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Implications of Experimental Mathematics for the Philosophy of Mathematics[1]

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From the Editors

When computers were first introduced, they were much more a tool for the other sciences than for mathematics. It was many years before more than a very small subset of mathematicians used them for anything beyond word-processing. Today, however, more and more mathematicians are using computers to actively assist their mathematical research in a range of ways. In this chapter, Jonathan Borwein, one of the leaders in this trend, discusses ways that computers can be used in the development of mathematics, both to assist in the discovery of mathematical facts and to assist in the development of their proofs. He suggests that what mathematics requires is secure *knowledge that mathematical claims are true, and an understanding of why they are true, and that proofs are not necessarily the only route to this security. For teachers of mathematics, computers are a very helpful, if not essential, component of a constructivist approach to the mathematics curriculum.*

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Christopher Koch [Koch 2004] accurately captures a great scientific distaste for philosophizing:

> "Whether we scientists are inspired, bored, or infuriated by philosophy, all our theorizing and experimentation depends on particular philosophical background assumptions. This hidden influence is an acute embarrassment to many researchers, and it is therefore not often acknowledged." (Christopher Koch, 2004)

That acknowledged, I am of the opinion that mathematical philosophy matters more now than it has in nearly a century. The power of modern computers matched with that of modern mathematical software and the sophistication of current mathematics is changing the way we do mathematics.

In my view it is now both necessary and possible to admit quasi-empirical inductive methods fully into mathematical argument. In doing so carefully we will enrich mathematics and yet preserve the mathematical literature's deserved reputation for reliability—even as the methods and criteria change. What do I mean by reliability? Well, research mathematicians still consult Euler or Riemann to be informed, anatomists only consult Harvey[4] for historical reasons. Mathematicians happily quote old papers as core steps of arguments, physical scientists expect to have to confirm results with another experiment.

1 Mathematical Knowledge as I View It

Somewhat unusually, I can exactly place the day at registration that I became a mathematician and I recall the reason why. I was about to deposit my punch cards in the 'honours history bin'. I remember thinking

[3] An earlier version of this chapter was taught in this short-course based book.

[4] William Harvey published the first accurate description of circulation, "An Anatomical Study of the Motion of the Heart and of the Blood in Animals," in 1628.

> "If I do study history, in ten years I shall have forgotten how to use the calculus properly. If I take mathematics, I shall still be able to read competently about the War of 1812 or the Papal schism." (Jonathan Borwein, 1968)

The inescapable reality of objective mathematical knowledge is still with me. Nonetheless, my view then of the edifice I was entering is not that close to my view of the one I inhabit forty years later.

I also know when I became a computer-assisted fallibilist. Reading Imre Lakatos' *Proofs and Refutations*, [Lakatos 1976], a few years later while a very new faculty member, I was suddenly absolved from the grave sin of error, as I began to understand that missteps, mistakes and errors are the grist of all creative work.[5] The book, his doctorate posthumously published in 1976, is a student conversation about the Euler characteristic. The students are of various philosophical stripes and the discourse benefits from his early work on Hegel with the Stalinist Lukács in Hungary and from later study with Karl Popper at the London School of Economics. I had been prepared for this dispensation by the opportunity to learn a variety of subjects from Michael Dummett. Dummett was at that time completing his study rehabilitating Frege's status, [Dummett 1973].

A decade later the appearance of the first 'portable' computers happily coincided with my desire to decode Srinivasa Ramanujan's (1887–1920) cryptic assertions about theta functions and elliptic integrals, [Borwein et al. 1989]. I realized that by coding his formulae and my own in the *APL* programming language[6], I was able to rapidly confirm and refute identities and conjectures and to travel much more rapidly and fearlessly down potential blind alleys. I had become a computer-assisted fallibilist, at first somewhat falteringly, but twenty years have certainly honed my abilities.

Today, while I appreciate fine proofs and aim to produce them when possible, I no longer view proof as the royal road to secure mathematical knowledge.

2 Introduction

I first discuss my views, and those of others, on the nature of mathematics, and then illustrate these views in a variety of mathematical contexts. A considerably more detailed treatment of many of these topics is to be found in my book with Dave Bailey entitled *Mathematics by Experiment: Plausible Reasoning in the 21st Century*—especially in Chapters One, Two and Seven, [Borwein/Bailey 2003]. Additionally, [Bailey et al. 2007] contains several pertinent case studies as well as a version of this current chapter.

Kurt Gödel may well have overturned the mathematical apple cart entirely deductively, but nonetheless he could hold quite different ideas about legitimate forms of mathematical reasoning, [Gödel 1995]:

> "If mathematics describes an objective world just like physics, there is no reason why inductive methods should not be applied in mathematics just the same as in physics." (Kurt Gödel[7], 1951)

[5] Gila Hanna [Hanna 2006] takes a more critical view placing more emphasis on the role of proof and certainty in mathematics; I do not disagree, so much as I place more value on the role of computer-assisted refutation. Also 'certainty' usually arrives late in the development of a proof.

[6] Known as a 'write only' very high level language, APL was a fine tool, albeit with a steep learning curve whose code is almost impossible to read later.

[7] Taken from a previously unpublished work, [Gödel 1995] originally given as the 1951 Gibbs lecture.

While we mathematicians have often separated ourselves from the sciences, they have tended to be more ecumenical. For example, a recent review of *Models. The Third Dimension of Science*, [Brown 2004], chose a mathematical plaster model of a Clebsch diagonal surface as its only illustration. Similarly, authors seeking examples of the aesthetic in science often choose iconic mathematics formulae such as $E = MC^2$.

Let me begin by fixing a few concepts before starting work in earnest. Above all, I hope to persuade you of the power of mathematical experimentation—it is also fun—and that the traditional accounting of mathematical learning and research is largely an ahistorical caricature. I recall three terms.

mathematics, n. *a group of related subjects, including algebra, geometry, trigonometry and calculus, concerned with the study of number, quantity, shape, and space, and their inter-relationships, applications, generalizations and abstractions.*

This definition—taken from my Collins Dictionary [Borowski/Borwein 2006]—makes no immediate mention of proof, nor of the means of reasoning to be allowed. The Webster's Dictionary [Webster's 1999] contrasts:

induction, n. *any form of reasoning in which the conclusion, though supported by the premises, does not follow from them necessarily.*; and

deduction, n. *a process of reasoning in which a conclusion follows necessarily from the premises presented, so that the conclusion cannot be false if the premises are true.*
b. a conclusion reached by this process.

Like Gödel, I suggest that both should be entertained in mathematics. This is certainly compatible with the general view of mathematicians that in some sense "mathematical stuff is out there" to be discovered. In this paper, I shall talk broadly about experimental and heuristic mathematics, giving accessible, primarily visual and symbolic, examples.

3 Philosophy of Experimental Mathematics

> "The computer has in turn changed the very nature of mathematical experience, suggesting for the first time that mathematics, like physics, may yet become an empirical discipline, a place where things are discovered because they are seen."
>
> (David Berlinski, [Berlinski 1997], p. 39)

The shift from *typographic* to *digital culture* is vexing for mathematicians. For example, there is still no truly satisfactory way of displaying mathematics on the web—and certainly not of asking mathematical questions. Also, we respect *authority*, [Grabiner 2004], but value *authorship* deeply—however much the two values are in conflict, [Borwein/Stanway 2005]. For example, the more I recast someone else's ideas in my own words, the more I enhance my authorship while undermining the original authority of the notions. Medieval scribes had the opposite concern and so took care to attribute their ideas to such as Aristotle or Plato.

And we care more about the *reliability* of our literature than does any other science. Indeed I would argue that we have over-subscribed to this notion and often pay lip-service, not real attention, to our older literature. How often does one see original sources sprinkled like holy water in papers that make no real use of them—the references offering a false sense of scholarship?

The traditional central role of proof in mathematics is arguably and perhaps appropriately under siege. Via examples, I intend to pose and answer various questions. I shall conclude with a variety of quotations from our progenitors and even contemporaries:

My Questions. What constitutes secure mathematical knowledge? When is computation convincing? Are humans less fallible? What tools are available? What methodologies? What of the 'law of the small numbers'? Who cares for certainty? What is the role of proof? How is mathematics actually done? How should it be? I mean these questions both about the apprehension (discovery) and the establishment (proving) of mathematics. This is presumably more controversial in the formal proof phase.

