Algebraicity and transcendence of power series: combinatorial and computational aspects

Alin Bostan

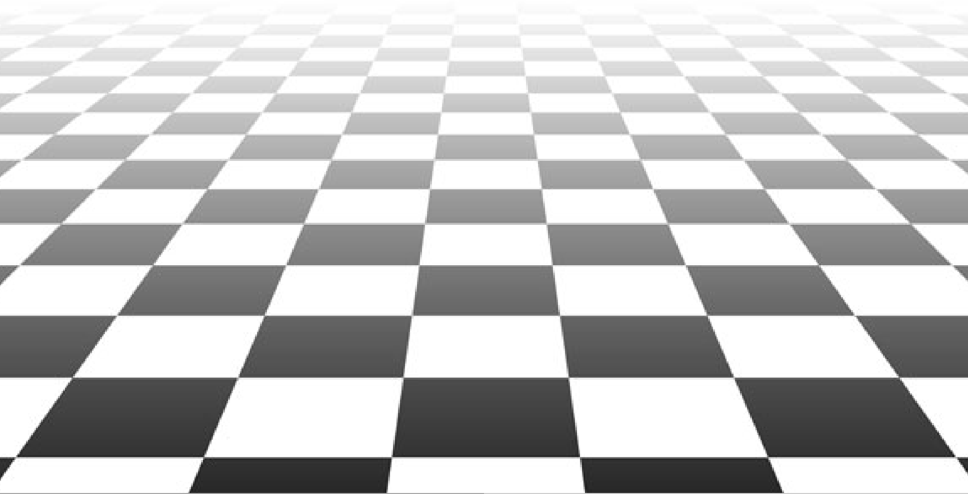
Inria — informatics mathematics

Algorithmic and Enumerative Combinatorics
RISC, Hagenberg, August 1–5, 2016

Overview

1. Monday: Context and Examples
2. Tuesday: Properties and Criteria (1)
3. Wednesday: Properties and Criteria (2)
4. Thursday: Algorithmic Proofs of Algebraicity
5. Friday: Transcendence in Lattice Path Combinatorics

Part V: Transcendence in Lattice Path Combinatorics

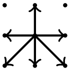


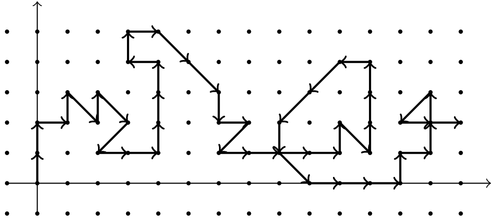
Lattice walks with small steps in the quarter plane

▷ We focus on nearest-neighbor walks in the quarter plane, i.e. walks in $\mathbb{N}^2$ starting at $(0,0)$ and using steps in a *fixed* subset $\mathfrak{S}$ of

$$\{\swarrow, \leftarrow, \nwarrow, \uparrow, \nearrow, \rightarrow, \searrow, \downarrow\}.$$

▷ Example with $n = 45$, $i = 14$, $j = 2$ for:

$\mathfrak{S} =$ 



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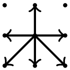
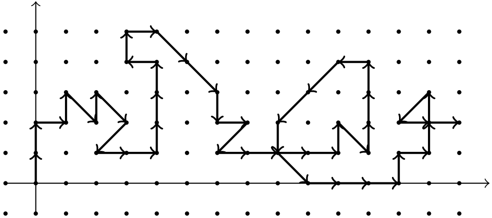
$\mathfrak{S} =$ 



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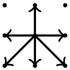
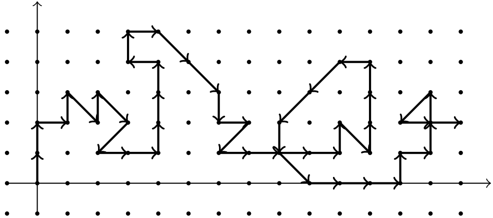
$\mathfrak{S} =$ 



▷ Counting sequence: $f_{n;i,j}$ = number of walks of length n ending at (i, j).

▷ Specializations:

- $f_{n;0,0}$ = number of walks of length n returning to origin ("excursions");
- $f_n = \sum_{i,j \geq 0} f_{n;i,j}$ = number of walks with prescribed length n.

Generating series and combinatorial problems

▷ Complete generating series:

$$F(t;x,y) = \sum_{n=0}^{\infty} \left(\sum_{i,j=0}^{\infty} f_{n;i,j} x^i y^j \right) t^n \ \in \mathbb{Q}[x,y][[t]].$$

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- Walks ending on the horizontal axis: $F(t;1,0)$;
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Combinatorial questions:

Given $\mathfrak{S}$, what can be said about $F(t;x,y)$, resp. $f_{n;i,j}$, and their variants?

- Structure of F: algebraic? transcendental?
- Explicit form: of F? of f?
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Our goal: Use computer algebra to give computational answers.

