Algebraicity and transcendence of power series: combinatorial and computational aspects

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Exercises

Exercises, Part I

1. Explain why the transcendence of π, $\sin(1)$, e^{π} follows from theorems by Hermite-Lindemann, Lindemann-Weierstrass, Gel'fond-Schneider.

2. Explain why $\sum_n F_n t^n$ is rational, where $F_{n+2} = F_{n+1} + F_n, F_0 = 0, F_1 = 1$. Find a general statement.

3. Apply the proof of Abel's theorem on the concrete example

$$f(t) = \sqrt{1-t} = 1 - \frac{1}{2}t - \frac{1}{8}t^2 - \frac{1}{16}t^3 - \frac{5}{128}t^4 - \cdots$$

4. Show that for any $b \in \mathbb{N}$ the series $f_b(t) = \sum_n \binom{bn}{n} t^n$ is algebraic.

5. Show that the power series
   - $E(t) = \sum_n \frac{t^n}{n!}$
   - $M(t) = \sum_n t^{2^n}$
   - $R_2(t) = \sum_n r_2(n) t^n,$ $\qquad r_2(n) = \#\{(a,b) \in \mathbb{Z}^2 : a^2 + b^2 = n\}$

   are transcendental.