Experimental Math for Math Monthly Problems

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Abstract. Experimental mathematics is a newly developed approach to discovering mathematical truths through the use of computers. In this paper, we look at how these techniques can be applied to help solve six problems that have appeared in the Problems section of the MONTHLY. The paper has examples of constant recognition, sequence recognition, and integer relation detection.

Experimental mathematics is a newly developed approach to discovering mathematical truths through the use of computers. Mathematicians have always calculated as part of their search for new facts. The computer makes this easier and has extended our range, but there are also new computer methods that are qualitatively different from what has gone before.

"Experimental math" is a very broad term. Borwein and Devlin state in [**10**] that "experimental mathematics is really an *approach* to mathematical discovery" (p. 115) and "Experimental mathematics is the use of a computer to run computations—sometimes no more than trial-and-error tests—to look for patterns, to identify particular numbers and sequences, to gather evidence in support of specific mathematical assertions that may themselves arise by computational means, including search." (p. 1) Experimental math is thus primarily heuristic; it guides us to an expression, but we still have to prove it.

In this paper, we will look at three particular computer methods that are important in mathematics research and illustrate their use on six problems that have appeared in the Problems section of THE AMERICAN MATHEMATICAL MONTHLY.

Experimental math has been very successful in mathematical research, but there are a couple of reasons why it might be even more successful in helping to solve MONTHLY problems. One reason is that MONTHLY problems tend to have short, neat answers (typical published solutions run about half a page to a page), so these methods, which lead directly to a final answer, might be an important shortcut in solving the problem. Another reason is that MONTHLY problems are always presented out of context so that we do not know where the problem came from or (usually) why it is interesting. The lookup methods are especially useful here because they do not require any context. The lack of context is more of a challenge for integer relation detection, as we will see in our example in Section 7.

The best place to start learning about experimental mathematics is the brief but wide-ranging survey and introduction [**10**]. The two-volume set [**6, 7**] contains an enormous number of worked examples and exercises from a wide variety of mathematics. The book [**3**] contains many lengthy and very difficult examples. The website [**2**] is a collection of much useful information and links to other sites. There is a research journal, *Experimental Mathematics*, published by Taylor & Francis.

For this paper, we will use *Mathematica* to perform the multiple-precision calculations needed, although any computer-algebra system or high-precision package would

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work. The calculations and timings shown in this paper were performed using *Mathematica* 10.0.2.0 on a 2.8 GHz Macintosh iMac computer.

1. THE METHODS. In this paper, we will look at three particular methods or techniques that are commonly used in experimental math and see examples of applying each to MONTHLY problems.

Other computer methods that are useful for many MONTHLY problems, but not covered in this paper, are the mechanical summation methods of Gosper, Wilf, Zeilberger, et al. These are especially useful for problems involving sums with binomial coefficients. They are illustrated in a very illuminating and entertaining article [**21**] published earlier in this MONTHLY. These methods are often included as part of experimental math, but they produce both the final answer and the proof, so they are not heuristic in the same way that methods we consider here are.

Constant recognition. The Inverse Symbolic Calculator Plus (ISC+) [**8**] is an online service that attempts to identify a constant, given a good numerical approximation to the constant. According to its website, ISC+ "uses a combination of lookup tables and integer relation algorithms in order to associate a closed form representation" with the given approximation. It is used to identify values that come up in research, such as definite integrals or infinite series, by calculating them to a high precision (the rule of thumb is that 15 digits are needed) and asking ISC+ for a closed-form candidate. Such problems are very common in the MONTHLY Problems section, and the value can often be discovered by this method. Even with computers, it is sometimes difficult to calculate a value to 15 digits, and we will see examples of this in this paper.

Even in the old days, we might have attempted to guess the value of a series by adding up several dozen terms. If we got a sum of 3.14159, we would probably guess that the series summed to π and attempt to prove this using known facts about π, including other series whose value included π. With computers we can get more digits; if the answer was 3.1415926535897932385, we would be even more confident that the answer was π and would be willing to work harder to prove this. Plugging in our 20-digit π suspect into ISC+ indeed produces π.