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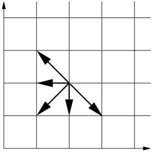

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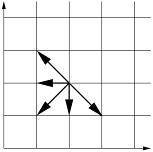

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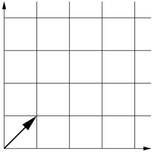

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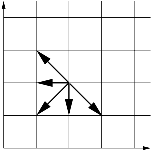

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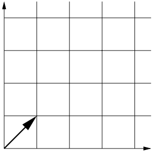

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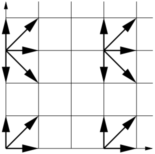

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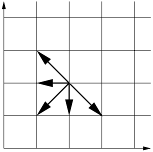

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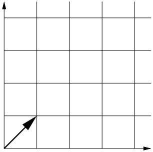

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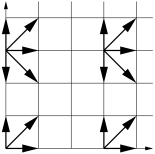

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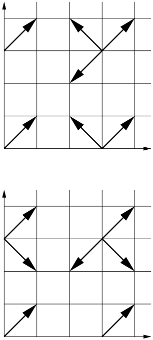

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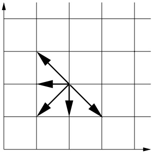

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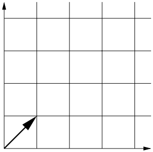

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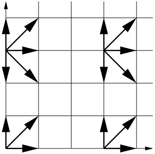

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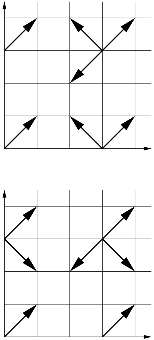

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One is left with 79 interesting distinct models.

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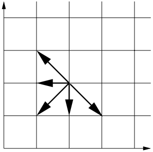

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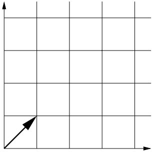

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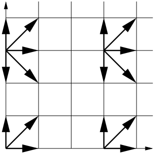

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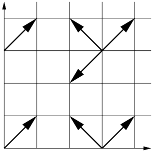

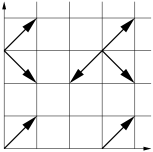

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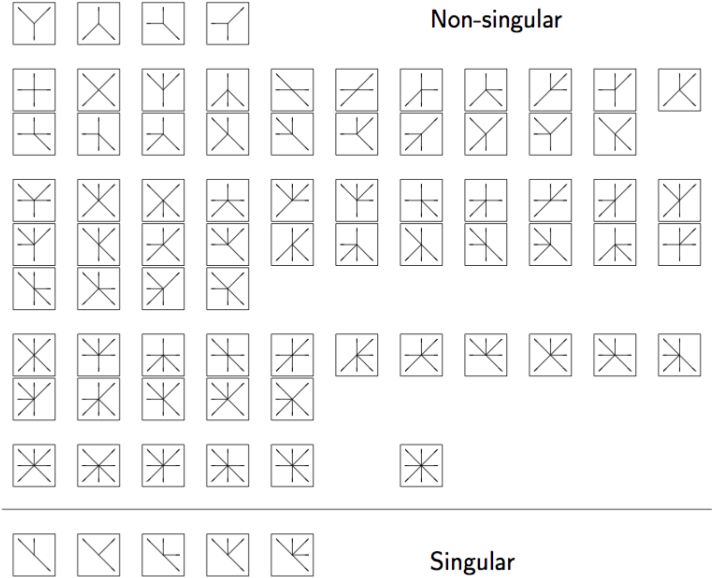
Is any further classification possible?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Singular

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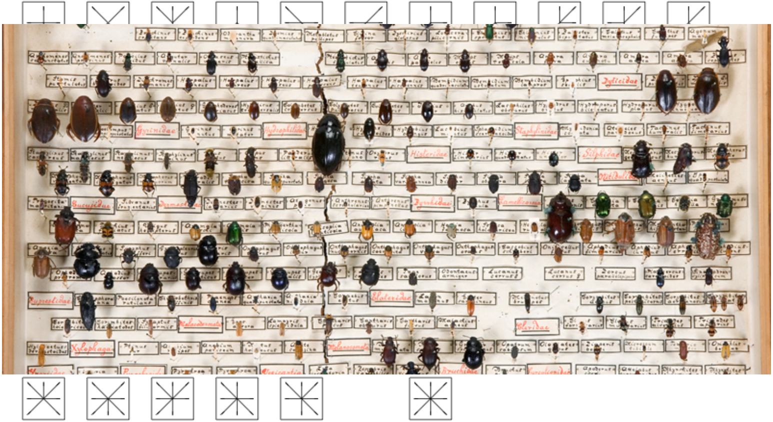

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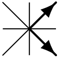
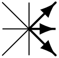
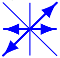
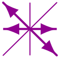
Dyck: 

Motzkin: 

Pólya: 

Kreweras: 

Gessel: 

Gouyou-Beauchamps: 

King: 

Algebraic reformulation: solving a functional equation

Generating function: $G(t;x,y) = \sum_{n=0}^{\infty} \sum_{i=0}^{n} \sum_{j=0}^{n} g(n;i,j) t^n x^i y^j \in \mathbb{Q}[x,y][[t]]$

"Kernel equation":

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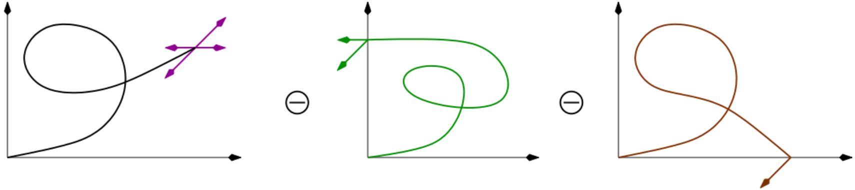

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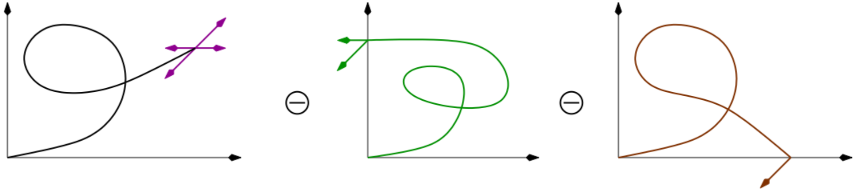

Task: Solve this functional equation!

