of degree n
- The operator L_i is not, because the constraints are inductive on the size
- $\deg\left(\frac{\partial}{\partial x_i}\right) = i \Rightarrow$ the KP equation is homogeneous
- Use the constraints to rewrite these terms as polynomials in F and its derivatives w.r.t. t to get an ODE
- Proposition

  Example $x_k = x\delta_{k,2}$. Denote $\overline{f} \equiv f_{|x_k = x\delta_{k,2}}$.

  $\overline{F_{3,1}}$, $\overline{F_{2,2}}$, $\overline{F_{1,1}}$ and $\overline{F_{1^4}}$ are differential polynomials in $\frac{d\overline{F}}{dt}$.
- Warning: take derivatives before evaluating!

Proof in the case $x_k = x\delta_{k,2}$

$$\text{homogeneity:}\quad t\frac{dF}{dt} = \sum_{i\geq 1} ix_iF_i, \qquad L_0:\quad F_1 = t^2\frac{dF}{dt} + tuv$$

$$L_1:\quad 2F_2 = t\sum_{i\geq 1}(i+1)x_iF_{i+1} + t(u+v)F_1$$

- Homogeneity implies $\overline{F_2} = \frac{t}{2x}\frac{d\overline{F}}{dt}$. Taking the x_2-derivative

$$t\frac{d\overline{F_2}}{dt} = 2\overline{F_2} + 2x\overline{F_{2,2}} \;\Rightarrow\; \overline{F_{2,2}} = \frac{t}{4x^2}\left(t\frac{d^2\overline{F}}{dt^2} - \frac{d\overline{F}}{dt}\right)$$

- L_0 gives by induction: $\overline{F_{1^k}} = t^2\frac{d\overline{F_{1^{k-1}}}}{dt} + tuv\delta_{k,1}$

$$\overline{F_{1^k}} = \left(t^2\frac{d}{dt}\right)^{k-1}\left(t^2\frac{d\overline{F}}{dt} + tuv\right) \;\Rightarrow\; \overline{F_{1,1}} \text{ and } \overline{F_{1,1,1,1}}$$

- x_1-derivative of homogeneity gives $\qquad t\frac{d\overline{F_1}}{dt} = \overline{F_1} + x\overline{F_{2,1}}$
- Take x_1-derivative of L_1 $\qquad 2\overline{F_{2,1}} = t\overline{F_2} + 3tx\overline{F_{3,1}} + t(u+v)\overline{F_{1,1}}$

What about bipartite hexangulations?

Claim (to be checked explicitly)
There is a closed recursive system for $x_k = x\delta_{k,3}$.

$$\text{homogeneity:} \quad t\frac{dF}{dt} = \sum_{i\geq 1} i x_i F_i, \qquad L_0: \quad F_1 = t^2\frac{dF}{dt} + tuv$$

$$L_1: \quad 2F_2 = t\sum_{i\geq 1}(i+1)x_i F_{i+1} + t(u+v)F_1$$

$$L_2: \quad 3F_3 = t\sum_{i\geq 1}(i+2)x_i F_{i+2} + t(u+v)F_2$$

- Homogeneity and L_0 give $\quad t\frac{d\overline{F}}{dt} = 3x\overline{F_3} = \frac{1}{t}\overline{F_1} - uv$
- L_1 gives $\quad 2\overline{F_2} = 4tx\overline{F_4} + t(u+v)\overline{F_1}$
- Not able to prove directly that $\overline{F_2}, \overline{F_{2,2}}$ are differential polynomials in $\frac{d\overline{F}}{dt}$...
- But able to write $\overline{F_{3,2}}, \overline{F_{4,2}}, \overline{F_{4,1}}, \overline{F_{5,1}}, \overline{F_{3^k,1^l}}$ as differential polynomials in $\overline{F_2}$ and $\overline{F_{2,2}}$ and $\frac{d\overline{F}}{dt}$

More KP equations!

- It is a bit like in Bousquet-Mélou–Jehanne with several unknown series
- "Need" to involve more equations to eliminate the dependence in $\overline{F_2}$ and $\overline{F_{2,2}}$
- The KP equation is accompanied by an infinite number of compatible PDEs

$$-F_{4,1} + F_{3,2} + F_{2,1}F_{1,1} + \frac{1}{6}F_{2,1^3} = 0$$

- They are labeled by partitions. Here is another one

$$-6F_{5,1} + 4F_{4,2} + 2F_{3,3} + 4F_{3,1}F_{1,1} + \frac{2}{3}F_{3,1^3} + 4F_{2,1}^2$$
$$+2F_{2,2}F_{1,1} + F_{2,2,1,1} + \frac{1}{3}F_{1,1}^3 + \frac{1}{6}F_{1^4}F_{1^2} + \frac{1}{180}F_{1^6} = 0$$

- Get a system of 3 ODEs involving $\overline{F}$, $\overline{F_2}$, $\overline{F_{2,2}}$. Resultants for ODEs?
- What is the algo for general d?
- Main idea: as d (x_d is last non-vanishing x_i) grows, Virasoro constraints create some inflation in the order of the derivatives $\overline{F_{\lambda_1,\lambda_2,\ldots}}$ required.
- Still, only require a finite number
- Use the KP equations which are homogeneous in $\lambda_1 + \lambda_2 + \cdots$ to close the system.

Plan

- All those KP equations are called the KP *hierarchy*
- What I propose now: re-reading textbooks adapted to CS
- Two typical approaches to the KP hierarchy
  - Algebraic combinatorics, very useful to prove that a GF satisfies those PDEs
    not today!
  - Lax pair and pseudo-differential operators, could be useful to extract recurrence formulas?
    today's proposal!
- Lax pair approach to integrable systems
- Toda lattice hierarchy, KdV hierarchy, KP hierarchy

Bibliography

- It's complicated. And nothing about FPS AFAIK
- *Classical integrable systems*, Babelon, Bernard, Talon
- *Solitons*, Jimbo, Miwa, Date
- *Infinite dimensional Lie algebras – Bombay lectures*, Kac, Raina

Warning!!