My Answers. To misquote D'Arcy Thompson (1860–1948) 'form follows function', [Thompson 1992]: rigour (proof) follows reason (discovery); indeed, excessive focus on rigour has driven us away from our wellsprings. Many good ideas are wrong. Not all truths are provable, and not all provable truths are worth proving. Gödel's incompleteness results certainly showed us the first two of these assertions while the third is the bane of editors who are frequently presented with correct but unexceptional and unmotivated generalizations of results in the literature. Moreover, near certainty is often as good as it gets—intellectual context (community) matters. Recent complex human proofs are often very long, extraordinarily subtle and fraught with error—consider Fermat's last theorem, the Poincaré conjecture, the classification of finite simple groups, presumably any proof of the Riemann hypothesis, [Economist 2005]. So while we mathematicians publicly talk of certainty we really settle for security.

In all these settings, modern computational tools dramatically change the nature and scale of available evidence. Given an interesting identity buried in a long and complicated paper on an unfamiliar subject, which would give you more confidence in its correctness: staring at the proof, or confirming computationally that it is correct to 10,000 decimal places?

Here is such a formula ([Bailey/Borwein 2005], p. 20):

$$\frac{24}{7\sqrt{7}} \int_{\pi/3}^{\pi/2} \log\left|\frac{\tan t + \sqrt{7}}{\tan t - \sqrt{7}}\right| dt \stackrel{?}{=} L_{-7}(2)$$

$$= \sum_{n=0}^{\infty} \left[\frac{1}{(7n+1)^2} + \frac{1}{(7n+2)^2} - \frac{1}{(7n+3)^2} + \frac{1}{(7n+4)^2} - \frac{1}{(7n+5)^2} - \frac{1}{(7n+6)^2}\right]. \quad (1)$$

This identity links a volume (the integral) to an arithmetic quantity (the sum). It arose out of some studies in quantum field theory, in analysis of the volumes of ideal tetrahedra in hyperbolic space. The question mark is used because, while no hint of a path to a formal proof is yet known, it has been verified numerically to 20,000 digit precision—using 45 minutes on 1024 processors at Virginia Tech.

A more inductive approach can have significant benefits. For example, as there is still some doubt about the proof of the classification of finite simple groups it is important to ask whether the result is true but the proof flawed, or rather if there is still perhaps an 'ogre' sporadic group even larger than the 'monster.' What heuristic, probabilistic or computational tools can increase our confidence that the ogre does or does not exist? Likewise, there are experts who still believe the *Riemann hypothesis*[8] (RH) may be false and that the billions of zeroes found so far are much too small to be representative.[9] In any event, our understanding of the complexity of various crypto-systems relies on (RH) and we should like secure knowledge that any counter-example is enormous.

Peter Medawar (1915–87)—a Nobel prize winning oncologist and a great expositor of science—writing in *Advice to a Young Scientist*, [Medawar 1979], identifies four forms of scientific experiment:

1. *The Kantian experiment: generating "the classical non-Euclidean geometries (hyperbolic, elliptic) by replacing Euclid's axiom of parallels (or something equivalent to it) with alternative forms."* All mathematicians perform such experiments while the majority of computer explorations are of the following Baconian form.
2. *The Baconian experiment is a contrived as opposed to a natural happening, it "is the consequence of 'trying things out' or even of merely messing about."* Baconian experiments are the explorations of a happy if disorganized beachcomber and carry little predictive power.
3. *Aristotelian demonstrations: "apply electrodes to a frog's sciatic nerve, and lo, the leg kicks; always precede the presentation of the dog's dinner with the ringing of a bell, and lo, the bell alone will soon make the dog dribble."* Arguably our 'Corollaries' and 'Examples' are Aristotelian, they reinforce but do not predict. Medawar then says the most important form of experiment is:
4. *The Galilean experiment is "a critical experiment—one that discriminates between possibilities and, in doing so, either gives us confidence in the view we are taking or makes us think it in need of correction."* The Galilean is the only form of experiment which stands to make Experimental Mathematics a serious enterprise. Performing careful, replicable Galilean experiments requires work and care.

Reuben Hersh's arguments for a humanist philosophy of mathematics, especially ([Hersh 1995], pp. 590–591), and ([Hersh 1999], p. 22), as paraphrased below, become even more convincing in our highly computational setting.

1. Mathematics is human. *It is part of and fits into human culture. It does not match Frege's concept of an abstract, timeless, tenseless, objective reality.*[10]
2. Mathematical knowledge is fallible. *As in science, mathematics can advance by making mistakes and then correcting or even re-correcting them. The "fallibilism" of mathematics is brilliantly argued in Lakatos' Proofs and Refutations.*
3. There are different versions of proof or rigor. *Standards of rigor can vary depending on time, place, and other things. The use of computers in formal proofs, exemplified by the*

[8] All non-trivial zeroes—not negative even integers—of the zeta function lie on the line with real part 1/2.

[9] See [Odlyzko 2001] and various of Andrew Odlyzko's unpublished but widely circulated works.

[10] That Frege's view of mathematics is wrong, for Hersh as for me, does not diminish its historical importance.

computer-assisted proof of the four color theorem in 1977,[11] *is just one example of an emerging nontraditional standard of rigor.*

4. Empirical evidence, numerical experimentation and probabilistic proof all can help us decide what to believe in mathematics. *Aristotelian logic isn't necessarily always the best way of deciding.*
5. Mathematical objects are a special variety of a social-cultural-historical object. *Contrary to the assertions of certain post-modern detractors, mathematics cannot be dismissed as merely a new form of literature or religion. Nevertheless, many mathematical objects can be seen as shared ideas, like Moby Dick in literature, or the Immaculate Conception in religion.*

I entirely subscribe to points 2., 3., 4., and with certain caveats about objective knowledge[12] to points 1. and 5. In any event mathematics is and will remain a uniquely human undertaking.

This version of humanism sits *fairly* comfortably along-side current versions of **social-constructivism** as described next.

> "The social constructivist thesis is that mathematics is a social construction, a cultural product, fallible like any other branch of knowledge." (Paul Ernest, [Ernest 1990], §3)

But only if I qualify this with "*Yes, but much-much less fallible than most branches of knowledge.*" Associated most notably with the writings of Paul Ernest—an English Mathematician and Professor in the Philosophy of Mathematics Education who in [Ernest 1998] traces the intellectual pedigree for his thesis, a pedigree that encompasses the writings of Wittgenstein, Lakatos, Davis, and Hersh among others—social constructivism seeks to define mathematical knowledge and epistemology through the social structure and interactions of the mathematical community and society as a whole.

This interaction often takes place over very long periods. Many of the ideas our students—and some colleagues—take for granted took a great deal of time to gel. The Greeks suspected the impossibility of the three *classical construction problems*[13] and the irrationality of the golden mean was well known to the Pythagoreans.

While concerns about potential and completed infinities are very old, until the advent of the calculus with Newton and Leibnitz and the need to handle fluxions or infinitesimals, the level of need for rigour remained modest. Certainly Euclid is in its geometric domain generally a model of rigour, while also Archimedes' numerical analysis was not equalled until the 19th century.

The need for rigour arrived in full force in the time of Cauchy and Fourier. The treacherous countably infinite processes of analysis and the limitations of formal manipulation came to the fore. It is difficult with a modern sensibility to understand how Cauchy's proof of the continuity

[11] Especially since a new implementation by Seymour, Robertson and Thomas in 1997 has produced a simpler, clearer and less troubling implementation.

[12] While it is not Hersh's intention, a superficial reading of point 5. hints at a cultural relativism to which I certainly do not subscribe.

[13] Trisection, circle squaring and cube doubling were taken by the educated to be impossible in antiquity. Already in 414 BCE, in his play *The Birds*, Aristophanes uses 'circle-squarers' as a term for those who attempt the impossible. Similarly, the French Academy stopped accepting claimed proofs a full two centuries before the 19th century achieved proofs of their impossibility.

of pointwise-limits could coexist in texts for a generation with clear counter-examples originating in Fourier's theory of heat.[14]

By the end of the 19th century Frege's (1848–1925) attempt to base mathematics in a linguistically based *logicism* had foundered on Russell and other's discoveries of the paradoxes of naive set theory. Within thirty five years Gödel—and then Turing's more algorithmic treatment[15]—had similarly damaged both Russell and Whitehead's and Hilbert's programs.