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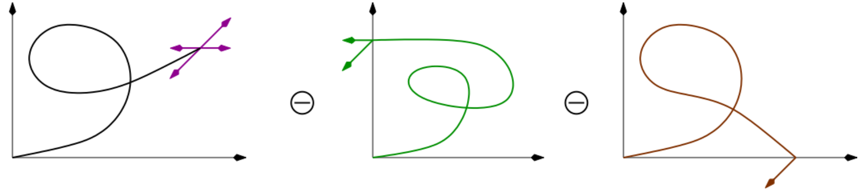

Task: For the other models: solve 78 similar equations!

Main results (I): algebraicity of Gessel walks

Theorem [Kreweras 1965; 100 pages long combinatorial proof!]

$$K(t;0,0) = {}_3F_2\left(\begin{matrix}1/3 & 2/3 & 1\\ & 3/2 & 2\end{matrix}\,\middle|\, 27\,t^3\right) = \sum_{n=0}^{\infty} \frac{4^n\binom{3n}{n}}{(n+1)(2n+1)}\,t^{3n}.$$

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▷ Guess'n'Prove method, using Hermite-Padé approximants† $\longrightarrow$ Yesterday

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▷ New (human) proofs [B., Kurkova & Raschel 2013], [Bousquet-Mélou 2015]

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Main results (II): Explicit form for $G(t;x,y)$

Theorem [B., Kauers & van Hoeij 2010]

Let $V = 1 + 4t^2 + 36t^4 + 396t^6 + \cdots$ be a root of

$$(V-1)(1+3/V)^3 = (16t)^2,$$

let $U = 1 + 2t^2 + 16t^4 + 2xt^5 + 2(x^2+83)t^6 + \cdots$ be a root of

$$x(V-1)(V+1)U^3 - 2V(3x+5xV-8Vt)U^2$$
$$-xV(V^2-24V-9)U + 2V^2(xV-9x-8Vt) = 0,$$

let $W = t^2 + (y+8)t^4 + 2(y^2+8y+41)t^6 + \cdots$ be a root of

$$y(1-V)W^3 + y(V+3)W^2 - (V+3)W + V - 1 = 0.$$

Then $G(t;x,y)$ is equal to

$$\frac{\frac{64(U(V+1)-2V)V^{3/2}}{x(U^2-V(U^2-8U+9-V))^2} - \frac{y(W-1)^4(1-Wy)V^{-3/2}}{t(y+1)(1-W)(W^2y+1)^2}}{(1+y+x^2y+x^2y^2)t - xy} - \frac{1}{tx(y+1)}.$$

▷ Computer-driven discovery and proof; no human proof yet.

Main results (II): Explicit form for $G(t;x,y)$

Theorem [B., Kauers & van Hoeij 2010]
Let $V = 1 + 4t^2 + 36t^4 + 396t^6 + \cdots$ be a root of

$$(V-1)(1+3/V)^3 = (16t)^2,$$

let $U = 1 + 2t^2 + 16t^4 + 2xt^5 + 2(x^2+83)t^6 + \cdots$ be a root of

$$x(V-1)(V+1)U^3 - 2V(3x+5xV-8Vt)U^2$$
$$-xV(V^2-24V-9)U + 2V^2(xV-9x-8Vt) = 0,$$

let $W = t^2 + (y+8)t^4 + 2(y^2+8y+41)t^6 + \cdots$ be a root of

$$y(1-V)W^3 + y(V+3)W^2 - (V+3)W + V - 1 = 0.$$

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▷ Computer-driven discovery and proof; no human proof yet.
▷ Proof uses guessed minimal polynomials for $G(t;x,0)$ and $G(t;0,y)$.

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▷ Computer-driven discovery and proof; ~~no human proof yet~~
▷ Recent (human) proofs [B., Kurkova, Raschel '13], [Bousquet-Mélou '15]

Main results (III): Conjectured D-Finite $F(t;1,1)$ [B. & Kauers 2009]

	OEIS	$\mathfrak{S}$	Pol size	ODE size		OEIS	$\mathfrak{S}$	Pol size	ODE size
1	A005566		—	3, 4	13	A151275		—	5, 24
2	A018224		—	3, 5	14	A151314		—	5, 24
3	A151312		—	3, 8	15	A151255		—	4, 16
4	A151331		—	3, 6	16	A151287		—	5, 19
5	A151266		—	5, 16	17	A001006		2, 2	2, 3
6	A151307		—	5, 20	18	A129400		2, 2	2, 3
7	A151291		—	5, 15	19	A005558		—	3, 5
8	A151326		—	5, 18					
9	A151302		—	5, 24	20	A151265		6, 8	4, 9
10	A151329		—	5, 24	21	A151278		6, 8	4, 12
11	A151261		—	4, 15	22	A151323		4, 4	2, 3
12	A151297		—	5, 18	23	A060900		8, 9	3, 5

Equation sizes = {order, degree}@(algeq, diffeq)

- ▷ Computerized discovery by enumeration + Hermite–Padé
- ▷ 1–22: Confirmed by human proofs in [Bousquet-Mélou & Mishna 2010]
- ▷ 23: Confirmed by a human proof in [B., Kurkova & Raschel 2015]

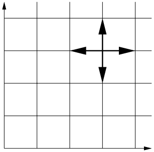
Main results (III): Conjectured D-Finite $F(t;1,1)$ [B. & Kauers 2009]