- Classical integrability is part of *symplectic geometry*
- Here, avoid symplectic geometry as much as possible
- So if anything unclear $\rightarrow$ symplectic geometry
- Classical system described by a set of "positions" $(q_1, \ldots, q_n)$ and momenta $(p_1, \ldots, p_n)$
- Time evolution given by *equations of motion* (EOM)

$$\frac{dq_i}{dt} = p_i, \qquad \frac{dp_i}{dt} = f_i(q_1, p_1, \ldots)$$

Example Two-body problem aka Kepler problem

- Two bodies in 3D space and gravitational attraction
- In center of mass frame, three coordinates x_1, x_2, x_3 and their momenta p_1, p_2, p_3
- Let $r = \sqrt{x_1^2 + x_2^2 + x_3^2}$ and $V(r) = C/r$, and $H = \frac{1}{2}(\sum_{i=1}^3 p_i^2) + V(r)$
- Equations of motion

$$\frac{dx_i}{dt} = p_i, \qquad \frac{dp_i}{dt} = -\frac{\partial V(r)}{\partial x_i}, \text{i.e.} \quad \frac{d^2\vec{x}}{dt^2} = -\vec{\nabla} V(r)$$

Liouville integrability

- Liouville/Classical integrability: Existence of n conserved quantities $I_1, \ldots, I_n$ which are indpdt and "Poisson commute"
- *Conservation*: for all $i = 1, \ldots, n$, $\frac{dI_i}{dt} = 0$
- *Independence*: the dI_i are linearly independent everywhere
- Liouville theorem: Solution by quadratures
- There exists a change of variables

$$(q_1, p_1, \ldots, q_n, p_n) \mapsto (I_1, \psi_1, I_2, \psi_2, \ldots, I_n, \psi_n)$$

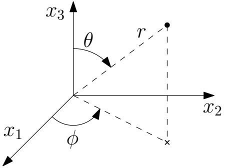
  where the equations of motion are

$$\frac{dI_i}{dt} = 0 \qquad \frac{d\psi_i}{dt} = f_i(I_1, \ldots, I_n) = \text{Const.}$$

- Space of solutions parametrized by $I_1, \ldots, I_n$

- Idea: “enough independent conserved quantities which commute”
- Calculating the ψ_i only involves solving algebraic systems and integrals
- In the two-body problem, three conserved quantities
- Introduce the angular momentum $\vec{J} = \vec{x} \times \vec{p}$

$$J_1 = x_2 p_3 - x_3 p_2, \quad J_2 = x_3 p_1 - x_1 p_3, \ldots$$
$$I_1 = H, \quad I_2 = \vec{J}^2, \quad I_3 = J_3$$



- In spherical coordinates, the action reads

$$S(r, \theta, \phi, I_1, I_2, I_3) = \int^r 2\sqrt{\Big(H - V(r)\Big) - \frac{\vec{J}^2}{r'^2}}dr' + \int^\theta \sqrt{\vec{J}^2 - \frac{J_z^2}{\sin^2\theta'}}d\theta' + \int^\phi J_z d\phi'$$

and

$$\psi_1 = \frac{\partial S}{\partial H}, \quad \psi_2 = \frac{\partial S}{\partial \vec{J}^2}, \quad \psi_3 = \frac{\partial S}{\partial J_z}$$

Lax pairs

- Modern and unifying approach to integrable systems
- Way to obtain conserved quantities directly
- Encode your degrees of freedom into a matrix or an operator L, such that there exists M such that

$$\frac{dL}{dt} = [M, L] := ML - LM$$

- If you have a notion of trace, satisfying cyclicity $\operatorname{tr} AB = \operatorname{tr} BA$, then

$$I_i := \operatorname{tr}(L^i) \;\Rightarrow\; \frac{dI_i}{dt} = 0$$

  "Isospectral flow": symmetric polynomials in eigenvalues are conserved
- Isospectral deformations

Example: the open Toda lattice

- N particles on the real line, positions $q_1, \ldots, q_N$, momenta $p_1, \ldots, p_N$
- Particle i interacts with $i-1$ and $i+1$ with exponential potential

$$\text{EOM} \qquad \frac{dq_i}{dt} = p_i, \qquad \frac{dp_i}{dt} = e^{q_{i-1}-q_i} - e^{q_i - q_{i+1}}$$

  and $\frac{dp_1}{dt} = -e^{q_1 - q_2}$ and $\frac{dp_N}{dt} = e^{q_{N-1}-q_N}$
- Other boundary conditions can be used and lead to different Lax pairs and solutions
- Initial configuration: values of $q_1, p_1, \ldots, q_N, p_N$ at time $t = 0$
- Energy is a conserved quantity $\frac{dH}{dt} = 0$

$$H = \frac{1}{2}\sum_{i=1}^{N} p_i^2 + \sum_{k=1}^{N-1} e^{q_i - q_{i+1}}$$

- Final configuration satisfies

$$q_{i+1} - q_i \to \infty \qquad \text{as } t \to \infty$$

  which is stationary $\frac{dp_i}{dt} = 0$, and each p_i converges.

Lax pair for open Toda lattice

- Change of variables: $a_i = \frac{1}{2} e^{(q_i - q_{i+1})/2}$ for $i = 1, \ldots, N-1$ and $b_i = -\frac{1}{2} p_i$ for $i = 1, \ldots, N$

$$\frac{da_i}{dt} = a_i(b_{i+1} - b_i), \qquad \frac{db_i}{dt} = 2(a_i^2 - a_{i-1}^2)$$

- Set L as a tridiagonal matrix

$$L = \begin{pmatrix} b_1 & a_1 & 0 & \ldots & 0 \\ a_1 & b_2 & a_2 & 0 & \vdots \\ 0 & a_2 & b_3 & a_3 & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix} \qquad M = L_+ - L_- = \begin{pmatrix} 0 & a_1 & 0 & \ldots & 0 \\ -a_1 & 0 & a_2 & 0 & \vdots \\ 0 & -a_2 & 0 & a_3 & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

- Proposition

  Lax equation $\frac{dL}{dt} = [M, L]$ reproduces the open Toda lattice EOM.