Throughout the twentieth century, bolstered by the armor of abstraction, the great ship Mathematics has sailed on largely unperturbed. During the last decade of the 19th and first few decades of the 20th century the following main streams of philosophy emerged explicitly within mathematics to replace logicism, but primarily as the domain of philosophers and logicians.

- *Platonism.* Everyman's idealist philosophy—stuff exists and we must find it. Despite being the oldest mathematical philosophy, Platonism—still predominant among working mathematicians—was only christened in 1934 by Paul Bernays.[16]
- *Formalism.* Associated mostly with Hilbert—it asserts that mathematics is invented and is best viewed as formal symbolic games without intrinsic meaning.
- *Intuitionism.* Invented by Brouwer and championed by Heyting, intuitionism asks for inarguable monadic components that can be fully analyzed and has many variants; this has interesting overlaps with recent work in cognitive psychology such as Lakoff and Nunez' work, [Lakoff/Nunez 2001], on 'embodied cognition'.[17]
- *Constructivism.* Originating with Markoff and especially Kronecker (1823–1891), and refined by Bishop it finds fault with significant parts of classical mathematics. Its 'I'm from Missouri, tell me how big it is' sensibility is not to be confused with Paul Ernest's 'social constructivism', [Ernest 1998].

The last two philosophies deny the principle of the *excluded middle*, "A or not A," and resonate with computer science—as does some of formalism. It is hard after all to run a deterministic program which does not know which disjunctive logic-gate to follow. By contrast the battle between a Platonic idealism (a 'deductive absolutism') and various forms of 'fallibilism' (a quasi-empirical 'relativism') plays out across all four, but fallibilism perhaps lives most easily within a restrained version of intuitionism which looks for 'intuitive arguments' and is willing to accept that 'a proof is what convinces'. As Lakatos shows, an argument that was convincing a hundred years ago may well now be viewed as inadequate. And one today trusted may be challenged in the next century.

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[14] Cauchy's proof appeared in his 1821 text on analysis. While counterexamples were pointed out almost immediately, Stokes and Seidel were still refining the missing uniformity conditions in the late 1840s.

[15] The modern treatment of incompleteness leans heavily on Turing's analysis of the *Halting problem* for so-called Turing machines.

[16] See Karlis Podnieks, "Platonism, Intuition and the Nature on Mathematics," available at `www.ltn.lv/podnieks/gt1.html`

[17] The cognate views of Henri Poincaré (1854–1912) ([Poincaré 2004], p. 23) on the role of the *subliminal* are reflected in "The mathematical facts that are worthy of study are those that, by their analogy with other facts are susceptible of leading us to knowledge of a mathematical law, in the same way that physical facts lead us to a physical law." He also wrote "It is by logic we prove, it is by intuition that we invent," [Poincaré 1904].

As we illustrate in the next section or two, it is only perhaps in the last twenty five years, with the emergence of powerful mathematical platforms, that any approach other than a largely undigested Platonism and a reliance on proof and abstraction has had the tools[18] to give it traction with working mathematicians.

In this light, Hales' proof of Kepler's conjecture that *the densest way to stack spheres is in a pyramid* resolves the oldest problem in discrete geometry. It also supplies the most interesting recent example of intensively computer-assisted proof, and after five years with the review process was published in the *Annals of Mathematics*—with an "only 99% checked" disclaimer, withdrawn very late in the process and after being widely reported.

This process has triggered very varied reactions [Kolata 2004] and has provoked Thomas Hales to attempt a formal computational proof which he expects to complete by 2011, [Economist 2005]. Famous earlier examples of fundamentally computer-assisted proof include the *Four color theorem* and proof of the *Non-existence of a projective plane of order 10*. The three raise and answer quite distinct questions about computer-assisted proof—both real and specious. For example, there were real concerns about the completeness of the search in the 1976 proof of the Four color theorem but there should be none about the 1997 reworking by Seymour, Robertson and Thomas.[19] Correspondingly, Lam deservedly won the 1992 *Lester R. Ford award* for his compelling explanation of why to trust his computer when it announced there was no plane of order ten, [Lam 1991]. Finally, while it is reasonable to be concerned about the certainty of Hales' conclusion, was it really the *Annal's* purpose to suggest all other articles have been more than 99% certified?

To make the case as to how far mathematical computation has come we trace the changes over the past half century. The 1949 computation of π to 2,037 places suggested by von Neumann, took 70 hours. A billion digits may now be computed in much less time on a laptop. Strikingly, it would have taken roughly 100,000 ENIAC's to store the Smithsonian's picture—as is possible thanks to *40 years of Moore's law* in action.[20]

This is an astounding record of sustained exponential progress without peer in the history of technology. Additionally, mathematical tools are now being implemented on parallel platforms, providing *much* greater power to the research mathematician. Amassing huge amounts of processing power will not alone solve many mathematical problems. There are very few mathematical 'Grand-challenge problems', [JBorwein/PBorwein 2001] where, as in the physical sciences, a few more orders of computational power will resolve a problem.

For example, an order of magnitude improvement in computational power currently translates into one more day of accurate weather forecasting, while it is now common for biomedical researchers to design experiments today whose outcome is predicated on 'peta-scale' computation being available by say 2010, [Rowe et al. 2005]. There is, however, much more value in *very rapid 'Aha's'* as can be obtained through "micro-parallelism;" that is, where we benefit by being able to compute many simultaneous answers on a neurologically-rapid scale and so can hold many parts of a problem in our mind at one time.

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[18] That is, to broadly implement Hersh's central points (2.-4.).

[19] See `www.math.gatech.edu/thomas/FC/fourcolor.html`.

[20] **Moore's Law** is now taken to be the assertion that *semiconductor technology approximately doubles in capacity and performance roughly every 18 to 24 months*.

To sum up, in light of the discussion and terms above, I now describe myself a sort-of social-constructivist, and as a computer-assisted fallibilist with constructivist leanings. I believe that more-and-more of the interesting parts of mathematics will be less-and-less susceptible to classical deductive analysis and that Hersh's 'non-traditional standard of rigor' must come to the fore.

4 Our Experimental Mathodology

Despite Picasso's complaint that "computers are useless, they only give answers," the main goal of computation in pure mathematics is arguably to yield *insight*. This demands speed or, equivalently, substantial *micro-parallelism* to provide answers on a cognitively relevant scale; so that we may ask and answer more questions while they remain in our consciousness. This is relevant for rapid verification; for validation; for *proofs* and *especially for refutations* which includes what Lakatos calls "monster barring," [Lakatos 1976]. Most of this goes on in the daily small-scale accretive level of mathematical discovery but insight is gained even in cases like the proof of the Four color theorem or the Non-existence of a plane of order ten. Such insight is not found in the case-enumeration of the proof, but rather in the algorithmic reasons for believing that one has at hand a tractable unavoidable set of configurations or another effective algorithmic strategy. For instance, Lam [Lam 1991] ran his algorithms on known cases in various subtle ways, and also explained why built-in redundancy made the probability of machine-generated error negligible. More generally, the act of programming—if well performed—always leads to more insight about the structure of the problem.

In this setting it is enough to equate *parallelism* with access to requisite *more* space and speed of computation. Also, we should be willing to consider all computations as 'exact' which provide truly reliable answers.[21] This now usually requires a careful *hybrid* of symbolic and numeric methods, such as achieved by *Maple*'s liaison with the *Numerical Algorithms Group* (NAG) Library[22], see [Bornemann et al. 2004], [Borwein 2005b]. There are now excellent tools for such purposes throughout analysis, algebra, geometry and topology, see [Borwein/Bailey 2003], [Borwein et al. 2004], [Bornemann et al. 2004], [JBorwein/PBorwein 2001], [Borwein/Corless 1999].

Along the way questions required by—or just made natural by—computing start to force out older questions and possibilities in the way beautifully described a century ago by Dewey regarding evolution.