The ISC+ table is enormous, and the lookup method almost always produces a candidate if you have enough digits. However, guessing a constant from a high-precision approximation is far from infallible. We like mathematical problems to have neat answers. For example, to 30 digits, we have

$$e^{\pi\sqrt{163}} = 262537412640768743.999999999999.$$

Anyone looking at the right-hand side would guess that it represents an integer, but to 35 digits, we have

$$e^{\pi\sqrt{163}} = 262537412640768743.99999999999925007.$$

Another example (from [**4**, pp. 498–503]) is the integral

$$\int_0^\infty \cos(2x) \prod_{n=1}^{\infty} \cos\left(\frac{x}{n}\right)\, dx.$$

To 42 digits, this agrees with $\pi/8$, but in fact, it is not $\pi/8$. A collection of even more spectacular examples of misleading near matches is in [**9**].

Sequence recognition. The On-Line Encyclopedia of Integer Sequences (OEIS) [**22**] is a large searchable table of integer sequences. It is used by calculating several terms of the sequence of interest and then using the table to see if it is a known sequence. This technique is especially valuable for combinatorial problems, where it is often easy to count the objects for small sizes but difficult to work out the general case. (The table lookup is not inherently a computer technique, as the Encyclopedia started as a collection of file cards in 1964 and became a print book in 1973, but the computer has extended its reach and made it easier to use.)

Let's suppose you become interested in the problem of how many slices a pizza can be cut into using n straight cuts, and you don't realize the problem has already been solved. After some experimentation, you decide the maximum number of pieces for one through four cuts is 2, 4, 7, 11. You can then try looking up this sequence in OEIS. Its top hit is the sequence A000124, which is described as the central polygonal numbers but also as the maximal number of pieces formed when slicing a pancake with n cuts, so you realize your problem has already been solved. Better yet, the OEIS gives you the formula for this number of pieces, $n(n+1)/2+1$, and numerous references where you can find proofs and further information.

Integer relation detection. The third method is integer relation detection, in which we seek to express a given constant as a rational linear combination of known constants. An ancient example is the greatest common divisor of two integers, which we know can be expressed as such a combination: $\gcd(a, b) = ax + by$ for some integers x, y.

The general integer relation detection problem is: Given a set of n numbers c_k, attempt to find an integer linear combination of them that is very nearly 0; that is, find integers a_k such that $\sum_{k=1}^{n} a_k c_k \approx 0$. If successful, and the combination is exactly 0, this means that any of the c_k that have a nonzero coefficient can be expressed as a rational linear combination of the others. Ferguson and Bailey's PSLQ algorithm [**15**] and the Lenstra–Lenstra–Lovász (LLL) lattice reduction algorithm [**19**] are two well-known integer relation detection algorithms. *Mathematica*'s solver is the function `FindIntegerNullVector`; the *Mathematica* documentation does not reveal which algorithm this uses.

Another (slightly disguised) example of integer relation detection is the question of whether a given number x is an algebraic number (that is, it is a zero of a polynomial with integer coefficients). We can recast this question as: For some n is there an integer relation between the numbers $1, x, \dots, x^n$? In other words, are there integers $a_0, \dots, a_n$ such that $a_n x^n + \cdots + a_0 = 0$? If we could show that a mystery number was a zero of particular polynomial, we would then know a lot about it, even if we could not get an explicit representation. Take the simple example that we are given a number x that is approximately

$$x \approx 3.14626436994197234232913 5.$$

Is x algebraic? (Clearly, the right-hand side is algebraic because it is rational, but the question is really whether x is the root of a polynomial with small coefficients.) This can be answered using `FindIntegerNullVector` and a suitable number of powers of x (say $n = 10$). *Mathematica* also has a function `RootApproximant` specifically for answering whether a number is algebraic, and it says that x satisfies

$$x^4 - 10x^2 + 1 = 0.$$

For most purposes, this would be almost as good as an explicit form. In this example, because the equation is so simple, we can in fact find the explicit form. A few keystrokes in *Mathematica* gives the roots, and looking at their numerical values shows $x = \sqrt{2} + \sqrt{3}$.