Example at $N = 3$

- $L = \begin{pmatrix} b_1 & a_1 & 0 \\ a_1 & b_2 & a_2 \\ 0 & a_2 & b_3 \end{pmatrix}$, $M = \begin{pmatrix} 0 & a_1 & 0 \\ -a_1 & 0 & a_2 \\ 0 & -a_2 & 0 \end{pmatrix}$

$$ML = \begin{pmatrix} 0 & a_1 & 0 \\ -a_1 & 0 & a_2 \\ 0 & -a_2 & 0 \end{pmatrix} \begin{pmatrix} b_1 & a_1 & 0 \\ a_1 & b_2 & a_2 \\ 0 & a_2 & b_3 \end{pmatrix} = \begin{pmatrix} a_1^2 & a_1 b_2 & a_1 a_2 \\ -a_1 b_1 & -a_1^2 + a_2^2 & a_2 b_3 \\ -a_1 a_2 & -a_2 b_2 & -a_2^2 \end{pmatrix}$$

$$LM = \begin{pmatrix} b_1 & a_1 & 0 \\ a_1 & b_2 & a_2 \\ 0 & a_2 & b_3 \end{pmatrix} \begin{pmatrix} 0 & a_1 & 0 \\ -a_1 & 0 & a_2 \\ 0 & -a_2 & 0 \end{pmatrix} = \begin{pmatrix} -a_1^2 & a_1 b_1 & a_1 a_2 \\ -a_1 b_2 & a_1^2 - a_2^2 & a_2 b_2 \\ -a_1 a_2 & -a_2 b_3 & a_2^2 \end{pmatrix}$$

- Hence

$$[M, L] = \begin{pmatrix} 2a_1^2 & a_1(b_2 - b_1) & 0 \\ a_1(b_2 - b_1) & 2(a_2^2 - a_1^2) & a_2(b_3 - b_2) \\ 0 & a_2(b_3 - b_2) & -2a_2^2 \end{pmatrix}$$

Toda flows

- L tridiagonal, $M = \text{skew}(L)$
- M "generates" the time evolution. Do other time evolutions exist?
- Consider $M_k = \text{skew}(L^k)$ and the "evolution equation" for L tridiagonal

$$\frac{\partial L}{\partial t_k} = [M_k, L] \quad \text{for } k = 1, \dots, N$$

- $t = t_1$ original time
- Are they consistent with one another?

$$\frac{\partial^2 L}{\partial t_l \partial t_k} = \frac{\partial^2 L}{\partial t_k \partial t_l} \quad \Leftrightarrow \quad \left[L, \frac{\partial M_k}{\partial t_l} - \frac{\partial M_l}{\partial t_k} + [M_k, M_l] \right] = 0$$

- Here for $M_k = \text{skew}(L^k)$

$$\frac{\partial M_k}{\partial t_l} - \frac{\partial M_l}{\partial t_k} + [M_k, M_l] = 0$$

Toda flows

- L tridiagonal, $M = \text{skew}(L)$
- M "generates" the time evolution. Do other time evolutions exist?
- Consider $M_k = \text{skew}(L^k)$ and the "evolution equation" for L tridiagonal

$$\frac{\partial L}{\partial t_k} = [M_k, L] \quad \text{for } k = 1, \ldots, N$$

- $t = t_1$ original time
- Given a solution to the original system, flow with respect to the other times to generate other solutions
- The I_k are conserved with respect to all Toda times
- Are the $I_k = \text{tr}(L^k)$ independent?
- If all $a_i = 0$, then the I_k are power-sums

$$I_k = \sum_{i=1}^{N} b_i^k$$

- Write the solutions "simply" in terms of the conserved quantities

With PDEs now

- Shift paradigm from conserved quantities to symmetries
- Conserved quantities: sum over particles $\rightarrow$ integrals
- Example: advection equation (describes propagation at speed c)

$$\frac{\partial u(x,t)}{\partial t} + c\frac{\partial u(x,t)}{\partial t} = 0$$

- Conserved quantities (assuming finitess) for $n \geq 1$

$$\frac{d}{dt}\int u(x,t)^n dx = 0$$

- How about formal power series?
- Use the notion of symmetry/infinitesimal transformation

$$\frac{\partial L}{\partial t_k} = [M_k, L] \quad \text{for } k \geq 1$$

Korteweg-de Vries (KdV)

- Let $u \equiv u(t,x)$ satisfying the KdV equation
$$\frac{\partial u}{\partial t} = 6u\frac{\partial u}{\partial x} - \frac{\partial^3 u}{\partial x^3}$$
- KdV hierarchy is an infinite set of non-linear, consistent PDEs for $u \equiv u(t, x, x_1, x_3, x_5, \dots)$
$$\frac{\partial u}{\partial x_k} = K_k[u], \qquad \frac{\partial K_k[u]}{\partial x_l} = \frac{\partial K_l[u]}{\partial x_k}$$
- $K_1[u] = (\partial u)$ with $\partial \equiv \frac{\partial}{\partial x}$ so x_1 is identified with x
- $K_3[u] = 6u(\partial u) - (\partial^3 u)$ so x_3 is identified with t
- $K_5[u] = 10u(\partial^3 u) - 20(\partial u)(\partial^2 u) - 30u^2(\partial u) - (\partial^5 u)$
- Infinite set of commuting symmetries
- Lax representation using pseudo-differential operators
- Example in combinatorics: Kontsevich-Witten's intersection numbers on moduli space of Riemann surfaces

