Positive harmonic functions and Martin compactification.

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Ballot problem.

In an election where candidate A receives p votes and candidate B receives q votes with $p > q$, what is the probability that A will be strictly ahead of B throughout the count?

This can be seen also as a number of positive Dyck paths from $(0, 0)$ to (n, y) with $n = p + q$ and $y = p - q$.

More general question: What is the number of Dyck paths from $(0, x)$ to (n, y) with some $x, y > 0$.

The answer is given by the reflection principle:

number of 'good paths'$=$ number of all paths $-$ number of bad paths.

Using this formula, we conclude that the number of positive Dyck paths between $(0, x)$ to (n, y) equals

$$\binom{n}{\frac{n+y-x}{2}} - \binom{n}{\frac{n+y+x}{2}}.$$

Letting $n \to \infty$, one can get, for all fixed $x, y > 0$,

$$\binom{n}{\frac{n+y-x}{2}} - \binom{n}{\frac{n+y+x}{2}} \sim \sqrt{\frac{2}{\pi}} xyn^{-3/2} 2^n.$$

Reformulation in terms of random walks.

Let $X(k)$ be independent, identically distributed, $\mathbb{Z}$-valued random walks. Set

$$S(n) := X(1) + X(2) + \ldots + X(n), \quad n \geq 1.$$

Set also

$$\tau_x := \inf\{n \geq 1 : x + S(n) \leq 0\}, \quad x \geq 0.$$

Then the number of positive Dyck paths of length n between x and y equals

$$2^n \mathbf{P}(x + S(n) = y, \tau_x > n).$$

In this case we choose $\mathbf{P}(X(k) = \pm 1) = 1/2$.

Conditioned random walks 1.

If the random variables have zero mean and finite variance then one has

$$\mathbf{P}(x + S(n) = y, \tau_x > n) \sim cV(x)V'(y)n^{-3/2}$$

and

$$\mathbf{P}(\tau_x > n) \sim \sqrt{\frac{2}{\pi}} V(x) n^{-1/2}.$$

The function $V(x)$ is called **discrete harmonic** function, i.e. it satisfies the following equation:

$$V(x) = \mathbf{E}[V(x + S(1)); \tau_x > 1], \quad x > 0.$$

If we define the transition kernel P by the equalities

$$P(x, y) = \mathbf{P}(x + S(1) = y, \tau_x > 1)$$

then we have

$$pV = V \quad \text{or} \quad (I - P)V = 0.$$

Conditioned random walks 2.

In the one-dimensional case one knows that the function $V(x)$ is given by

$$V(x) = x - \mathbf{E}[x + S(\tau_x)].$$

This function is asymptotically linear, $V(x) \sim x$. There is only a 'very bounded' number of particular cases where one can have a closed form expression for $V(x)$.

The function $V'(y)$ is also a discrete harmonic function, but for the walk $\{-S(n)\}$.

Planar walks in positive quadrant is a quite popular object in the enumerative combinatorics. Given a set of possible steps, one is interested in the number (exact or asymptotic) of walks between x and y of length n.

Also in this case the dependence on the endpoints x and y is described by the harmonic functions for the corresponding random walks.

Typical mathematical questions:

- How can one construct a positive harmonic function for a discrete time stochastic process confined to some subset?
- How many such functions do exist?
- Can one describe the set of all harmonic functions?

If our primer interest comes from some boundary crossing problems in probability or from some enumeration problems in combinatorics then we need to construct the 'right' harmonic function, which describes the asymptotic behaviour of numbers we are interested in.

This function can be seen as a 'natural' one.

Random walk conditioned to stay positive at all times.