> "Old ideas give way slowly; for they are more than abstract logical forms and categories. They are habits, predispositions, deeply engrained attitudes of aversion and preference. Moreover, the conviction persists—though history shows it to be a hallucination—that all the questions that the human mind has asked are questions that can be answered in terms of the alternatives that the questions themselves present. But in fact intellectual progress usually occurs through sheer abandonment of questions together with both of the alternatives they assume; an abandonment that results from their decreasing vitality and a change of urgent interest. We do not solve them: we get over them. Old questions

[21] If careful interval analysis can certify that a number known to be integer is larger than 2.5 and less than 3.5, this constitutes an exact computational proof that it is 3.

[22] See `www.nag.co.uk/`.

are solved by disappearing, evaporating, while new questions corresponding to the changed attitude of endeavor and preference take their place. Doubtless the greatest dissolvent in contemporary thought of old questions, the greatest precipitant of new methods, new intentions, new problems, is the one effected by the scientific revolution that found its climax in the 'Origin of Species.'" (John Dewey, [Dewey 1997])

Lest one think this a feature of the humanities and the human sciences, consider the artisanal chemical processes that have been lost as they were replaced by cheaper industrial versions. And mathematics is far from immune. Felix Klein, quoted at length in the introduction to [JBorwein/PBorwein 1987], laments that "now the younger generation hardly knows abelian functions." He goes on to explain that:

"In mathematics as in the other sciences, the same processes can be observed again and again. First, new questions arise, for internal or external reasons, and draw researchers away from the old questions. And the old questions, just because they have been worked on so much, need ever more comprehensive study for their mastery. This is unpleasant, and so one is glad to turn to problems that have been less developed and therefore require less foreknowledge—even if it is only a matter of axiomatics, or set theory, or some such thing." (Felix Klein, [Klein 1928], p. 294)

Freeman Dyson has likewise gracefully described how taste changes:

"I see some parallels between the shifts of fashion in mathematics and in music. In music, the popular new styles of jazz and rock became fashionable a little earlier than the new mathematical styles of chaos and complexity theory. Jazz and rock were long despised by classical musicians, but have emerged as art-forms more accessible than classical music to a wide section of the public. Jazz and rock are no longer to be despised as passing fads. Neither are chaos and complexity theory. But still, classical music and classical mathematics are not dead. Mozart lives, and so does Euler. When the wheel of fashion turns once more, quantum mechanics and hard analysis will once again be in style." (Freeman Dyson, [Dyson 1996])

For example recursively defined objects were once anathema—Ramanujan worked very hard to replace lovely iterations by sometimes-obscure closed-form approximations. Additionally, what is "easy" changes: high performance computing and networking are blurring, merging disciplines and collaborators. This is democratizing mathematics but further challenging authentication—consider how easy it is to find information on *Wikipedia*[23] and how hard it is to validate it.

Moving towards a well articulated Experimental *Mathodology*—both in theory and practice—will take much effort. The need is premised on the assertions that intuition is acquired—we can and must better mesh computation and mathematics, and that visualization is of growing importance—in many settings even three is a lot of dimensions.

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[23] *Wikipedia* is an open source project at `en.wikipedia.org/wiki/Main_Page`; "wiki-wiki" is Hawaiian for "quickly."

"Monster-barring" (Lakatos's term, [Lakatos 1976], for refining hypotheses to rule out nasty counter-examples[24]) and "caging" (Nathalie Sinclair tells me this is my own term for imposing needed restrictions in a conjecture) are often easy to enhance computationally, as for example with randomized checks of equations, linear algebra, and primality or graphic checks of equalities, inequalities, areas, etc. Moreover, our mathodology fits well with the kind of pedagogy espoused at a more elementary level (and without the computer) by John Mason in [Mason 2006].

4.1 Eight Roles for Computation

I next recapitulate eight roles for computation that Bailey and I discuss in our two recent books [Borwein/Bailey 2003], [Borwein et al. 2004]:

#1. Gaining insight and intuition or just knowledge. Working algorithmically with mathematical objects almost inevitably adds insight to the processes one is studying. At some point even just the careful aggregation of data leads to better understanding.

#2. Discovering new facts, patterns and relationships. The number of *additive partitions* of a positive integer n, $p(n)$, is *generated* by

$$P(q) := 1 + \sum_{n\geq 1} p(n)q^n = \frac{1}{\prod_{n=1}^{\infty}(1-q^n)}. \tag{2}$$

Thus, $p(5) = 7$ since

$$5 = 4+1 = 3+2 = 3+1+1 = 2+2+1 = 2+1+1+1 = 1+1+1+1+1.$$

Developing (2) is a fine introduction to enumeration via *generating functions*. Additive partitions are harder to handle than multiplicative factorizations, but they are very interesting ([Borwein et al. 2004], Chapter 4). Ramanujan used Major MacMahon's table of $p(n)$ to intuit remarkable deep congruences such as

$$p(\mathbf{5n+4}) \equiv 0 \mod \mathbf{5}, \quad p(\mathbf{7n+5}) \equiv 0 \mod \mathbf{7}, \quad p(\mathbf{11n+6}) \equiv 0 \mod \mathbf{11},$$

from relatively limited data like

$$\begin{aligned} P(q) = {} & 1 + q + 2\,q^2 + 3\,q^3 + \underline{5}\,q^{\mathbf{4}} + \overline{7}\,q^{\mathbf{5}} + 11\,q^6 + 15\,q^7 \\ & + 22\,q^8 + \underline{30}\,q^{\mathbf{9}} + 42\,q^{10} + 56\,q^{11} + \overline{77}\,q^{\mathbf{12}} + 101\,q^{13} + \underline{135}\,q^{\mathbf{14}} \\ & + 176\,q^{15} + 231\,q^{16} + 297\,q^{17} + 385\,q^{18} + \underline{\overline{490}}\,q^{\mathbf{19}} \\ & + 627\,q^{20}b + 792\,q^{21} + 1002\,q^{22} + \cdots + p(200)q^{200} + \cdots \end{aligned} \tag{3}$$

Cases $5n+4$ and $7n+5$ are flagged in (3). Of course, it is markedly easier to (heuristically) confirm than find these fine examples of *Mathematics: the science of patterns*.[25] The study of such congruences—much assisted by symbolic computation—is very active today.

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[24] Is, for example, a polyhedron always convex? Is a curve intended to be simple? Is a topology assumed Hausdorff, a group commutative?

[25] The title of Keith Devlin's 1996 book, [Devlin 1996].

#3. Graphing to expose mathematical facts, structures or principles. Consider Nick Trefethen's fourth challenge problem as described in [Bornemann et al. 2004], [Borwein 2005b]. It requires one to find ten good digits of:

4. What is the global minimum of the function

$$\exp(\sin(50x)) + \sin(60e^y) + \sin(70 \sin x) + \sin(\sin(80y))$$
$$- \sin(10(x + y)) + (x^2 + y^2)/4?$$

As a foretaste of future graphic tools, one can solve this problem graphically and interactively using current *adaptive 3-D plotting* routines which can catch all the bumps. This does admittedly rely on trusting a good deal of software.

#4. Rigourously testing and especially falsifying conjectures. I hew to the Popperian scientific view that we primarily falsify; but that as we perform more and more testing experiments without such falsification we draw closer to firm belief in the truth of a conjecture such as: *the polynomial* $P(n) = n^2 - n + p$ *has prime values for all* $n = 0, 1, \ldots, p - 2$, *exactly for* Euler's lucky prime numbers, *that is,* $p = 2, 3, 5, 11, 17,$ *and* 41.[26]

#5. Exploring a possible result to see if it *merits* formal proof. A conventional deductive approach to a hard multi-step problem really requires establishing all the subordinate lemmas and propositions needed along the way—especially if they are highly technical and un-intuitive. Now some may be independently interesting or useful, but many are only worth proving if the entire expedition pans out. Computational experimental mathematics provides tools to survey the landscape with little risk of error: only if the view from the summit is worthwhile, does one lay out the route carefully. I discuss this further at the end of the next Section.