Pseudo-differential operators

- Let R be an algebra of functions of x, stable under derivatives
- Typically for us $R = \mathbb{Q}[x, x_1, x_2, x_3, \ldots][[t]]$ (not very typical in integrable systems though)
- Consider the algebra $R[\partial]$, product being defined via the usual $\partial f = (\partial f) + f\partial$
- Consider the symbol ∂^{-1} defined by

$$\partial^{-1}\partial = \partial\partial^{-1} = 1, \qquad \partial^{-1} f = \sum_{i=0}^{\infty} (-1)^i (\partial^i f)\partial^{-i-1}$$

- $\partial^{-1} c = c\partial^{-1}$ $\qquad \partial^{-1} x = x\partial^{-1} - \partial^{-2}$
- $\partial^{-1} x^2 = x^2\partial^{-1} - 2x\partial^{-2} + 2\partial^{-3}$
- Consider $R((\partial^{-1}))$, formal Laurent series in ∂^{-1}

$$A = \sum_{i\geq 0} a_i(x)\partial^{m-i}$$

- It is an associative algebra and $\quad \partial^k f = \sum_{i\geq 0} \binom{k}{i} (\partial^i f)\partial^{k-i}$

Properties of formal Laurent pseudo-differential operators

- Monic elements are invertible

$$A = \partial^m + \sum_{i\geq 1} a_i(x)\partial^{m-i}, \qquad A^{-1} = \partial^{-m} + \sum_{j\geq 1} \bar{a}_j(x)\partial^{-m-j}$$

  then $A^{-1}A = 1$ gives

$$A^{-1}A = \sum_{i,j,l\geq 0} \binom{-m-i}{l} \bar{a}_i(x)(\partial^l a_j(x))\partial^{-i-j-l}$$

  hence $\bar{a}_1 = -a_1$, $\bar{a}_2 = a_1^2 - a_2 + m(\partial a_1)$

- More generally, set degrees as $\deg a_i = \deg \bar{a}_i = \deg \partial^i = i$

$$\begin{aligned} \bar{a}_i &= -a_i + \text{diff. pol}_i(a_1, \bar{a}_1, \ldots, a_{i-1}, \bar{a}_{i-1}) \\ &= -a_i + p_i(a_1, \ldots, a_{i-1}, (\partial a_1), \ldots) \end{aligned}$$

- $G = 1 + \bigoplus_{n\geq 1} R\partial^{-n}$ is a group

- Monic elements of degree m have m-th roots. Set

$$B = \partial + \sum_{i \geq 1} b_i \partial^{1-i}$$

then $\quad B^2 = \partial^2 + 2b_1\partial + (2b_2 + b_1^2 + \partial b_1) + (2b_3 + 2b_1 b_2 + \partial b_2)\partial^{-1} + \cdots$

- If $A = B^2$, then

$$a_i = 2b_i + \text{diff. pol}_i(b_1, \ldots, b_{i-1})$$
$$2b_i = a_i + p_i^l(a_1, \ldots, a_{i-1}, (\partial a_1), \ldots)$$

- Example $A = \partial^2 + \sum_{i \geq 1} a_i(x)\partial^{2-i}$

$$A^{\frac{1}{2}} = \partial + \frac{a_1}{2} + \left(a_2 - \frac{a_1^2}{4} - \frac{(\partial a_1)}{2}\right)\frac{\partial^{-1}}{2} + \left(a_3 - \frac{a_1 a_2}{2} + \frac{a_1^3}{8} + \frac{a_1 \partial a_1}{2} - \frac{\partial a_2}{2} + \frac{(\partial^2 a_1)}{4}\right)\frac{\partial^{-2}}{2} + \cdots$$

Back to KdV

- Lax pair for KdV lives on $R((\partial^{-1}))$. Let $u \in R$

$$L = \partial^2 + u, \qquad M_k = (L^{k/2})_+$$

where M_+ is the differential part.

- Let us go directly to KP...
- $L^{1/2} = (\partial^2 + u)^{1/2}$ as a series in ∂^{-1}

$$\begin{aligned} L^{1/2} &= \partial + \sum_{i=1}^{\infty} b_i \partial^{-i+1} \\ &= \partial + \frac{u}{2}\partial^{-1} - \frac{1}{4}(\partial u)\partial^{-2} + \frac{1}{8}((\partial^2 u) - u^2)\partial^{-3} + \mathcal{O}(\partial^{-5}) \end{aligned}$$

- Gives

$$L_+^{1/2} = \partial, \quad L_+^{3/2} = \partial^3 + \frac{3}{2}u\partial + \frac{3}{4}(\partial u)$$

- Prove that the symmetries commute!
- Express all derivatives $\frac{\partial u}{\partial x_k}$ wrt x_k as polynomials in $u, (\partial u), (\partial^2 u), \ldots$

Kadomtsev-Petviashvili (KP) hierarchy

- This is where things get a little dicey... For $i \geq 2$, let $q_i \equiv q_i(x, x_1, x_2, \dots) \in R$ and

$$L = \partial + \sum_{i \geq 1} q_{i+1} \partial^{-i}, \qquad \frac{\partial L}{\partial x_k} := \sum_{i \geq 1} \frac{\partial q_{i+1}}{\partial x_k} \partial^{-i} = [(L^k)_+, L]$$

which means

$$\frac{\partial q_{i+1}}{\partial x_k} = [\partial^{-i}][(L^k)_+, L]$$

- Example: $(L^1)_+ = \partial$ then

$$\frac{\partial L}{\partial x_1} = [L_+, L] = [\partial, L] = \sum_{i \geq 1} (\partial q_{i+1}) \partial^{-i} \Rightarrow \frac{\partial q_{i+1}}{\partial x_1} = (\partial q_{i+1})$$

identifies x_1 with x

- Evolution with respect to x_2 and x_3

$$(L^2)_+ = \partial^2 + 2q_2, \quad (L^3)_+ = \partial^3 + 3q_2 \partial + 3(\partial q_2) + 3q_3$$