One early and spectacular example of integer relation detection is the Bailey–Borwein–Plouffe formula for π in base 16 (see, for example, [**10**, Chapter 2]):

$$\pi = \sum_{i=0}^{\infty} \frac{1}{16^i} \left(\frac{4}{8i+1} - \frac{2}{8i+4} - \frac{1}{8i+5} - \frac{1}{8i+6} \right).$$

This formula allows the calculation of any base-16 digit of π with a moderate amount of effort and without calculating the preceding digits. The formula was hard to discover but once discovered can be proved easily using only calculus. More examples of integer relation detection are in an article [**4**] published earlier in this MONTHLY.

2. A RAPIDLY CONVERGING SUM. We start with an easy example. MONTHLY problem 11853 [**23**] asks for the value of

$$\sum_{n=1}^{\infty} \frac{1}{\sinh 2^n}.$$

This series converges extremely rapidly, so it is easy to get a good numerical approximation: The first five terms give about 28 digits of accuracy. *Mathematica* gives to 15 digits that

$$K = \sum_{n=1}^{5} \frac{1}{\sinh 2^n} = 0.313035285499331.$$

The ISC+ "standard lookup" does not identify this constant, but the "advanced lookup" yields the transformed value $1/(1+K) = \tanh(1)$, in other words, $K = 1/\tanh 1 - 1$. We are prompted to conjecture

$$\sum_{n=1}^{\infty} \frac{1}{\sinh 2^n} = \frac{1}{\tanh 1} - 1, \tag{1}$$

which is plausible because of the hyperbolic functions on both sides and checks out numerically: If we sum the first 10 terms, the two sides agree to about 900 decimals. This is strong evidence but not a proof; we still use traditional hand methods to get a proof.

Because the hyperbolic functions have expressions in terms of the exponential function, we might try expanding both sides of (1) as power series in e^{-1} and see if they match. We have on the left, using the geometric series, that

$$\sum_{n=1}^{\infty} \frac{1}{\sinh 2^n} = \sum_{n=1}^{\infty} \frac{2}{\exp(2^n) - \exp(-2^n)} = 2\sum_{n=1}^{\infty} \frac{\exp(-2^n)}{1 - \exp(-2 \cdot 2^n)}$$
$$= 2\sum_{n=1}^{\infty}\sum_{m=0}^{\infty} \exp(-2^n(2m+1)).$$

Some thought shows that this double series can be rearranged to $2\sum_{k=1}^{\infty} e^{-2k}$: Each positive integer can be written in exactly one way as the product of a power of 2 and an odd integer, so the expression $2^n(2m+1) = 2 \cdot 2^{n-1}(2m+1)$ in the sum takes on each even positive integer value exactly once. Meanwhile, expanding the right-hand side of (1) using the geometric series gives

$$\frac{1}{\tanh 1} - 1 = \frac{e^1 + e^{-1}}{e^1 - e^{-1}} - 1 = \frac{2e^{-1}}{e^1 - e^{-1}} = \frac{2e^{-2}}{1 - e^{-2}} = 2\sum_{k=1}^{\infty} e^{-2k},$$

so the two sides are equal, and our numerically inspired conjecture is proved.

3. A NUMBER-THEORETIC DETERMINANT. Let's try a discrete problem which does not require any high-precision calculation. MONTHLY problem 11179 [**5**] asks: For positive integers i and j, let

$$m_{ij} = \begin{cases} -1 & \text{if } j \mid (i+1) \\ 0 & \text{if } j \nmid (i+1) \end{cases},$$

and when $n \geq 2$ let M_n be the $(n-1) \times (n-1)$ matrix with (i, j)-entry m_{ij}. Evaluate $\det M_n$. (For integers a, b the notation $a \mid b$ means that a divides b, that is, b/a is an integer.)