#6. Suggesting approaches for formal proof. The proof of the *cubic theta function identity* discussed in ([Borwein et al. 2004], p. 210ff), shows how a fully intelligible human proof can be obtained entirely by careful symbolic computation.

#7. Computing replacing lengthy hand derivations. Who would wish to verify the following prime factorization by hand?

$$\begin{aligned}&6422607578676942838792549775208734746307\\ =\ &(2140992015395526641)(1963506722254397)(1527791).\end{aligned}$$

Surely, what we value is understanding the underlying algorithm, not the human work?

#8. Confirming analytically derived results. This is a wonderful and frequently accessible way of confirming results. Even if the result itself is not computationally checkable, there is often an accessible corollary. An assertion about bounded operators on Hilbert space may have a useful consequence for three-by-three matrices. It is also an excellent way to error correct, or to check calculus examples before giving a class.

5 Finding Things versus Proving Things

I now illuminate these eight roles with eight mathematical examples. At the end of each I note some of the roles illustrated.

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[26] See [Weisstein WWW] for the answer.

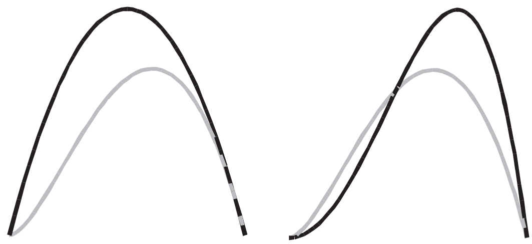

Figure 2.1 **(Ex. 1.): Graphical comparison of** $-x^2 \ln(x)$ (lower local maximum in both graphs) with $x - x^2$ (left graph) and $x^2 - x^4$ (right graph)

1. **Pictorial comparison** of $y - y^2$ and $y^2 - y^4$ to $-y^2 \ln(y)$, when y lies in the unit interval, is a much more rapid way to divine which function is larger than by using traditional analytic methods.

   Figure 2.1 below shows that it is clear in the latter case that the functions cross, and so it is futile to try to prove one majorizes the other. In the first case, evidence is provided to motivate attempting a proof and often the picture serves to guide such a proof—by showing monotonicity or convexity or some other salient property. ■

This certainly illustrates roles #3 and #4, and perhaps role #5.

2. **A proof and a disproof.** Any modern computer algebra can tell one that

$$0 < \int_0^1 \frac{(1-x)^4 x^4}{1+x^2}\, dx = \frac{22}{7} - \pi, \tag{4}$$

   since the integral may be interpreted as the area under a positive curve. We are however no wiser as to why! If however we ask the same system to compute the indefinite integral, we are likely to be told that

$$\int_0^t \cdot = \frac{1}{7} t^7 - \frac{2}{3} t^6 + t^5 - \frac{4}{3} t^3 + 4\,t - 4 \arctan(t)\,.$$

   Then (4) is now rigourously established by differentiation and an appeal to the Fundamental theorem of calculus. ■

This illustrates roles #1 and #6. It also falsifies the bad conjecture that $\pi = 22/7$ and so illustrates #4 again. Finally, the computer's proof is easier (#7) and very nice, though probably it is not the one we would have developed by ourselves. The fact that $22/7$ is a continued fraction approximation to π has led to many hunts for generalizations of (4), see [Borwein et al. 2004], Chapter 1. None so far are entirely successful.

3. **A computer discovery and a 'proof' of the series for** $\arcsin^2(x)$. We compute a few coefficients and observe that there is a regular power of 4 in the numerator, and integers in the denominator; or equivalently we look at $\arcsin(x/2)^2$. The generating function package 'gfun' in *Maple*, then predicts a recursion, r, for the denominators and solves it, as R.

```
>with(gfun):
>s:=[seq(1/coeff(series(arcsin(x/2)^2,x,25),x,2*n),n=1..6)]:
>R:=unapply(rsolve(op(1, listtorec(s,r(m))),r(m)),m);[seq(R(m),m=0..8)];
```

yields, s := [4, 48, 360, 2240, 12600, 66528],

$$R := m \mapsto 8\,\frac{4^m\,\Gamma(3/2+m)(m+1)}{\pi^{1/2}\Gamma(1+m)},$$

where Γ is the Gamma function, and then returns the sequence of values

$$[4, 48, 360, 2240, 12600, 66528, 336336, 1647360, 7876440].$$

We may now use Sloane's *Online Encyclopedia of Integer Sequences*[27] to reveal that the coefficients are $R(n) = 2n^2\binom{2n}{n}$. More precisely, sequence A002544 identifies $R(n+1)/4 = \binom{2n+1}{n}(n+1)^2$.

```
> [seq(2*n^2*binomial(2*n,n),n=1..8)];
```

confirms this with

$$[4, 48, 360, 2240, 12600, 66528, 336336, 1647360].$$

Next we write

```
> S:=Sum((2*x)^(2*n)/(2*n^2*binomial(2*n,n)),n=1..infinity):S=values(S);
```

which returns

$$\frac{1}{2}\sum_{n=1}^{\infty}\frac{(2\,x)^{2n}}{n^2\binom{2n}{n}} = \arcsin^2(x).$$

That is, we have discovered—and proven if we trust or verify *Maple*'s summation algorithm—the desired Maclaurin series.

As prefigured by Ramanujan, it transpires that there is a beautiful closed form for $\arcsin^{2m}(x)$ for all $m = 1, 2, \ldots$. In [Borwein/Chamberland 2007] there is a discussion of the use of *integer relation methods*, [Borwein/Bailey 2003], Chapter 6, to find this closed form and associated proofs are presented. ■

Here we see an admixture of all of the roles save #3, but above all #2 and #5.

4. Discovery without proof. Donald Knuth[28] asked for a closed form evaluation of:

$$\sum_{k=1}^{\infty}\left\{\frac{k^k}{k!\,e^k} - \frac{1}{\sqrt{2\,\pi\,k}}\right\} = -0.084069508727655\ \ldots. \tag{5}$$

Since about 2000 CE it has been easy to compute 20—or 200—digits of this sum in *Maple* or *Mathematica*; and then to use the 'smart lookup' facility in the *Inverse Symbolic Calculator*(ISC). The ISC at `oldweb.cecm.sfu.ca/projects/ISC` uses a variety of search algorithms and heuristics to predict what a number might actually be. Similar ideas are now implemented as 'identify' in *Maple* and (for algebraic numbers only) as 'Recognize' in *Mathematica*, and are described in [Borwein 2005b], [Borwein/Bailey 2003],

[27] At `www.research.att.com/~njas/sequences/index.html`.

[28] Posed as an MAA Problem [Knuth 2002].

[Borwein/Corless 1999], [Bailey/Borwein 2000]. In this case it *rapidly* returns

$$0.084069508727655 \approx \frac{2}{3} + \frac{\zeta(1/2)}{\sqrt{2\pi}}.$$

We thus have a prediction which *Maple* 9.5 on a 2004 laptop *confirms* to 100 places in under 6 seconds and to 500 in 40 seconds. Arguably we are done. After all we were asked to *evaluate* the series and we now know a closed-form answer.

Notice also that the 'divergent' $\zeta(1/2)$ term is formally to be expected in that while $\sum_{n=1}^{\infty} 1/n^{1/2} = \infty$, the *analytic continuation* of $\zeta(s) := \sum_{n=1}^{\infty} 1/n^s$ for $s > 1$ evaluated at $1/2$ does occur! ■

We have discovered and tested the result and in so doing gained insight and knowledge while illustrating roles #1, #2 and #4. Moreover, as described in [Borwein et al. 2004], p. 15, one can also be led by the computer to a very satisfactory computer-assisted but also very human proof, thus illustrating role #6. Indeed, the first hint is that the computer algebra system returned the value in (5) very quickly even though the series is very slowly convergent. This suggests the program is doing something intelligent—and it is! Such a use of computing is termed "instrumental" in that the computer is fundamental to the process, see [Lagrange 2005].