- In general $(L^j)_+ = \partial^j + j q_2 \partial^{j-2} + \mathcal{O}(\partial^{j-3})$

Kadomtsev-Petviashvili (KP) hierarchy

- Evolution with respect to x_2

$$\frac{\partial q_2}{\partial x_2} = \partial^2 q_2 + 2\partial q_3, \quad \frac{\partial q_3}{\partial x_2} = \partial^2 q_3 + 2\partial q_4 + 2q_2 \partial q_2$$

- Evolution with respect to x_3

$$\frac{\partial q_2}{\partial x_3} = \partial^3 q_2 + 3\partial^2 q_3 + 3\partial q_4 + 6q_2 \partial q_2$$

- Set degrees as $\deg \partial = 1$, $\deg q_i = i$
- Then $\frac{\partial q_i}{\partial x_j}$ is homogeneous of degree $i+j$

$$\frac{\partial q_i}{\partial x_j} = \text{homogeneous polynomial of degree } i+j, \text{ in } (\partial^k q_l)$$

$$= \cancel{q_{i+j}} + j\partial q_{i+j-1}$$

$$+ \text{ homogeneous polynomial of degree } i+j, \text{ in } (\partial^k q_l) \text{ with } l < i+j-1$$

- Please someone help generate them!

Deriving the KP equation

- Evolution with respect to x_2

$$\frac{\partial q_2}{\partial x_2} = 2\partial q_3 + \partial^2 q_2, \quad \frac{\partial q_3}{\partial x_2} = 2\partial q_4 + \partial^2 q_3 + 2q_2\partial q_2$$

- Evolution with respect to x_3

$$\frac{\partial q_2}{\partial x_3} = 3\partial q_4 + 3\partial^2 q_3 + \partial^3 q_2 + 6q_2\partial q_2$$

- Look at $\frac{\partial^2 q_2}{\partial x_2^2}$

$$\frac{\partial^2 q_2}{\partial x_2^2} = 4\partial^2 q_4 + 4\partial^3 q_3 + \partial^4 q_2 + 4\partial(q_2\partial q_2)$$

- Eliminate $4\partial^2 q_4 + 4\partial^3 q_3$ using $\frac{\partial^2 q_2}{\partial x_3 \partial x_1}$
- Let $u := -2q_2$, then this is the KP equation

$$3\frac{\partial^2 u}{\partial x_2^2} = \frac{\partial}{\partial x_1}\left(4\frac{\partial u}{\partial x_3} + 6u\frac{\partial u}{\partial x_1} - \frac{\partial^3 u}{\partial x_1^3}\right)$$

Commuting symmetries

- Want to prove
$$\frac{\partial M_i}{\partial x_j} - \frac{\partial M_j}{\partial x_i} + [M_j, M_i] = 0 \qquad \text{for } M_i = (L^i)_+$$
- For all polynomials P, $\dfrac{\partial P(L)}{\partial x_k} = [(L^k)_+, P(L)]$. Then
$$\frac{\partial (L^k)_+}{\partial x_l} = \left(\frac{\partial L^k}{\partial x_l}\right)_+ = [(L^l)_+, L^k]_+$$
so that
$$\frac{\partial (L^i)_+}{\partial x_j} - \frac{\partial (L^j)_+}{\partial x_i} = [(L^j)_+, L^i]_+ + [L^j, (L^i)_+]_+$$
- Use $L^i = (L^i)_+ + (L^i)_-$
$$\frac{\partial (L^i)_+}{\partial x_j} - \frac{\partial (L^j)_+}{\partial x_i} = [(L^j)_+, (L^i)_+]_+ + [(L^j)_+, (L^i)_-]_+ + [L^j, (L^i)_+]_+$$
$$= [(L^j)_+, (L^i)_+] + [(L^j)_+, (L^i)_-]_+ - [L^j, (L^i)_-]_+$$
$$= [(L^j)_+, (L^i)_+] + [(L^j)_-, (L^i)_-]_+$$
$$= [(L^j)_+, (L^i)_+]$$

Wave function

- Let $\Phi \in R((\partial^{-1}))$ such that

$$L = \Phi \partial \Phi^{-1}, \qquad \Phi = 1 + \sum_{i \geq 1} w_i \partial^{-i}$$

  called a dressing transformation
- This gives

$$q_{i+1} = (\partial w_i) + \text{diff. pol}_i(w_1, \ldots, w_{i-1})$$

- L determines Φ up to $\Phi \to \Phi C$ with $C = 1 + \sum_{i \geq 1} c_i \partial^{-i}$
- KP-flows for Φ

$$\frac{\partial \Phi}{\partial x_i} = -(L^i)_- \Phi$$

- Extract $[\partial^{-j}]$ to get $\frac{\partial w_j}{\partial x_i}$
- It is a homogeneous polynomial of degree $i + j$ in $(\partial^k w_l)$

$$\frac{\partial w_j}{\partial x_i} = \cancel{w_{i+j}} + a(\partial w_{i+j-1}) + b w_1 w_{i+j-1} + \cdots$$

Tau functions and generating series

- *Sato's formula* There exists a function $\tau(x_1, x_2, \ldots) \in R$ such that

$$\psi(z) := 1 + \sum_{i \geq 1} w_i z^{-i} = \frac{\tau(x_1 - z^{-1}, x_2 - z^{-2}, x_3 - z^{-3}, \ldots)}{\tau(x_1, x_2, x_3, \ldots)}$$

  z-dependence is related to x_i-dependences
- Write $\log \psi(z) = \sum_{i \geq 1} \gamma_i z^{-i}$ then

$$\frac{\partial \log \tau}{\partial x_i} = -i\gamma_i - \sum_{j=1}^{i-1} \frac{\partial \gamma_{i-j}}{\partial x_j}$$

- Consistent definition of τ thanks to the KP flows
- Still leaves some constraints on τ
- Thm (in which space?)
  Φ satisfies the KP flows iff τ satisfies Hirota's bilinear equations.