	OEIS	$\mathfrak{S}$	alg	asympt		OEIS	$\mathfrak{S}$	alg	asympt
1	A005566		N	$\frac{4}{\pi}\frac{4^n}{n}$	13	A151275		N	$\frac{12\sqrt{30}}{\pi}\frac{(2\sqrt{6})^n}{n^2}$
2	A018224		N	$\frac{2}{\pi}\frac{4^n}{n}$	14	A151314		N	$\frac{\sqrt{6}\lambda\mu C^{5/2}}{5\pi}\frac{(2C)^n}{n^2}$
3	A151312		N	$\frac{\sqrt{6}}{\pi}\frac{6^n}{n}$	15	A151255		N	$\frac{24\sqrt{2}}{\pi}\frac{(2\sqrt{2})^n}{n^2}$
4	A151331		N	$\frac{8}{3\pi}\frac{8^n}{n}$	16	A151287		N	$\frac{2\sqrt{2}A^{7/2}}{\pi}\frac{(2A)^n}{n^2}$
5	A151266		N	$\frac{1}{2}\sqrt{\frac{3}{\pi}}\frac{3^n}{n^{1/2}}$	17	A001006		Y	$\frac{3}{2}\sqrt{\frac{3}{\pi}}\frac{3^n}{n^{3/2}}$
6	A151307		N	$\frac{1}{2}\sqrt{\frac{5}{2\pi}}\frac{5^n}{n^{1/2}}$	18	A129400		Y	$\frac{3}{2}\sqrt{\frac{3}{\pi}}\frac{6^n}{n^{3/2}}$
7	A151291		N	$\frac{4}{3\sqrt{\pi}}\frac{4^n}{n^{1/2}}$	19	A005558		N	$\frac{8}{\pi}\frac{4^n}{n^2}$
8	A151326		N	$\frac{2}{\sqrt{3\pi}}\frac{6^n}{n^{1/2}}$					
9	A151302		N	$\frac{1}{3}\sqrt{\frac{5}{2\pi}}\frac{5^n}{n^{1/2}}$	20	A151265		Y	$\frac{2\sqrt{2}}{\Gamma(1/4)}\frac{3^n}{n^{3/4}}$
10	A151329		N	$\frac{1}{3}\sqrt{\frac{7}{3\pi}}\frac{7^n}{n^{1/2}}$	21	A151278		Y	$\frac{3\sqrt{3}}{\sqrt{2}\Gamma(1/4)}\frac{3^n}{n^{3/4}}$
11	A151261		N	$\frac{12\sqrt{3}}{\pi}\frac{(2\sqrt{3})^n}{n^2}$	22	A151323		Y	$\frac{\sqrt{2}\,3^{3/4}}{\Gamma(1/4)}\frac{6^n}{n^{3/4}}$
12	A151297		N	$\frac{\sqrt{3}B^{7/2}}{2\pi}\frac{(2B)^n}{n^2}$	23	A060900		Y	$\frac{4\sqrt{3}}{3\Gamma(1/3)}\frac{4^n}{n^{2/3}}$

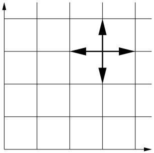
$A=1+\sqrt{2},\ B=1+\sqrt{3},\ C=1+\sqrt{6},\ \lambda=7+3\sqrt{6},\ \mu=\sqrt{\frac{4\sqrt{6}-1}{19}}$

- ▷ Computerized discovery by enumeration + Hermite–Padé + LLL/PSLQ.
- ▷ Confirmed by human proofs in [Melczer & Wilson, 2015]

The group of a model: the simple walk case



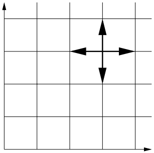
The characteristic polynomial $\chi_{\mathfrak{S}} := x + \frac{1}{x} + y + \frac{1}{y}$



The characteristic polynomial $\chi_{\mathfrak{S}} := x + \frac{1}{x} + y + \frac{1}{y}$ is left invariant under

$$\psi(x,y) = \left(x, \frac{1}{y}\right), \quad \phi(x,y) = \left(\frac{1}{x}, y\right),$$

The group of a model: the simple walk case



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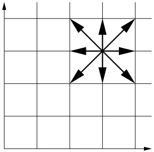
$$\psi(x,y) = \left(x, \frac{1}{y}\right), \quad \phi(x,y) = \left(\frac{1}{x}, y\right),$$

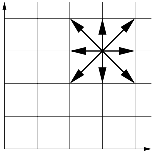
and thus under any element of the group

$$\langle \psi, \phi \rangle = \left\{ (x,y), \left(x, \frac{1}{y}\right), \left(\frac{1}{x}, \frac{1}{y}\right), \left(\frac{1}{x}, y\right) \right\}.$$

The group of a model: the general case



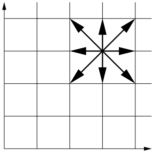
The polynomial $\chi_{\mathfrak{S}} := \sum_{(i,j)\in\mathfrak{S}} x^i y^j = \sum_{i=-1}^{1} B_i(y) x^i = \sum_{j=-1}^{1} A_j(x) y^j$



The polynomial $\chi_{\mathfrak{S}} := \sum_{(i,j)\in\mathfrak{S}} x^i y^j = \sum_{i=-1}^{1} B_i(y) x^i = \sum_{j=-1}^{1} A_j(x) y^j$ is left invariant under

$$\psi(x,y) = \left(x, \frac{A_{-1}(x)}{A_{+1}(x)} \frac{1}{y}\right), \quad \phi(x,y) = \left(\frac{B_{-1}(y)}{B_{+1}(y)} \frac{1}{x}, y\right),$$

The group of a model: the general case



The polynomial $\chi_{\mathfrak{S}} := \sum_{(i,j)\in\mathfrak{S}} x^i y^j = \sum_{i=-1}^{1} B_i(y) x^i = \sum_{j=-1}^{1} A_j(x) y^j$ is left invariant under