5. A striking conjecture with no known proof strategy (as of spring 2007)[29] given in [Borwein et al. 2004], p. 162, is: for $n = 1, 2, 3 \ldots$

$$8^n \, \zeta\left(\{\overline{2}, 1\}_n\right) \stackrel{?}{=} \zeta\left(\{2, 1\}_n\right). \tag{6}$$

Explicitly, the first two cases are

$$8 \sum_{n>m>0} \frac{(-1)^n}{n^2 m} = \sum_{n>0} \frac{1}{n^3} \qquad \text{and} \qquad 64 \sum_{n>m>o>p>0} \frac{(-1)^{n+o}}{n^2 m \, o^2 p} = \sum_{n>m>0} \frac{1}{n^3 m^3}.$$

The notation should now be clear—we use the 'overbar' to denote an alternation. Such alternating sums are called *multi-zeta values* (MZV) and positive ones are called *Euler sums* after Euler who first studied them seriously. They arise naturally in a variety of modern fields from combinatorics to mathematical physics and knot theory.

There is abundant evidence amassed since 'identity' (6) was found in 1996. For example, very recently Petr Lisonek checked the first 85 cases to 1000 places in about 41 HP hours with only the *predicted round-off error*. And the case $n = 163$ was checked in about ten hours. These objects are very hard to compute naively and require substantial computation as a precursor to their analysis.

Formula (6) is the *only* identification of its type of an Euler sum with a distinct MZV and we have no idea why it is true. Any similar MZV proof has been both highly non-trivial and illuminating. To illustrate how far we are from proof: can just the case $n = 2$ be proven *symbolically* as has been the case for $n = 1$? ■

This identity was discovered by the British quantum field theorist David Broadhurst and me during a large hunt for such objects in the mid-nineties. In this process we discovered and proved many lovely results (see [Borwein/Bailey 2003], Chapter 2, and [Borwein et al. 2004], Chapter 4), thereby illustrating #1,#2, #4, #5 and #7. In the case of 'identity' (6) we have failed with

[29] A quite subtle proof has now been found by Zhao and is described in the second edition of [Borwein/Bailey 2003].

 https://doi.org/10.5948/UPO9781614445050.004

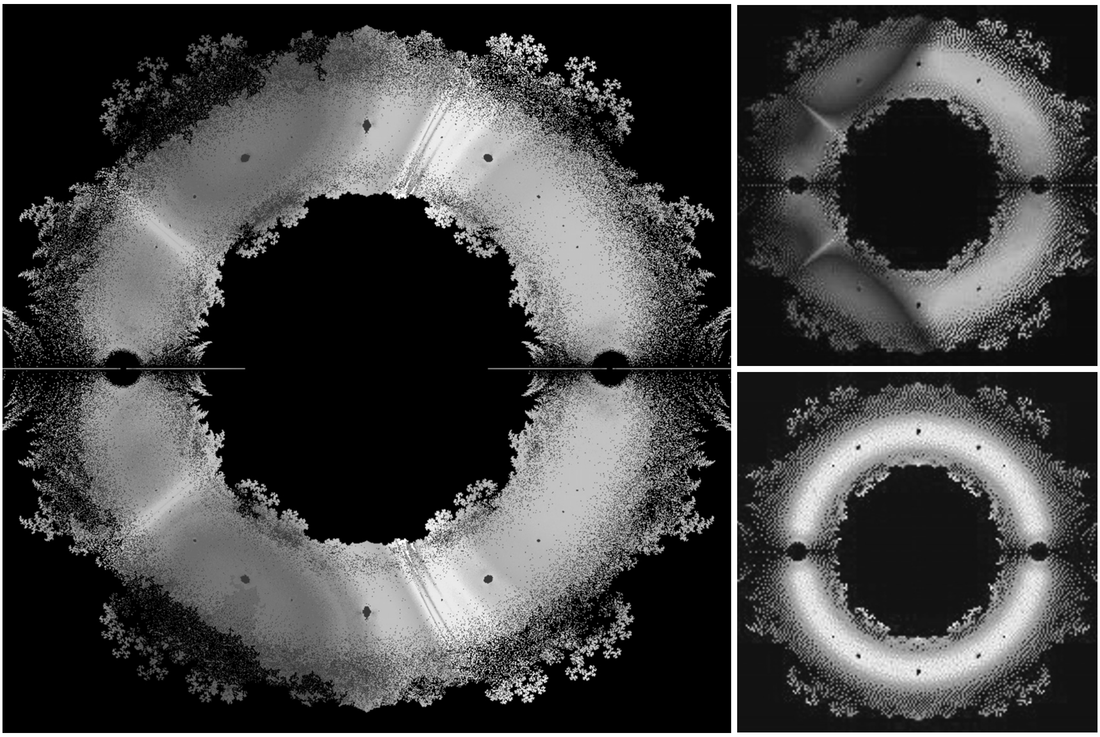

Figure 2.2 **(Ex. 6.): "The price of metaphor is eternal vigilance."** (Arturo Rosenblueth & Norbert Wiener, [Lewontin 2001])

#6, but we have ruled out many sterile approaches. It is one of many examples where we can now have (near) certainty without proof. Another was shown in equation (1) above.

6. **What you draw *is* what you see.** *Roots of polynomials with coefficients 1 or* -1 *up to degree 18.*

   As the quote suggests, pictures are highly metaphorical. The shading in Figure 2.2 is determined by a normalized sensitivity of the coefficients of the polynomials to slight variations around the values of the zeros with red indicating low sensitivity and violet indicating high sensitivity.[30] It is hard to see how the structure revealed in the pictures above[31] would be seen other than through graphically data-mining. Note the different shapes—now proven by P. Borwein and colleagues—of the holes around the various roots of unity.

   The striations are unexplained but all re-computations expose them! And the fractal structure is provably there. Nonetheless different ways of measuring the stability of the calculations reveal somewhat different features. This is very much analogous to a chemist discovering an unexplained but robust spectral line. ■

This certainly illustrates #2 and #7, but also #1 and #3.

[30] Colour versions may be seen at `oldweb.cecm.sfu.ca/personal/loki/Projects/Roots/Book/` and on the cover of this book.

[31] We plot all complex zeroes of polynomials with only -1 and 1 as coefficients up to a given degree. As the degree increases some of the holes fill in—at different rates.

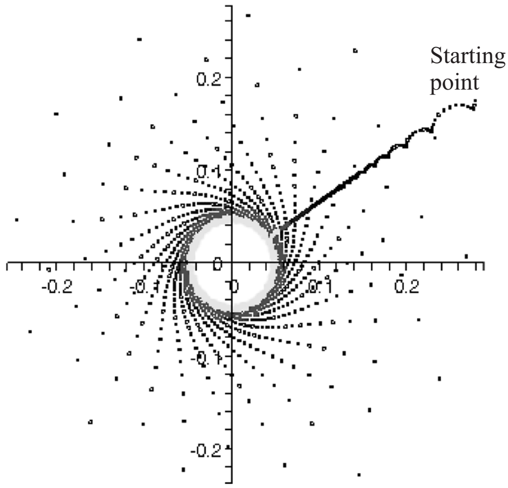

Figure 2.3 **(Ex. 7.): "Visual convergence in the complex plane"**

7. Visual Dynamics. In recent continued fraction work, Crandall and I needed to study the *dynamical system* $t_0 := t_1 := 1$:

$$t_n := \frac{1}{n}\, t_{n-1} + \omega_{n-1}\left(1 - \frac{1}{n}\right) t_{n-2},$$

where $\omega_n = a^2, b^2$ for n even, odd respectively, are two unit vectors. Think of this as a **black box** which we wish to examine scientifically. Numerically, all one *sees* is $t_n \to 0$ slowly. Pictorially, in Figure 2.3, we *learn* significantly more.[32] If the iterates are plotted with colour changing after every few hundred iterates,[33] it is clear that they spiral roman-candle like in to the origin:

Scaling by $\sqrt{n}$, and distinguishing even and odd iterates, *fine structure* appears in Figure 2.4. We now observe, predict and validate that the outcomes depend on whether or not one or both of a and b are roots of unity (that is, rational multiples of π). Input a pth root of unity and out come p spirals, input a non-root of unity and we see a circle. ■

This forceably illustrates role #2 but also roles #1, #3, #4. It took my coauthors and me, over a year and 100 pages to convert this intuition into a rigorous formal proof, [Bailey/Borwein 2005]. Indeed, the results are technical and delicate enough that I have more faith in the facts than in the finished argument. In this sentiment, I am not entirely alone.