Hirota's bilinear equations

Consider two sets of indeterminates $x_1, y_1, x_2, y_2, \ldots$

$$[z^{-1}]e^{-2\sum_{i\geq 1}\frac{z^i}{i}y_i}e^{\sum_{i\geq 1}z^{-i}\frac{\partial}{\partial y_i}}\tau(x_1-y_1, x_2-y_2,\ldots)\tau(x_1+y_1,x_2+y_2,\ldots)=0$$

- Looks non-local (translations by y_i and z^i)
- Extract coefficients w.r.t. $y_1, y_2, \ldots$ gives a finite number of derivatives

$$[y_3]\,[z^{-1}]e^{-2\sum_{i\geq 1}\frac{z^i}{i}y_i}e^{\sum_{i\geq 1}z^{-i}\frac{\partial}{\partial y_i}}\tau(x_1+y_1,x_2+y_2,\ldots)\tau(x_1-y_1,x_2-y_2,\ldots)$$

$$=\left(\frac{\partial^4}{\partial u_1^4}+3\frac{\partial^2}{\partial u_2^2}-4\frac{\partial^2}{\partial u_1\partial u_3}\right)\tau(x_1+u_1,x_2+u_2,\ldots)\tau(x_1-u_1,x_2-u_2,\ldots)_{|u_1=u_2=\cdots=0}$$

- Set $u = 2\frac{\partial^2}{\partial x_1^2}\log\tau$ to recover the KP equation
- The other two equations I showed before are from $[y_4]$ and $[y_5]$.
- In general, extract $[y_{\lambda_1}y_{\lambda_2}\cdots]$ $\to$ partitions
- Question: How come that they are quadratic while the KP flows are not?

From KP flows to Hirota

$$[z^{-1}]e^{-2\sum_{i\geq 1}\frac{z^i}{i}y_i}e^{\sum_{i\geq 1}z^{-i}\frac{\partial}{\partial y_i}}\tau(x_1-y_1,x_2-y_2,\dots)\tau(x_1+y_1,x_2+y_2,\dots)$$

- Set $p_i = x_i + y_i, q_i = x_i - y_i$,

$$[z^{-1}]e^{\sum_{i\geq 1}\frac{z^i}{i}(q_i-p_i)}\tau(q_1-z^{-1},q_2-z^{-2},\dots)\tau(p_1+z^{-1},p_2+z^{-2},\dots)$$
$$\sim [z^{-1}]\psi(z,q_1,q_2,\dots)e^{\sum_{i\geq 1}\frac{z^i}{i}q_i}\psi^*(z,p_1,p_2,\dots)e^{-\sum_{i\geq 1}\frac{z^i}{i}p_i}$$

- The function $\Psi \equiv \psi(z,q_1,q_2,\dots)e^{\sum_{i\geq 1}\frac{z^i}{i}q_i}$ satisfies $\frac{\partial \Psi}{\partial x_i} = (L^i)_+\Psi$
- It is enough to check

$$[z^{-1}]\partial^i\left(\psi(z)e^{\sum_{i\geq 1}\frac{z^i}{i}q_i}\right)\psi^*(z)e^{-\sum_{i\geq 1}\frac{z^i}{i}p_i} = 0$$

From KP flows to Hirota

- It is enough to check

$$[z^{-1}]\partial^i\left(\psi(z)e^{\sum_{i\geq 1}\frac{z^i}{i}q_i}\right)\psi^*(z)e^{-\sum_{i\geq 1}\frac{z^i}{i}p_i}=0$$

- How to transform this into pseudo-differential operators?
- Define $\partial^{-k}\cdot e^{xz}=z^{-k}e^{xz}$, then $\partial^i(\psi(z)e^{xz})=(\partial^i\Phi)\cdot e^{xz}$
- Moreover, define the antihomomorphism $*$ by $(a(x)\partial^i)^*=(-\partial)^i a(x)$, then

$$[z^{-1}]\left(\sum_i \alpha_i z^i\right)\left(\sum_j \beta_j(-z)^j\right)=[\partial^{-1}]\left(\sum_i \alpha_i z^i\right)\left(\sum_j \beta_j z^j\right)^*$$

- Eventually

$$[z^{-1}]\partial^i\left(\psi(z)e^{\sum_{i\geq 1}\frac{z^i}{i}q_i}\right)\psi^*(z)e^{-\sum_{i\geq 1}\frac{z^i}{i}p_i}=[\partial^{-1}]\partial^i\Phi\Phi^{-1}=0$$

What now?

- Where are our generating series? If I give you a combinatorial problem, how do you may find the KP hierarchy?
- Testing the KP equation is a good start
- The Japanese school came with new objects and a new point of view!
- There exists a geometric approach to τ which in practice is useful to prove KP