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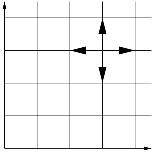
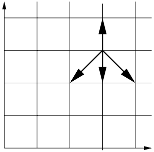
and thus under any element of the group

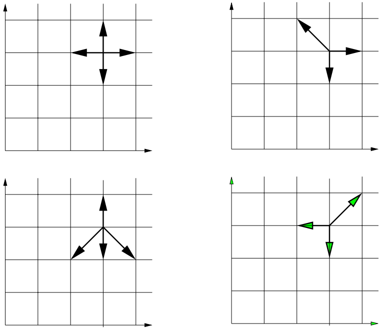
$$\mathcal{G}_{\mathfrak{S}} := \langle \psi, \phi \rangle.$$

Examples of groups



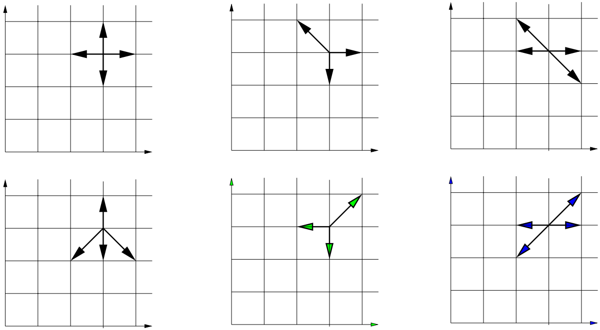


Order 4,

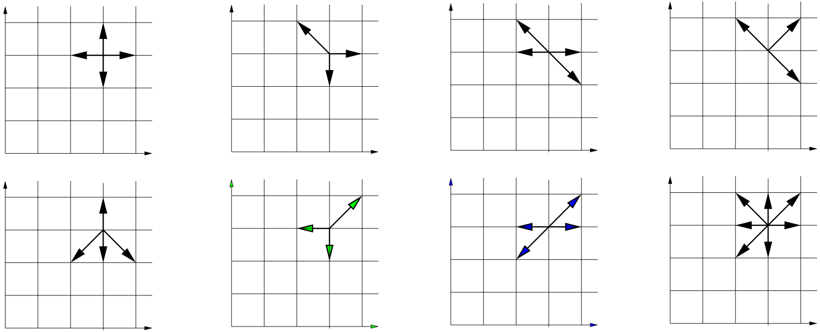


Order 4, order 6,

Examples of groups



Order 4, order 6, order 8,



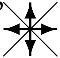
Order 4, order 6, order 8, order ∞.

An important concept: the orbit sum (OS)

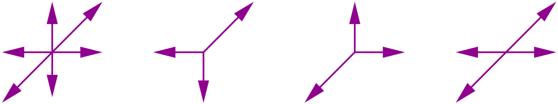
The orbit sum of a model $\mathfrak{S}$ is the following polynomial in $\mathbb{Q}[x, x^{-1}, y, y^{-1}]$:

$$\mathrm{OrbitSum}(\mathfrak{S}) := \sum_{\theta \in \mathcal{G}_{\mathfrak{S}}} (-1)^{\theta} \theta(xy)$$

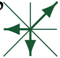
▷ E.g., for the simple walk:

OS  $= x \cdot y - \frac{1}{x} \cdot y + \frac{1}{x} \cdot \frac{1}{y} - x \cdot \frac{1}{y}$

▷ For 4 models, the orbit sum is zero:



E.g. for the Kreweras model:

OS  $= x \cdot y - \frac{1}{xy} \cdot y + \frac{1}{xy} \cdot x - y \cdot x + y \cdot \frac{1}{xy} - x \cdot \frac{1}{xy} \quad = 0$

The 79 models: finite and infinite groups

79 models

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23 admit a finite group

[Mishna'07]

79 models

56 have an infinite group

[Bousquet-Mélou & Mishna'10]

The 79 models: finite and infinite groups

all $F(t;x,y)$ D-finite

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19 transcendental

(OS $\neq 0$)

[Gessel & Zeilberger'92]

[Bousquet-Mélou'02]

4 algebraic (OS $= 0$)

(3 Kreweras-type + Gessel)

[BMM'10] + [B. & Kauers'10]

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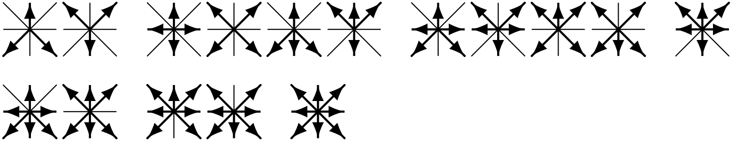
$\longrightarrow$ all non-D-finite

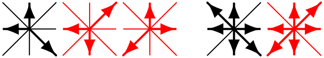
- [Mishna & Rechnitzer'07] and [Melczer & Mishna'13] for 5 singular models
- [Kurkova & Raschel'13] and [B., Raschel & Salvy'13] for all others

The 23 models with a finite group

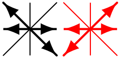
(i) 16 with a vertical symmetry, and group isomorphic to D_2



(ii) 5 with a diagonal or anti-diagonal symmetry, and group isomorphic to D_3



(iii) 2 with group isomorphic to D_4



(i): vertical symmetry; (ii)+(iii): zero drift $\sum_{s \in \mathfrak{S}} s$

In red, models with OS $= 0$ and algebraic GF

Main results (IV): explicit expressions for the 19 D-finite transcendental models

Theorem [B., Chyzak, van Hoeij, Kauers & Pech, 2016]

Let $\mathfrak{S}$ be one of the 19 models with finite group $\mathcal{G}_{\mathfrak{S}}$, and non-zero orbit sum. Then