Carl Friedrich Gauss, who drew (carefully) and computed a great deal, is said to have noted, *I **have** the result, but I do not yet know how to get it.*[34] An excited young Gauss writes: "A new field of analysis has appeared to us, self-evidently, in the study of functions etc." (October 1798,

[32] . . . "Then felt I like a watcher of the skies, when a new planet swims into his ken." From John Keats (1795–1821) poem *On first looking into Chapman's Homer*.

[33] A colour version may be seen on the cover of [Bailey et al. 2007].

[34] Like so many attributions, the quote has so far escaped exact isolation!

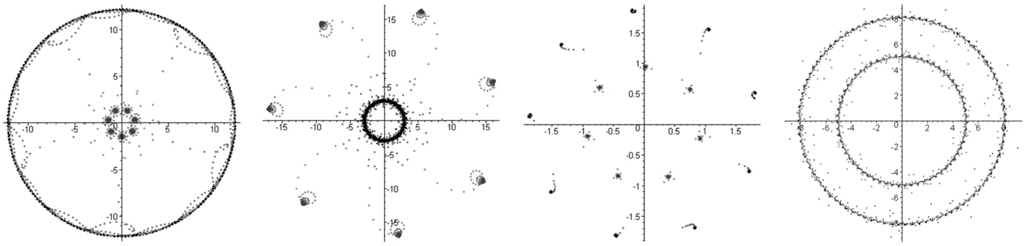

Figure 2.4 **(Ex. 7.): The attractors for various** $|a| = |b| = 1$

reproduced in [Borwein/Bailey 2003], Fig. 1.2, p.15). It had and the consequent proofs pried open the doors of much modern elliptic function and number theory.

My penultimate and more comprehensive example is more sophisticated and I beg the less-expert analyst's indulgence. Please consider its structure and not the details.

8. A full run. Consider the *unsolved* **Problem 10738** from the 1999 American Mathematical Monthly, [Borwein et al. 2004]:

Problem: For $t > 0$ let

$$m_n(t) = \sum_{k=0}^{\infty} k^n \exp(-t)\,\frac{t^k}{k!}$$

be the nth moment of a *Poisson distribution* with parameter t. Let $c_n(t) = m_n(t)/n!$. Show

a) $\{m_n(t)\}_{n=0}^{\infty}$ is log-convex[35] for all $t > 0$.
b) $\{c_n(t)\}_{n=0}^{\infty}$ is not log-concave for $t < 1$.
c*) $\{c_n(t)\}_{n=0}^{\infty}$ is log-concave for $t \geq 1$.

Solution. (a) Neglecting the factor of $\exp(-t)$ as we may, this reduces to

$$\sum_{k,j\geq 0} \frac{(jk)^{n+1} t^{k+j}}{k!\,j!} \leq \sum_{k,j\geq 0} \frac{(jk)^n t^{k+j}}{k!\,j!}\,k^2 = \sum_{k,j\geq 0} \frac{(jk)^n t^{k+j}}{k!\,j!}\,\frac{k^2+j^2}{2},$$

and this now follows from $2jk \leq k^2 + j^2$.
(b) As

$$m_{n+1}(t) = t\sum_{k=0}^{\infty}(k+1)^n \exp(-t)\,\frac{t^k}{k!},$$

on applying the binomial theorem to $(k+1)^n$, we see that $m_n(t)$ satisfies the recurrence

$$m_{n+1}(t) = t\sum_{k=0}^{n}\binom{n}{k} m_k(t), \qquad m_0(t) = 1.$$

In particular for $t = 1$, we computationally obtain as many terms of the sequence

$$1, 1, 2, 5, 15, 52, 203, 877, 4140 \ldots$$

[35] A sequence $\{a_n\}$ is *log-convex* if $a_{n+1}a_{n-1} \geq a_n^2$, for $n \geq 1$ and log-concave when the inequality is reversed.

as we wish. These are the *Bell numbers* as was discovered again by consulting *Sloane's Encyclopedia* which can also tell us that, for $t = 2$, we have the *generalized Bell numbers*, and gives the exponential generating functions.[36] Inter alia, an explicit computation shows that

$$t\,\frac{1+t}{2} = c_0(t)\,c_2(t) \leq c_1(t)^2 = t^2$$

exactly if $t \geq 1$, which completes (b).

Also, preparatory to the next part, a simple calculation shows that

$$\sum_{n\geq 0} c_n u^n = \exp\left(t(e^u - 1)\right). \tag{7}$$

(c*)[37] We appeal to a recent theorem, [Borwein et al. 2004], p. 42, due to E. Rodney Canfield which proves the lovely and quite difficult result below. A self-contained proof would be very fine.

Theorem 1: *If a sequence* $1, b_1, b_2, \ldots$ *is non-negative and log-concave then so is the sequence* $1, c_1, c_2, \ldots$ *determined by the generating function equation*

$$\sum_{n\geq 0} c_n u^n = \exp\left(\sum_{j\geq 1} b_j \frac{u^j}{j}\right).$$

Using equation (7) above, we apply this to the sequence $\mathbf{b_j} = \mathbf{t}/(\mathbf{j}-\mathbf{1})!$ which is log-concave exactly for $t \geq 1$. ■

A search in 2001 on *MathSciNet* for "Bell numbers" since 1995 turned up 18 items. Canfield's paper showed up as number 10. Later, *Google* found it immediately!

Quite unusually, the given solution to (c) was the only one received by the Monthly. The reason might well be that it relied on the following sequence of steps:

A **(Question Posed)** ⇒ Computer Algebra System ⇒ Interface ⇒
Search Engine ⇒ Digital Library ⇒ Hard New Paper ⇒ **(Answer)**

Without going into detail, we have visited most of the points elaborated in Section 4.1. Now if only we could already automate this process!

Jacques Hadamard, describes the role of proof as well as anyone—and most persuasively given that his 1896 proof of the Prime number theorem is an inarguable apex of rigorous analysis.

> "The object of mathematical rigor is to sanction and legitimize the conquests of intuition, and there was never any other object for it." (Jacques Hadamard[38])

Of the eight uses of computers instanced above, let me reiterate the central importance of heuristic methods for determining what is true and whether it merits proof. I tentatively offer the

[36] Bell numbers were known earlier to Ramanujan—an example of *Stigler's Law of Eponymy*, [Borwein et al. 2004], p. 60. Combinatorially they count the number of nonempty subsets of a finite set.

[37] The '*' indicates this was the unsolved component.

[38] J. Hadamard, in E. Borel, Lecons sur la theorie des fonctions, 3rd ed. 1928, quoted in ([Polya 1981](2), p. 127). See also [Poincaré 2004].

following surprising example which is very very likely to be true, offers no suggestion of a proof and indeed may have no reasonable proof.

9. **Conjecture.** *Consider*

$$x_n = \left\{16x_{n-1} + \frac{120n^2 - 89n + 16}{512n^4 - 1024n^3 + 712n^2 - 206n + 21}\right\}. \tag{8}$$

The sequence $\beta_n = (\lfloor 16x_n \rfloor)$, *where* (x_n) *is the sequence of iterates defined in equation (8), precisely generates the hexadecimal expansion of* $\pi - 3$.