Let us determine the limit

$$\lim_{n\to\infty} \mathbf{P}(x+S(k+1)=z|x+S(k)=y,\tau_x>n)$$

for all fixed k, $y, z>0$. Clearly,

$$\begin{aligned}\mathbf{P}&(x+S(k+1)=z|x+S(k)=y,\tau_x>n)\\ &=\frac{\mathbf{P}(x+S(k)=y,x+S(k+1)=z,\tau_x>n)}{\mathbf{P}(x+S(k)=y,\tau_x>n)}\end{aligned}$$

By the Markov property at time k,

$$\begin{aligned}\mathbf{P}(x+S(k)=y,\tau_x>n)&=\mathbf{P}(x+S(k)=y,\tau_x>k)\mathbf{P}(\tau_y>n-k)\\ &\sim \mathbf{P}(x+S(k)=y,\tau_x>k)c_0V(y)n^{-1/2}.\end{aligned}$$

Similarly,

$$\begin{aligned}
&\mathbf{P}(x+S(k)=y, x+S(k+1)=z, \tau_x>n)\\
&=\mathbf{P}(x+S(k)=y,\tau_x>k)\mathbf{P}(X(k+1)=z-y)\mathbf{P}(\tau_z>n-k)\\
&\sim \mathbf{P}(x+S(k)=y,\tau_x>k)c_0\mathbf{P}(X(k+1)=z-y)V(z)n^{-1/2}.
\end{aligned}$$

As a result we have

$$\begin{aligned}
&\lim_{n\to\infty}\mathbf{P}(x+S(k+1)=z|x+S(k)=y,\tau_x>n)\\
&=\frac{V(z)}{V(y)}\mathbf{P}(X(k+1)=z-y)=\frac{V(z)}{V(y)}P(y,z),\quad y,z>0.
\end{aligned}$$

The transition kernel $\widehat{P}(y,z)=\frac{V(z)}{V(y)}P(y,z)$ is stochastic and lives on positive numbers.

This procedure is called **Doob's h-transform**. Clearly, one can perform such a change of measure with any positive harmonic function, but only V gives the connection to 'physical' interpretation of conditioning.

Universality approach to the construction of harmonic functions.

Let $P(x, dy)$ be a transition kernel for some (killed) Markov chain $\{U_n\}$, that is,

$$\mathbf{P}(U_{n+1} \in A | U_n = x) = P(x, A).$$

The kernel P is assumed to be substochastic, i.e. $P(x, \mathbb{R}) \le 1$.

We want to find a positive solution to the fixed point equation

$$PV(x) = \int P(x, dy)V(y) = V(x) \quad \text{for all } x.$$

Considering this as a fixed point equation, we can try to use the standard method of iterations: take some function $f(x)$ and determine the limit $\lim_{n\to\infty} P^n f(x)$. But for almost all 'starting points' f one will get 0 or ∞ in the limit.

A quite good choice of the starting point is a 'known' harmonic function for a different process, which belongs to the same class of universality.

For example, if we look at random walks with zero mean and finite second moments then we can take a harmonic function of the Brownian motion. Due to the central limit theorem, random walks and Brownian motion are in the same class of universality.

In the case of processes in continuous time one can use the powerful machinery of differential calculus. So, in many cases one knows the corresponding harmonic function (solution to some diff.equation).

Random walks in cones. Let $X(k)$ be now independent copies of a random vector X and let K be an open cone in $\mathbb{R}^d$, satisfying some additional geometric properties.

We are interested in a harmonic function for $S(n)$ killed at leaving the cone K. In other words, we want to find a solution to

$$V(x) = \mathbf{E}[V(x+S(1)); \tau_x > 1], \quad x \in K,$$

where

$$\tau_x := \inf\{n \geq 1 : x + S(n) \notin K\}.$$

A harmonic function for the Brownian motion is a solution to

$$\Delta u(x) = 0, \; x \in K \quad \text{and} \quad u\Big|_{\partial K} = 0.$$

The function $u(x)$ is homogeneous:

$$u(x) = |x|^p u\left(\frac{x}{|x|}\right).$$

If we apply the transition kernel of our random walk to the function u then we get

$$\begin{aligned}\mathbf{E}[u(x+X)] - u(x) &= \nabla u(x)\mathbf{E}[X_1] + \sum \frac{\partial^2}{\partial x_i \partial x_j} u(x)\mathbf{E}[X_i X_j] + R(x) \\ &= R(x)\end{aligned}$$

and $R(x)$ is of order $o(u(x)/|x|^2)$. This shows how the universality works on the technical level.