- $F_{\mathfrak{S}}$ is expressible using iterated integrals of ${}_2F_1$ expressions.
- Among the 19×4 specializations of $F_{\mathfrak{S}}(t; x, y)$ at $(x, y) \in \{0, 1\}^2$, only 4 are algebraic: for $\mathfrak{S} =$  at $(1, 1)$, and $\mathfrak{S} =$  at $(1, 0), (0, 1), (1, 1)$

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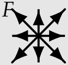
Example (King walks in the quarter plane, A025595)

F  $(t; 1, 1) = \frac{1}{t} \int_0^t \frac{1}{(1+4x)^3} \cdot {}_2F_1\left(\begin{matrix} \frac{3}{2} \ \frac{3}{2} \\ 2 \end{matrix} \middle| \frac{16x(1+x)}{(1+4x)^2}\right) dx$

$$= 1 + 3t + 18t^2 + 105t^3 + 684t^4 + 4550t^5 + 31340t^6 + 219555t^7 + \cdots$$

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Example (King walks in the quarter plane, A025595)

$$F_{\mathfrak{S}}(t; 1, 1) = \frac{1}{t} \int_0^t \frac{1}{(1+4x)^3} \cdot {}_2F_1\left(\left. \begin{matrix} \frac{3}{2} \;\; \frac{3}{2} \\ 2 \end{matrix} \right| \frac{16x(1+x)}{(1+4x)^2} \right) dx$$

where $\mathfrak{S} =$ 

$$= 1 + 3t + 18t^2 + 105t^3 + 684t^4 + 4550t^5 + 31340t^6 + 219555t^7 + \cdots$$

▷ Computer-driven discovery and proof; no human proof yet.

▷ Proof uses creative telescoping, ODE factorization, ODE solving.

Hypergeometric Series Occurring in Explicit Expressions for $F(t;1,1)$

	hyp$_1$	hyp$_2$	w		hyp$_1$	hyp$_2$	w
1	${}_2F_1\left(\begin{matrix}\frac{1}{2} & \frac{1}{2}\\ 1\end{matrix}\middle\| w\right)$	${}_2F_1\left(\begin{matrix}\frac{1}{2} & \frac{3}{2}\\ 2\end{matrix}\middle\| w\right)$	$16t^2$	10	${}_2F_1\left(\begin{matrix}\frac{7}{4} & \frac{9}{4}\\ 2\end{matrix}\middle\| w\right)$	${}_2F_1\left(\begin{matrix}\frac{9}{4} & \frac{11}{4}\\ 3\end{matrix}\middle\| w\right)$	$\frac{64(t^2+t+1)t^2}{(12t^2+1)^2}$
2	${}_2F_1\left(\begin{matrix}\frac{1}{2} & \frac{1}{2}\\ 1\end{matrix}\middle\| w\right)$		$16t^2$	11	${}_2F_1\left(\begin{matrix}\frac{1}{2} & \frac{3}{2}\\ 2\end{matrix}\middle\| w\right)$	${}_2F_1\left(\begin{matrix}\frac{1}{2} & \frac{5}{2}\\ 3\end{matrix}\middle\| w\right)$	$\frac{16t^2}{4t^2+1}$
3	${}_2F_1\left(\begin{matrix}\frac{3}{2} & \frac{3}{2}\\ 2\end{matrix}\middle\| w\right)$		$\frac{16t}{(2t+1)(6t+1)}$	12	${}_2F_1\left(\begin{matrix}\frac{5}{4} & \frac{7}{4}\\ 1\end{matrix}\middle\| w\right)$	${}_2F_1\left(\begin{matrix}\frac{5}{4} & \frac{7}{4}\\ 2\end{matrix}\middle\| w\right)$	$\frac{64t^3(2t+1)}{(8t^2-1)^2}$
4	${}_2F_1\left(\begin{matrix}\frac{3}{2} & \frac{3}{2}\\ 2\end{matrix}\middle\| w\right)$		$\frac{16t(1+t)}{(1+4t)^2}$	13	${}_2F_1\left(\begin{matrix}\frac{7}{4} & \frac{9}{4}\\ 2\end{matrix}\middle\| w\right)$	${}_2F_1\left(\begin{matrix}\frac{7}{4} & \frac{9}{4}\\ 3\end{matrix}\middle\| w\right)$	$\frac{64t^2(t^2+1)}{(16t^2+1)^2}$
5	${}_2F_1\left(\begin{matrix}\frac{3}{4} & \frac{5}{4}\\ 1\end{matrix}\middle\| w\right)$	${}_2F_1\left(\begin{matrix}\frac{5}{4} & \frac{7}{4}\\ 2\end{matrix}\middle\| w\right)$	$64t^4$	14	${}_2F_1\left(\begin{matrix}\frac{7}{4} & \frac{9}{4}\\ 2\end{matrix}\middle\| w\right)$	${}_2F_1\left(\begin{matrix}\frac{9}{4} & \frac{11}{4}\\ 3\end{matrix}\middle\| w\right)$	$\frac{64(t^2+t+1)t^2}{(12t^2+1)^2}$
6	${}_2F_1\left(\begin{matrix}\frac{7}{4} & \frac{9}{4}\\ 2\end{matrix}\middle\| w\right)$	${}_2F_1\left(\begin{matrix}\frac{7}{4} & \frac{9}{4}\\ 3\end{matrix}\middle\| w\right)$	$\frac{64t^3(1+t)}{(1-4t^2)^2}$	15	${}_2F_1\left(\begin{matrix}\frac{1}{4} & \frac{3}{4}\\ 1\end{matrix}\middle\| w\right)$	${}_2F_1\left(\begin{matrix}\frac{3}{4} & \frac{5}{4}\\ 2\end{matrix}\middle\| w\right)$	$64t^4$
7	${}_2F_1\left(\begin{matrix}\frac{1}{2} & \frac{1}{2}\\ 1\end{matrix}\middle\| w\right)$	${}_2F_1\left(\begin{matrix}\frac{1}{2} & \frac{3}{2}\\ 1\end{matrix}\middle\| w\right)$	$\frac{16t^2}{4t^2+1}$	16	${}_2F_1\left(\begin{matrix}\frac{7}{4} & \frac{9}{4}\\ 2\end{matrix}\middle\| w\right)$	${}_2F_1\left(\begin{matrix}\frac{9}{4} & \frac{11}{4}\\ 3\end{matrix}\middle\| w\right)$	$\frac{64t^3(1+t)}{(1-4t^2)^2}$
8	${}_2F_1\left(\begin{matrix}\frac{5}{4} & \frac{7}{4}\\ 2\end{matrix}\middle\| w\right)$	${}_2F_1\left(\begin{matrix}\frac{7}{4} & \frac{9}{4}\\ 2\end{matrix}\middle\| w\right)$	$\frac{64t^3(2t+1)}{(8t^2-1)^2}$				
9	${}_2F_1\left(\begin{matrix}\frac{7}{4} & \frac{9}{4}\\ 2\end{matrix}\middle\| w\right)$	${}_2F_1\left(\begin{matrix}\frac{7}{4} & \frac{9}{4}\\ 3\end{matrix}\middle\| w\right)$	$\frac{64t^2(t^2+1)}{(16t^2+1)^2}$	19	${}_2F_1\left(\begin{matrix}-\frac{1}{2} & \frac{1}{2}\\ 1\end{matrix}\middle\| w\right)$	${}_2F_1\left(\begin{matrix}\frac{1}{2} & \frac{1}{2}\\ 2\end{matrix}\middle\| w\right)$	$16t^2$