(Here $\{\cdot\}$ denotes the fractional part and $(\lfloor\cdot\rfloor)$ denotes the integer part.) In fact, we know from [Borwein/Bailey 2003], Chapter 4, that the first million iterates are correct and in consequence:

$$\sum_{n=1}^{\infty} \|x_n - \{16^n \pi\}\| \leq 1.46 \times 10^{-8} \ldots . \tag{9}$$

where $\|a\| = \min(a, 1-a)$. By the first Borel-Cantelli lemma this shows that the hexadecimal expansion of π only finitely differs from (β_n). Heuristically, the probability of any error is very low. ■

6 Conclusions

To summarize, I do argue that reimposing the primacy of mathematical knowledge over proof is appropriate. So I return to the matter of what it takes to persuade an individual to adopt new methods and drop time honoured ones. Aptly, we may start by consulting Kuhn on the matter of paradigm shift:

> "The issue of paradigm choice can never be unequivocally settled by logic and experiment alone. . . . in these matters neither proof nor error is at issue. The transfer of allegiance from paradigm to paradigm is a conversion experience that cannot be forced."
> (Thomas Kuhn[39])

As we have seen, the pragmatist philosopher John Dewey eloquently agrees, while Max Planck, [Planck 1949], has also famously remarked on the difficulty of such paradigm shifts. This is Kuhn's version[40]:

> "And Max Planck, surveying his own career in his Scientific Autobiography, sadly remarked that 'a new scientific truth does not triumph by convincing its opponents and making them see the light, but rather because its opponents eventually die, and a new generation grows up that is familiar with it.'"
> (Albert Einstein, [Kuhn 1996], [Planck 1949])

This transition is certainly already apparent. It is certainly rarer to find a mathematician under thirty who is unfamiliar with at least one of *Maple*, *Mathematica* or *MatLab*, than it is to one

[39] In [Regis 1988], *Who Got Einstein's Office?* The answer is Arne Beurling.

[40] Kuhn is quoting Einstein quoting Planck. There are various renderings of this second-hand German quotation.

over sixty five who is really fluent. As such fluency becomes ubiquitous, I expect a re-balancing of our community's valuing of deductive proof over inductive knowledge.

In his famous lecture to the Paris International Congress in 1900, Hilbert writes[41]

> "Moreover a mathematical problem should be difficult in order to entice us, yet not completely inaccessible, lest it mock our efforts. It should be to us a guidepost on the mazy path to hidden truths, and ultimately a reminder of our pleasure in the successful solution." (David Hilbert, [Yandell 2002])

Note the primacy given by a most exacting researcher to discovery and to truth over proof and rigor. More controversially and most of a century later, Greg Chaitin invites us to be bolder and act more like physicists.

> "I believe that elementary number theory and the rest of mathematics should be pursued more in the spirit of experimental science, and that you should be willing to adopt new principles. . . . *And the Riemann Hypothesis isn't self-evident either, but it's very useful*. A physicist would say that there is ample experimental evidence for the Riemann Hypothesis and would go ahead and take it as a working assumption. . . . We may want to introduce it formally into our mathematical system."
> (Greg Chaitin, [Borwein/Bailey 2003], p. 254)

Ten years later:

> "[Chaitin's] "Opinion" article proposes that the Riemann hypothesis (RH) be adopted as a new axiom for mathematics. Normally one could only countenance such a suggestion if one were assured that the RH was undecidable. However, a proof of undecidability is a logical impossibility in this case, since if RH is false it is provably false. Thus, the author contends, one may either wait for a proof, or disproof, of RH—both of which could be impossible—or one may take the bull by the horns and accept the RH as an axiom. He prefers this latter course as the more positive one." (Roger Heath Brown[42])

Much as I admire the challenge of Greg Chaitin's statements, I am not yet convinced that it is helpful to add axioms as opposed to proving conditional results that start "Assuming the continuum hypothesis" or emphasize that "without assuming the Riemann hypothesis we are able to show. . . . " Most important is that we lay our cards on the table. We should explicitly and honestly indicate when we believe our tools to be heuristic, we should carefully indicate why we have confidence in our computations—and where our uncertainty lies—and the like.

On that note, Hardy is supposed to have commented—somewhat dismissively—that Landau, a great German number theorist, would never be the first to prove the Riemann Hypothesis, but that if someone else did so then Landau would have the best possible proof shortly after. I certainly hope that a more experimental methodology will better value independent replication and honour

[41] See the late Ben Yandell's fine account of the twenty-three "*Mathematische Probleme*" lecture, Hilbert Problems and their solvers, [Yandell 2002]. The written lecture (given in [Yandell 2002]) is considerably longer and further ranging that the one delivered in person.

[42] Roger Heath-Brown's *Mathematical Review* of [Chaitin 2004], 2004.

the first transparent proof[43] of Fermat's last theorem as much as Andrew Wiles' monumental proof. Hardy also commented that he did his best work past forty. Inductive, accretive, tool-assisted mathematics certainly allows brilliance to be supplemented by experience and—as in my case—stands to further undermine the notion that one necessarily does one's best mathematics young.

6.1 As for Education

The main consequence for me is that a *constructivist educational curriculum*—supported by both good technology and reliable content—is both possible and highly desirable. In a traditional instructivist mathematics classroom there are few opportunities for realistic discovery. The current sophistication of dynamic geometry software such as *Geometer's Sketchpad*, *Cabri* or *Cinderella*, of many fine web-interfaces, and of broad mathematical computation platforms like *Maple* and *Mathematica* has changed this greatly—though in my opinion both *Maple* and *Mathematica* are unsuitable until late in high-school, as they presume too much of both the student and the teacher. A thoughtful and detailed discussion of many of the central issues can be found in J.P. Lagrange's article [Lagrange 2005] on teaching functions in such a milieu.

Another important lesson is that we need to teach procedural or *algorithmic thinking*. Although some vague notion of a computer program as a repeated procedure is probably ubiquitous today, this does not carry much water in practice. For example, five years or so ago, while teaching future elementary school teachers (in their final year), I introduced only one topic not in the text: extraction of roots by Newton's method. I taught this in class, tested it on an assignment and repeated it during the review period. About half of the students participated in both sessions. On the final exam, I asked the students to compute $\sqrt{3}$ using Newton's method starting at $x_0 = 3$ to estimate $\sqrt{3} = \underline{1.732}050808\ldots$ so that the first three digits after the decimal point were correct. I hoped to see $x_1 = 2, x_2 = 7/4$ and $x_3 = 97/56 = \underline{1.732}142857\ldots$. I gave the students the exact iteration in the form

$$x_{\mathrm{NEW}} = \frac{x + 3/x_{\mathrm{OLD}}}{2}, \qquad (10)$$

and some other details. The half of the class that had been taught the method had no trouble with the question. The rest almost without exception "guessed and checked." They tried $x_{\mathrm{OLD}} = 3$ and then rather randomly substituted many other values in (10). If they were lucky they found some x_{OLD} such that x_{NEW} did the job.

My own recent experiences with technology-mediated curriculum are described in Jen Chang's 2006 MPub, [Chang 2006]. There is a concurrent commercial implementation of such a middle-school *Interactive School Mathematics* currently being completed by *MathResources*.[44] Many of the examples I have given, or similar ones more tailored to school [Borwein 2005a], are easily introduced into the curriculum, but only if the teacher is not left alone to do so. Technology also allows the same teacher to provide enriched material (say, on fractions, binomials, irrationality, fractals or chaos) to the brightest in the class while allowing more practice for those still

[43] Should such exist and as you prefer be discovered or invented.

[44] See `www.mathresources.com/products/ism/index.html`. I am a co-founder of this ten-year old company. Such a venture is very expensive and thus relies on commercial underpinning.

struggling with the basics. That said, successful mathematical education relies on active participation of the learner and the teacher and my own goal has been to produce technological resources to support not supplant this process; and I hope to make learning or teaching mathematics more rewarding and often more fun.

6.2 Last Words

To reprise, I hope to have made convincing arguments that the traditional deductive accounting of Mathematics is a largely ahistorical caricature—Euclid's millennial sway not withstanding.[45] Above all, mathematics is primarily about *secure knowledge* not proof, and that while the aesthetic is central, we must put much more emphasis on notions of supporting evidence and attend more closely to the reliability of witnesses.

Proofs are often out of reach—but understanding, even certainty, is not. Clearly, computer packages can make concepts more accessible. A short list includes linear relation algorithms, Galois theory, Groebner bases, etc. While progress is made "*one funeral at a time,*"[46] in Thomas Wolfe's words "*you can't go home again*" and as the co-inventor of the Fast Fourier transform properly observed, in [Tukey 1962][47]

> "Far better an approximate answer to the right question, which is often vague, than the exact answer to the wrong question, which can always be made precise."

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[45] Most of the cited quotations are stored at `jborwein/quotations.html`

[46] This grim version of Planck's comment is sometimes attributed to Niels Bohr but this seems specious. It is also spuriously attributed on the web to Michael Milken, and I imagine many others.

[47] Ironically, despite often being cited as in that article, I can not locate it!

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