Theorem. (Denisov and W.) Assume that $\mathbf{E}X_1 = 0$, $\mathbf{E}X_i X_j = \delta_{ij}$ and $\mathbf{E}|X|^p < \infty$, $\mathbf{E}|X|^2 \log|X| < \infty$. Then the function

$$V(x) := \lim_{n\to\infty} \mathbf{E}[u(x+S(n)); \tau_x > n]$$

is well-defined. This function is harmonic for $S(n)$ killed at leaving K and satisfies

$$V(x) \sim u(x) \quad \text{as } \mathrm{dist}(x, \partial K) \to \infty.$$

Having constructed V one can prove various limit theorems for $S(n)$ conditioned to stay in cone:

$$\mathbf{P}(\tau_x > n) \sim \kappa \frac{V(x)}{n^{p/2}},$$

$$\mathbf{P}(x + S(n) = y, \tau_x > n) \sim c \frac{V(x)V'(y)}{n^{p+d/2}}.$$

Consequence for enumeration of lattice paths in the positive quadrant: number of paths between x and y is asymptotically equivalent to

$$c \frac{V(x)V'(y)}{n^{p+d/2}} R^n.$$

Example: Gambler's problem with many players.

Classical gambler's problem:

Two players are playing a fair coin-tossing game, the result of each round consists in the transfer of one Euro.

This game can be modelled as a random walk on the interval $[0, N]$ with jumps ± 1. (N denotes the total amount of money in the game.)

If the starting capital of the first player is x then the probability that he wins the whole game is equal to x/N.

Gambler's problem with many players (Diaconis and Ethier):

Now we assume that we have more than two players. More precisely, we shall assume that the number of players is 3. In this case every round consists of two steps:

1. choose randomly a pair of players,

2. toss a fair coin and transfer one Euro accordingly.

Again, the total amount of money remains constant and it is sufficient to notice the capitals of two players. As a result one has a Markov chain on the simplex

$$\{(x,y) : x > 0, y > \text{ and } x + y < N\}$$

and with equidistributed jumps $(1,0), (-1,0), (0,1), (0,-1), (1,-1), (-1,1)$.

If one of the boundaries is reached then we have a fair game with just two players.

In this example one has, as $N \to \infty$,

$$P_{y_1,y_2}(3,2,1) \sim \frac{\Gamma(1/3)^9}{32\pi^4} \frac{y_1 y_2 (y_1 + y_2)}{N^3},$$

$$\mathbf{P}_{y_1,y_2}(\text{third player gets eliminated first}) \sim \frac{\Gamma(1/3)^9}{16\pi^4} \frac{y_1 y_2 (y_1 + y_2)}{N^3}.$$

How many positive harmonic functions do exist?

Martin compactification.

Let $P(x,y)$ be a substochastic transition kernel on a discrete state space U.

We shall always assume that this kernel is transient:

$$G(x,y) := \sum_{n\geq 0} P^n(x,y) < \infty.$$

The function

$$K(x,y) := \frac{G(x,y)}{G(x_0,y)} \leq C(x)$$

is called **Martin kernel.**

Now one can define a metric

$$\rho(x,y) = \sum_{z\in U} T(z) \frac{|K(z,x) - K(z,y)| + |\delta_{zx} - \delta_{zy}|}{1 + C(z)},$$

with some summable weights $T(z)$.

One can embed (U, ρ) into a compact space $(\widehat{U}, \rho)$.

Also the function $y \mapsto K(x, y)$ can be extended, by continuity, to the whole compactification $\widehat{U}$.

The set $\partial_M U = \widehat{U} \setminus U$ is then called **Martin boundary.**

Representation theorem: For every non-negative harmonic function $v(x)$ there exists a measure μ such that

$$v(x) = \int_{\partial_M U} K(x, \alpha) \mu(d\alpha).$$

If the set $\partial_M U$ contains only one point then $v(x)$ is unique.

Theorem (Duraj, Raschel, Tarrago and W.)

For random walks in cones one has

$$K(x,y) \to V(x) \quad \text{as } y \to \infty.$$

In particular, the harmonic function $V(x)$ is unique.