For example, for $n = 6$ we have the 5×5 matrix

$$M_6 = \begin{pmatrix} -1 & -1 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 & 0 \\ -1 & -1 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 & -1 \\ -1 & -1 & -1 & 0 & 0 \end{pmatrix}. \tag{2}$$

We work out the first few terms as examples and get that for $n = 2$ through $n = 25$ the values of $\det M_n$ are

$$-1, -1, 0, -1, 1, -1, 0, 0, 1, -1, 0, -1, 1, 1, 0, -1, 0, -1, 0, 1, 1, -1, 0, 0.$$

A number theorist might recognize this sequence, but anyone can ask the OEIS about it. One of the OEIS hints is "enter about 6 terms, starting with the second term," so we ask about the subsequence $-1, 0, -1, 1, -1, 0$. OEIS immediately replies with 1,399 matches, of which the one rated most relevant is its sequence A008683, the Möbius function $\mu(n)$. This sequence in fact matches all 24 of our calculated values, so we conjecture $\det M_n = \mu(n)$.

The matrices M_n have an obvious recursive structure in the sense that m_{ij} does not depend on n, and so the upper left $(k-1) \times (k-1)$ submatrix is always M_k. The determinants have a further recursive structure: If we expand by minors along the bottom row, the minor determinant for column j is $\pm \det M_j$. This is because, in forming the minor, the M_j at the upper left is preserved while the -1 terms in the superdiagonal slide into the diagonal. For example, the minor for column 3 and row 5 in (2) is

$$\begin{vmatrix} -1 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ -1 & -1 & -1 & 0 \\ -1 & 0 & 0 & -1 \end{vmatrix}.$$

In this example, M_3 is in the upper left, and the matrix in the lower right has all -1 along the diagonal and all 0 above the diagonal.

The minor thus has determinant $(-1)^{n-j} \det M_j$. If we make the convention that $\det M_1 = 1$ for the empty matrix M_1, this evaluation is still true for $j = 1$. Therefore, expanding $\det M_n$ by minors along the bottom row gives us a recurrence: We have for $n > 1$ that

$$\det M_n = \sum_{j<n,\, j|n} (-1)(-1)^{n-1+j-1}(-1)^{n-j} \det M_j = -\sum_{j<n,\, j|n} \det M_j.$$

This rearranges as

$$\sum_{j|n} \det M_j = 0.$$

The Möbius function also has a recursive structure. It satisfies a recurrence

$$\sum_{d|n} \mu(d) = \begin{cases} 1, & n = 1; \\ 0, & n > 1. \end{cases}$$

(This is the first formula in the OEIS entry A008683.) This is the same recurrence satisfied by $\det M_n$, and $\det M_n$ and $\mu(n)$ have the same starting value of 1, so we have by induction that $\det M_n = \mu(n)$.

4. A PARAMETRIC SERIES DEFINED BY RECURRENCE. We look at a more difficult series that depends on a parameter and whose terms are given by a recurrence rather than explicitly. MONTHLY problem 11604 [**13**] asks: Given $0 \le a \le 2$, let $\langle a_n \rangle$ be the sequence defined by $a_1 = a$ and

$$a_{n+1} = 2^n - \sqrt{2^n(2^n - a_n)} \quad \text{for } n \ge 1. \tag{3}$$

Find $\sum_{n=1}^{\infty} a_n^2$.