Grassmanianns in finite dimensions

- Consider $\mathrm{Gr}(k,n)$ the set of k-dimensional vector spaces in $\mathbb{C}^n$ like
$$P(v_1,\ldots,v_k) = \mathrm{span}(v_1,\ldots,v_k) \text{ for } k \text{ linearly indpt vectors}$$
- Recall the exterior product $v_1 \wedge v_2 = v_1 \otimes v_2 - v_2 \otimes v_1 \in \mathbb{C} \otimes \mathbb{C}$
- It is non-zero iff v_1 and v_2 are linearly independent
- Think of elements of $\mathrm{Gr}(k,n)$ via the map
$$\Sigma: \ P(v_1,\ldots,v_k) \ \rightarrow \ [v_1 \wedge v_2 \wedge \cdots \wedge v_k] \ \in \mathbb{P}\Lambda^k\mathbb{C}^n$$
- E.g. $v_1 \wedge (v_2 + v_1) \wedge \cdots \wedge v_k = v_1 \wedge v_2 \wedge \cdots \wedge v_k$
- How to identify $\mathrm{Gr}(k,n) \subset \mathbb{P}\Lambda^k\mathbb{C}^n$? *Plücker embedding*
- Notice that if $v \in P(v_1,\ldots,v_k)$ then
$$v \wedge (v_1 \wedge v_2 \wedge \cdots \wedge v_k) = 0$$
- If $u \in P(v_1,\ldots,v_k)^\perp$ then
$$\iota_u(v_1 \wedge v_2 \wedge \cdots \wedge v_k) = 0$$
where $\iota_u v_1 \wedge v_2 \wedge \cdots = \langle u, v_1\rangle v_2 \wedge \cdots - \langle u, v_2\rangle v_1 \wedge \cdots + \cdots$

Two characterizations

- Let $(e_1, \dots, e_n)$ be the can. basis of $\mathbb{C}^n$ and denote $\psi_i w = e_i \wedge w$ and $\psi_i^* w = \iota_{e_i} w$
- Let $\omega \in \mathbb{P}\Lambda^k\mathbb{C}^n$. There exists $P \in \mathrm{Gr}(k,n)$ such that $\omega = \Sigma(P)$ iff

$$\sum_{i=1}^{n} \psi_i \omega \otimes \psi_i^* \omega = 0$$

  In coordinates, Plücker relations
- Representation of GL_n on $\mathbb{P}\Lambda^k\mathbb{C}^n$

$$\rho(A)(v_1 \wedge \cdots \wedge v_k) = (Av_1) \wedge (Av_2) \wedge \cdots \wedge (Av_k)$$

  Extend linearly.
- Let $\omega \in \mathbb{P}\Lambda^k\mathbb{C}^n$. There exists $P \in \mathrm{Gr}(k,n)$ such that $\omega = \Sigma(P)$ iff

$$\exists A \in GL_n \quad \omega = [\rho(A)\underbrace{(e_1 \wedge \cdots \wedge e_k)}_{\text{reference vector}}]$$

  i.e. ω is in the orbit of GL_n.

- Consider $V = C^\infty = \bigoplus_{n\in\mathbb{Z}} \mathbb{C}$ and GL_∞ its group of automorphisms

$$GL_\infty = \{(a_{ij})_{i,j\in\mathbb{Z}}, \text{invertible and only a finite number of diagonal elements not 1 and off diag. not 0}\}$$

- Plücker relations for Sato's Grassmaniann

$$\sum_{i\in\mathbb{Z}} \psi_i\omega \otimes \psi_i^*\omega = 0$$

- Equivalence between this and being in the orbit of a reference vector under GL_∞
- Correspondence boson-fermion maps

$$\mathcal{S} : \text{ Sato's Grassmaniann } \to \mathbb{C}[x_1, x_2, \ldots]$$

  and maps Plücker relations on ω to Hirota equations on τ
- Gives rise to the bosonic representation ρ_B of GL_∞ on $\mathbb{C}[x_1, x_2, \ldots]$

$$\mathcal{S} \circ \rho = \rho_B \circ \mathcal{S}$$

In short

- Theorem – $\tau \in \mathbb{C}[x_1, x_2, \dots]$ satisfies the Hirota equations iff it comes from an element of GL_∞

$$\exists A \in GL_\infty \quad \tau(x_1, x_2, \dots) = \rho_B(A) \cdot 1$$

- Extension to FPS in my HDR dissertation: $\overline{GL_\infty} \to$ KP
- Prove KP in for a specific problem $\leftarrow$ Find an element of $\overline{GL_\infty}$ as above
- In combinatorics, $\tau(x_1, x_2, \dots)$ is a GF of objects which are connected or not and $F(x_1, x_2, \dots) = \log \tau(x_1, x_2, \dots)$ is the GF of same, connected objects
- $\psi(z)$ is the GF of derivatives of F of fixed order

$$\psi(z) := 1 + \sum_{i \geq 1} w_i z^{-i} = \frac{\tau(x_1 - z^{-1}, x_2 - z^{-2}, x_3 - z^{-3}, \dots)}{\tau(x_1, x_2, x_3, \dots)} \in R[[z^{-1}]]$$

GF of bipartite maps!

- Thm
  $\tau(t, u, v, x_1, x_2, \dots)$ of bipartite maps satisfies the bilinear Hirota equation.
- What is $\psi(z)$? By Sato's formula

$$\psi(z) = \frac{\tau(t, u, v, x_1 - z^{-1}, x_2 - z^{-2}, \dots)}{\tau(t, u, v, x_1, x_2, \dots)}$$
$$= \tau^{-1} e^{-\sum_{i\geq 1} z^{-i} \frac{\partial}{\partial x_i}} \tau$$
$$= \tau^{-1} \sum_{(\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_l)} (-1)^l \frac{z^{-\lambda_1 - \lambda_2 - \dots - \lambda_l}}{\text{Combi. factor}} x^*_{\lambda_1} x^*_{\lambda_2} \cdots x^*_{\lambda_l} \tau$$

- Turn the constraints into an equation on $\psi(z)$

$$L_i \tau = \left(-x^*_{i+1} + t \sum_{j+k=i} x^*_j x^*_k + t \sum_{j\geq 1} x_j x^*_{i+j} + t(u+v) x^*_i + tuv\delta_{i,0} \right) \tau = 0$$

All genera equation aka quantum spectral curve

- Differential, or "quantum" version of the planar equation!

$$tzW(z)^2 + \left(t\sum_{i=1}^{d} x_i z^{-i+1} + tz(u+v) - 1\right)W(z) + tuv - t\sum_{i=2}^{d}\sum_{j=1}^{i-1} x_i z^{-(i-1-j)} x_j^* F^{(0)} = 0$$