▷ All related to complete elliptic integrals!

Theorem [B., Rachel & Salvy 2013]

Let $\mathfrak{S}$ be one of the 51 non-singular models with infinite group $\mathcal{G}_{\mathfrak{S}}$. Then $F_{\mathfrak{S}}(t;0,0)$, and in particular $F_{\mathfrak{S}}(t;x,y)$, are non-D-finite.

Main results (V): non-D-finiteness in models with an infinite group

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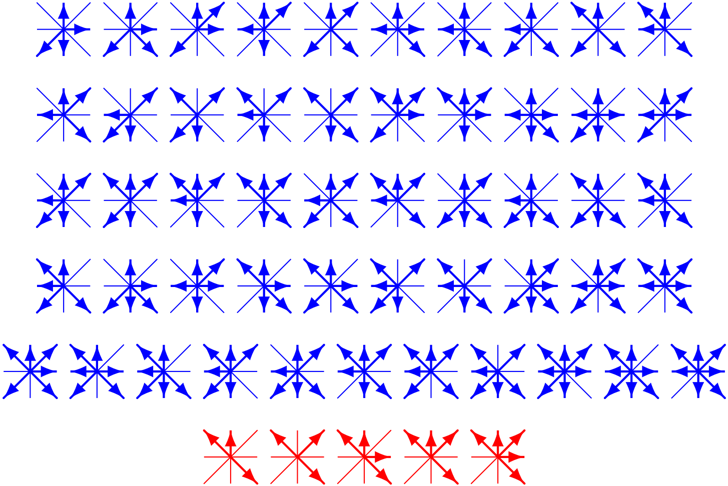
▷ Algorithmic proof. Uses Gröbner basis computations, polynomial factorization, cyclotomy testing.

▷ Based on two ingredients: asymptotics + irrationality.

▷ [Kurkova & Raschel 2013] Human proof that $F_{\mathfrak{S}}(t;x,y)$ is non-D-finite.

▷ No human proof yet for $F_{\mathfrak{S}}(t;0,0)$ non-D-finite.

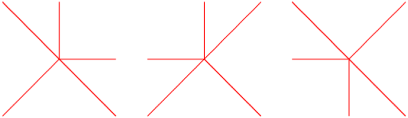
The 56 models with infinite group



In blue, non-singular models, solved by [B., Raschel & Salvy 2013]
In red, singular models, solved by [Melczer & Mishna 2013]

Example: the scarecrows

[B., Raschel & Salvy 2013]: $F_{\mathfrak{S}}(t;0,0)$ is not D-finite for the models



For the 1st and the 3rd, the excursions sequence $[t^n]\, F_{\mathfrak{S}}(t;0,0)$

$$1, 0, 0, 2, 4, 8, 28, 108, 372, \ldots$$

is $\sim K \cdot 5^n \cdot n^{-\alpha}$, with $\alpha = 1 + \pi / \arccos(1/4) = 3.383396\ldots$

The irrationality of α prevents $F_{\mathfrak{S}}(t;0,0)$ from being D-finite.

Summary: Classification of 2D non-singular walks

The Main Theorem Let $\mathfrak{S}$ be one of the 74 non-singular models. The following assertions are equivalent:

(1) The full generating series $F_{\mathfrak{S}}(t; x, y)$ is D-finite

(2) the excursions generating series $F_{\mathfrak{S}}(t; 0, 0)$ is D-finite

(3) the excursions sequence $[t^n]\, F_{\mathfrak{S}}(t; 0, 0)$ is $\sim K \cdot \rho^n \cdot n^\alpha$, with $\alpha \in \mathbb{Q}$

(4) the group $\mathcal{G}_{\mathfrak{S}}$ is finite (and $|\mathcal{G}_{\mathfrak{S}}| = 2 \cdot \min\{\ell \in \mathbb{N}^\star \mid \frac{\ell}{\alpha+1} \in \mathbb{Z}\}$)

(5) the step set $\mathfrak{S}$ has either an axial symmetry, or zero drift and cardinal different from 5.