The sequence depends on the parameter a, so we are being asked for a function and not a single number, but we will try to work out the value for particular values of a and then try to guess the general result. Try the endpoints first: The case $a = 0$ is easy but uninformative (all terms are 0). For the other endpoint, $a = 2$, the first few terms a_n are

$$2.0,\ 2.0,\ 1.17157,\ 0.608964,\ 0.307436,\ 0.154089,\ 0.0770908,\ 0.0385512,$$

and we see that each term is roughly half the preceding term. In the sum $\sum a_n^2$, each term is about $1/4$ the previous term, so to get 15 decimals, 50 terms should be plenty. To 25 decimals, *Mathematica* gives

$$\sum_{n=1}^{50} a_n^2 = 9.869604401089358618834491,$$

which does not look like anything in particular, but ISC+ immediately identifies it as π^2. We do not know how π got into a problem with only square roots, but we press on. Trying some additional values, we get

a	$\sum_{n=1}^{50} a_n^2$	ISC+ identifies as
0	0	0
1/2	0.346622711232150957648277	$\pi^2/9 - \frac{3}{4}$
1	1.467401100272339654708623	$\pi^2/4 - 1$
2	9.869604401089358618834491	π^2

We are not asked anything about the individual values a_n, and even if we had some information, it might not help with the value of the sum. But we will make a detour and see if a_n has any interesting properties. We suspect from the $a = 2$ example above that $2^n a_n$ goes to a nonzero finite limit. We make a wild guess that the value for $n = 50$ gives a result close to the true limit, and calculate some examples:

a	$2^{50}a_{50} \approx \lim_{n\to\infty} 2^n a_n$	ISC+ identifies as
0	0	0
1/2	1.096622711232150957648277	$\pi^2/9$
1	2.467401100272339654708623	$\pi^2/4$
2	9.869604401089358618834491	π^2

Surprisingly, the same π^2 values turn up! We still do not know where the π^2 comes from, but comparing the tables, we conjecture that

$$\sum_{n=1}^{\infty} a_n^2 = \lim_{n\to\infty} 2^n a_n + \text{simple function of } a.$$

In fact, this is easy to prove now that we have thought of it. Rearrange, square, and rearrange the recurrence (3) to get

$$a_{n+1}^2 = 2^{n+1}a_{n+1} - 2^n a_n.$$

When this is summed, the right-hand side telescopes, and we get

$$\sum_{n=1}^{\infty} a_n^2 = a_1^2 + \sum_{n=1}^{\infty} a_{n+1}^2 = a_1^2 + \lim_{n\to\infty} 2^n a_n - 2a_1 = \lim_{n\to\infty} 2^n a_n + a^2 - 2a,$$

so the "simple function" is $a^2 - 2a$, and this gives the right answer for the four examples we tried. (We are assuming temporarily that $\lim 2^n a_n$ exists; this will be proved later when we evaluate it.)

So we have reduced the sum problem to an asymptotic problem for the general term, which should be easier. The recurrence (3) has a lot of 2^n in it, and it should be easier to think about if we reparameterize to get rid of them. If we define $b_n = a_n/2^n$, we get

$$2^{n+1}b_{n+1} = 2^n - \sqrt{2^n(2^n - 2^n b_n)} = 2^n - 2^n\sqrt{1 - b_n},$$

which rearranges to

$$b_{n+1} = \frac{1 - \sqrt{1 - b_n}}{2} \quad \text{with } b_1 = a/2, \text{ so } 0 \le b_1 \le 1. \tag{4}$$

This is much simpler: Not only is the 2^n gone, but each b value depends only on the previous value and not on n. That is, it is an iteration, $b_{n+1} = f(b_n)$ where $f(x) = \frac{1}{2}\left(1 - \sqrt{1-x}\right)$. This is attractive not only because it is simpler but also because there is a systematic (but complicated) theory to get asymptotic values of sequences defined by an iteration (see [**14**, Chapter 8]).