- The constraints $L_i\tau = 0$ for $i \geq 0$ give

$$z^2 t\frac{d^2\psi}{dz^2} - \left(t\sum_{i=1}^{d} p_i z^{i+1} + tz(u+v-1) - z^2\right)\frac{d\psi}{dz} + tuv\psi - t\sum_{i=2}^{d} p_i \sum_{j=1}^{i-1} z^{i-j}(x_j^*\psi + \psi x_j^* F) = 0$$

- All genera version of the unknown series in Bousquet-Mélou–Jehanne

All genera equation aka quantum spectral curve

- Differential, or "quantum" version of the planar equation!

$$tzW(z)^2 + \left(t\sum_{i=1}^{d} x_i z^{-i+1} + tz(u+v) - 1\right)W(z) + tuv - t\sum_{i=2}^{d}\sum_{j=1}^{i-1} x_i z^{-(i-1-j)} x_j^* F^{(0)} = 0$$

- Then recursion for $i \geq 0$

$$(ti(i+u+v) + tuv)w_i + t\sum_{k=1}^{d}(k+i)x_k w_{k+i} - (i+1)w_{i+1} - t\sum_{k=2}^{d}\sum_{j=1}^{k-1} x_k\left(x_j^* w_{k-j+i} + w_{k-j+i} x_j^* F\right) = 0$$

- Lemma $x_j^* w_{k-j+i}$ and $x_j^* F$ are polynomials in $\partial^m w_l$ of degree $k+i$ and j respectively.
- Example: $F_2 = w_2 - \frac{1}{2}(w_1^2 + (\partial w_1))$

Revisit the "trivial" case $x_k = x\delta_{k,2}$

$$2txw_2 - w_1 + tuv - tx(w_1^2 + (\partial w_1)) = 0$$

$$3txw_3 + (t(u+v+1) + tuv)w_1 - 2w_2 - tx(w_1 w_2 + (\partial w_2)) = 0$$

$$(i+1)txw_{i+1} + (t(i-1)(u+v+i-1) + tuv)w_{i-1} - iw_i - tx(w_i w_1 + (\partial w_i)) = 0$$

- Express all w_is for $i \geq 2$ as a polynomial in $w_1, \partial w_1, \partial^2 w_1, \ldots$
- Take the flow

$$\frac{\partial w_2}{\partial x_2} = -[\partial^{-2}](\Phi \partial^2 \Phi^{-1})_- \Phi = w_1(\partial^2 w_1) - w_1^2(\partial w_1) + (\partial^2 w_2) + 2(\partial w_3) - 2w_2(\partial w_1) + w_1 w_3$$

and replace all w_2, w_3 in terms of $w_1, (\partial w_1), \ldots$

- Take $\frac{\partial}{\partial x_2}$ of the first equation $\quad 2tx\frac{\partial w_2}{\partial x_2} = 2tw_2 - \frac{\partial w_1}{\partial x_2} + \cdots$
- Use the flow $\frac{\partial w_1}{\partial x_2} = (\partial^2 w_1) + 2(\partial w_2) - 3w_1(\partial w_1)$ and the first equation to express everything in terms of $w_1, (\partial w_1), \ldots$
- Equating those two ways of evaluating $\frac{\partial w_2}{\partial x_2}$ produces an ODE of order 3 and degree 4.

Example: $x_k = x\delta_{k,3}$

$$3txw_3 - w_1 - tx(\text{things in } w_1\partial w_1, \partial^2 w_1, \partial w_2, w_1 w_2) = 0$$

$$4txw_4 - 2w_2 + t(uv + u + v + 1)w_1$$
$$- tx((\partial w_3) + w_3 w_1 + x_2^* w_2 + w_2(w_2 - \frac{1}{2}(w_1^2 + (\partial w_1)))) = 0$$

- $(\partial w_3), w_3$ in the 2nd eq. are given by the first eq.
- Inductively all $w_3, w_4, w_5, \ldots$ are given as differential polynomials in w_1, w_2
- Take x_2^* of first eq.

$$3txx_2^* w_3 \underset{\text{KP}}{=} 2(\partial w_4) + 2(\partial^2 w_3) + \ldots = 2\frac{\partial w_1}{\partial x_2} + \cdots$$

- Take x_3^* of first eq. $x_3^* w_3 = (\partial w_5) + \ldots$

$$3txx_3^* w_3 \underset{\text{KP}}{=} 3(\partial w_5) + \ldots = 3\frac{\partial w_1}{\partial x_3} + \cdots$$

- This gives an infinite number of equations involving $\partial^k w_1, \partial^l w_2$ only

Conclusion

- KP flows as a tool for some combinatorial systems
- Infinite number of commuting symmetries, generated by a Lax pair
- Application to maps still w.i.p., devise general algorithm
- Close the Virasoro constraints which have growing number of derivatives using KP flows which are homogeneous
- All genera analog of the unknown series of BMJ, with diff. eq. instead of algebraic
- Did not find a handbook of KP flows, nor a program writing the equations
- Other systems?
- Maps decorated with the Ising model → M. Albenque's talk!
- Revisit some "old" (Tutte's) recurrence for q-properly colored *planar* maps

$$q(n+1)(n+2)h_{n+2} = q(q-4)(3n-1)(3n-2)h_{n+1} + 2\sum_{i=1}^{n} i(i+1)(3n-3i+1)h_{i+1}h_{n+2-i}$$

Not treated here

- Lax pair with spectral parameter: rational function with matrix coefficients
- How to identify systems satisfying KP?
- Reduction of KP to more specific hierarchies like KdV, Boussinesq, etc. (combinatorial examples?)
- B-type for non-oriented maps [VB-Chapuy-Dołęga]
- Modern works on (q, t)-deformation, etc.