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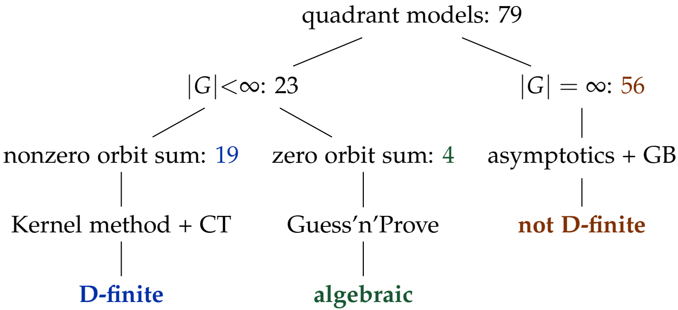
In this case, $F_{\mathfrak{S}}(t;x,y)$ is expressible using nested radicals.
If not, $F_{\mathfrak{S}}(t;x,y)$ is expressible using iterated integrals of ${}_2F_1$ expressions.

Summary: Walks with unit steps in $\mathbb{N}^2$



Extensions: Walks with unit steps in $\mathbb{N}^3$

$2^{3^3-1} \approx 67$ millions models, of which ≈ 11 million inherently 3D

- 3D octant models with ≤ 6 steps: 20804
  - $|G| < \infty$: 170
    - orbit sum $\neq 0$: 108
      - kernel method
        - **D-finite**
    - orbit sum $= 0$: 62
      - 2D-reducible: 43
        - **D-finite**
      - not 2D-reducible: 19
        - **not D-finite?**
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[B., Bousquet-Mélou, Kauers, Melczer 2015]

▷ Open question: are there non-D-finite models with a finite group?

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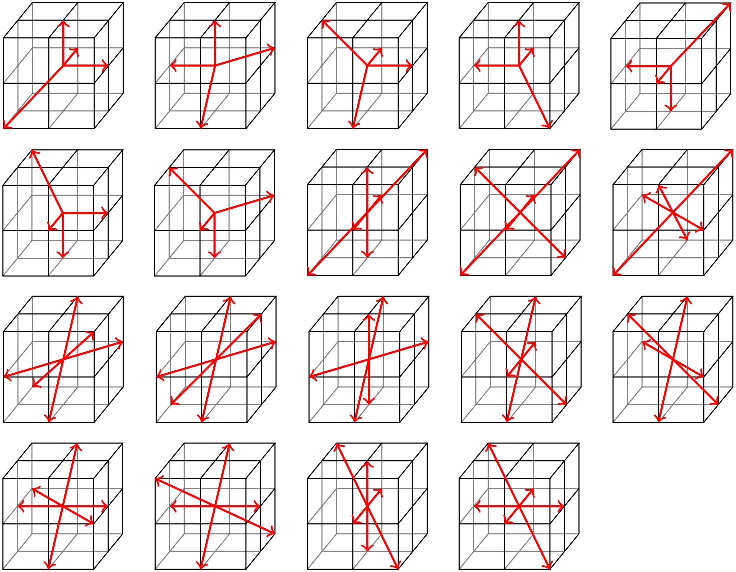
[B., Bousquet-Mélou, Kauers, Melczer 2015]

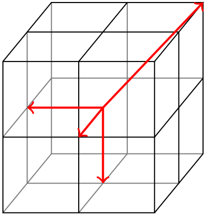
▷ Open question: are there non-D-finite models with a finite group?

▷ [Du, Hu, Wang, 2015]: proofs that groups are infinite in the 20634 cases

▷ [Bacher, Kauers, Yatchak, 2016]: extension to all 3D models; 170 models found with $|G| < \infty$ and orbit sum 0 (instead of 19)

The 19 mysterious 3D-models





Two different computations suggest:

$$k_n \approx C \cdot 256^n / n^{3.3257570041744...},$$

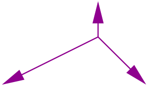
so excursions are very probably transcendental (and even non-D-finite)

Extensions: Walks in $\mathbb{N}^2$ with longer steps

- Define (and use) a group $\mathcal{G}$ for models with larger steps?

- Example: When $\mathfrak{S} = \{(0,1),(1,-1),(-2,-1)\}$, there is an underlying group that is finite and

$$xyF(t;x,y) = [x^{>0}y^{>0}]\,\frac{(x-2x^{-2})(y-(x-x^{-2})y^{-1})}{1-t(xy^{-1}+y+x^{-2}y^{-1})}$$



[B., Bousquet-Mélou & Melczer, in preparation]

▷ Current status:

- 680 models with one large step, 643 proved non D-finite, 32 of 37 have differential equations guessed.

- 5910 models with two large steps, 5754 proved non D-finite, 69 of 156 have differential equations guessed.

Conclusion

Computer algebra may solve difficult combinatorial problems

Classification of $F(t;x,y)$ fully completed for 2D small step walks

Robust algorithmic methods, based on efficient algorithms:

- Guess'n'Prove
- Creative Telescoping

Brute-force and/or use of naive algorithms = hopeless.
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Lack of "purely human" proofs for some results.

Still missing a unified proof of: finite group $\leftrightarrow$ D-finite.

Open: is $F(t;1,1)$ non-D-finite for all 56 models with infinite group?

Many open questions in dimension > 2.

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Thanks for your attention!