In our case, rather than apply the systematic theory, we will make an observation that leads to a quick solution. Remembering the π^2 and the square roots, we might be reminded of the half-angle formulas for the trigonometric functions, of which the most common are

$$\cos\frac{\theta}{2} = \sqrt{\frac{1+\cos\theta}{2}} \text{ and } \sin\frac{\theta}{2} = \sqrt{\frac{1-\cos\theta}{2}}.$$

Neither of these has exactly the same form as our recurrence, but if we square the second one, we can get a half-angle formula for $\sin^2$ that does have the right format, namely

$$\sin^2\frac{\theta}{2} = \frac{1-\cos\theta}{2} = \frac{1-\sqrt{1-\sin^2\theta}}{2}.$$

Therefore, we define

$$\theta_1 = \arcsin\sqrt{b_1} \quad \text{ and } \quad \theta_{n+1} = \tfrac{1}{2}\theta_n$$

so that $b_n = \sin^2\theta_n$ is the solution of the recurrence (4), and

$$a_n = 2^n \sin^2\frac{\theta_1}{2^{n-1}}$$

is the solution to the recurrence (3). Then using $\lim_{x\to\infty} x^2\sin^2(c/x) = c$ we calculate

$$\lim_{n\to\infty} 2^n a_n = \lim_{n\to\infty} 2^{2n}\sin^2\frac{\theta_1}{2^{n-1}} = 4\theta_1^2 = 4\arcsin^2\sqrt{\frac{a}{2}}.$$

The final formula is then

$$\sum_{n=1}^{\infty} a_n^2 = 4\arcsin^2\sqrt{\frac{a}{2}} + a^2 - 2a,$$

which matches the calculated values.

5. A STIRLING SERIES. Even with today's fast computers, it is often difficult to get enough digits to feed to the lookup program (recall that our rule of thumb is that we need 15 digits). Traditional methods of numerical analysis are still very useful, particularly methods for transforming series and integrals and methods for accelerating convergence of series. In this example, we will approximate a slowly convergent series with a combination of other series that converge just as slowly but for which we know the sum explicitly. This method is sometimes called Kummer's transformation of series (see, for example, [**17**, p. 247]).

MONTHLY problem 10832 [**18**] asks for an explicit form for the sum

$$\sum_{k=1}^{\infty}\left(\frac{k^k}{k!e^k}-\frac{1}{\sqrt{2\pi k}}\right). \tag{5}$$

Today, *Mathematica* can identify the sum immediately and directly, but back in 2000, when this problem was posed, *Mathematica* was not as smart. Let's see how experimental math can help us identify the sum.

The sum converges slowly (the general term is about $1/k^{3/2}$), so brute force does not work. There is a very precise asymptotic formula (an extension of Stirling's formula; see, for example, [**24**, p. 140, formula 5.11.1]) for $\ln k!$, which begins

$$\ln k! = \left(k+\tfrac{1}{2}\right)\ln k - k + \tfrac{1}{2}\ln(2\pi) + \frac{1}{12k} - \frac{1}{360k^3} + \frac{1}{1260k^5} - \cdots.$$

We therefore have

$$\begin{aligned}\frac{k^k}{k!e^k} &= \frac{1}{\sqrt{2\pi k}}\exp\left(-\frac{1}{12k}+\frac{1}{360k^3}-\frac{1}{1260k^5}+\cdots\right)\\ &= \frac{1}{\sqrt{2\pi k}}\left(1-\frac{1}{12k}+\frac{1}{288k^2}+\frac{139}{51840k^3}-\frac{571}{2488320k^4}+\cdots\right).\end{aligned}$$

The first term of this will cancel with the other term in the sum (5), and we can use as many of the remaining terms as we think is useful. Using three terms reduces the required numerical work to a reasonable level for a computer. Write

$$a_k = -\frac{1}{12k}+\frac{1}{288k^2}+\frac{139}{51840k^3}$$

so that we have

$$\begin{aligned}\sum_{k=1}^{\infty}\left(\frac{k^k}{k!e^k}-\frac{1}{\sqrt{2\pi k}}\right) &= \sum_{k=1}^{\infty}\left(\frac{k^k}{k!e^k}-\frac{1}{\sqrt{2\pi k}}(1+a_k)\right)+\frac{1}{\sqrt{2\pi}}\sum_{k=1}^{\infty}\frac{a_k}{\sqrt{k}}\\ &= \sum_{k=1}^{\infty}\left(\frac{k^k}{k!e^k}-\frac{1}{\sqrt{2\pi k}}(1+a_k)\right) \qquad (6)\\ &\quad+\frac{1}{\sqrt{2\pi}}\left(-\frac{1}{12}\zeta(\tfrac{3}{2})+\frac{1}{288}\zeta(\tfrac{5}{2})+\frac{139}{51840}\zeta(\tfrac{7}{2})\right), \qquad (7)\end{aligned}$$

where ζ is the Riemann zeta function. The zeta series converge slowly too, but a lot is known about them and how to calculate them more quickly, and we can let *Mathematica* figure them for us. To 25 decimals, we get

$$(7) = -0.08378540362877196918178047.$$

We estimate (6) by truncating it at some point and summing numerically. The general term is about the first omitted term from the asymptotic expansion, i.e., $571/(\sqrt{2\pi k}\cdot 2488320k^4)$, so truncating the series at N introduces an error of about

$$\frac{571}{2488320\sqrt{2\pi}}\sum_{k=N}^{\infty}\frac{1}{k^{9/2}} \approx \frac{571}{(7/2)2488320\sqrt{2\pi}}\frac{1}{N^{7/2}} < 10^{-4}\frac{1}{N^{7/2}}.$$

Therefore, if we take $N = 10^4$, we will get about 18 decimals of accuracy. (Remember that we are doing heuristics. If we misestimate the error, there is no logical problem, but we may misidentify the number.) To 25 digits, *Mathematica* gives the truncated sum as $-0.0002841050988840270181249700$. This takes about 30 seconds, which is reasonable; if it had taken too long, we could have sped up the convergence some more by using more terms in the asymptotic expansion and fewer terms in the numerical part.

The whole sum is therefore approximately

$$(5) \approx -0.08406950872765599619 99.$$

The ISC+ identifies this as

$$-\frac{2}{3} - \frac{\zeta\left(\frac{1}{2}\right)}{\sqrt{2\pi}} = -0.0840695087276559964614 8950,$$

which matches to 18 decimals.

This result is plausible and encouraging; looking at the second term in the original problem, we have that the $1/\sqrt{2\pi}$ matches, and the $\sum 1/\sqrt{k}$ "sort of" matches the $\zeta\left(\frac{1}{2}\right)$, although we know the series does not converge and is not really $\zeta\left(\frac{1}{2}\right)$. Having detected the zeta function, we might look through books and find expressions such as

$$\zeta(s) = s\int_0^{\infty}\frac{\lfloor x\rfloor - x}{x^{s+1}}\,dx \quad (0 < \operatorname{Re} s < 1),$$

which is one way to analytically continue the zeta function to the left of the line $\operatorname{Re} s = 1$ (see, for example, [**26**], p. 14). For $s = \frac{1}{2}$, we rearrange and evaluate this to get

$$\lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{\sqrt{k}} - 2\sqrt{n}\right) = \zeta\left(\tfrac{1}{2}\right).$$

This explains the second part of the answer, so we would now need to show

$$\lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{k^k}{k!e^k} - \frac{2}{\sqrt{2\pi}}\sqrt{n}\right) = -\frac{2}{3}.$$

The sum here is also a limiting case of a known function, in this case the Lambert W-function (see, for example, [**24**], section 4.13, p. 111), whose power series expansion is

$$W(z) = \sum_{k=1}^{\infty}(-1)^{k-1}\frac{k^{k-1}}{k!}z^k.$$

Formally, we want to study the derivative at $z = -1/e$, but this point is on the circle of convergence and the series does not converge there, so we have to work inside the circle and take a limit. This can be done by appealing to properties of this function; the details are in the published solution [**18**]. Another experimental math treatment of this problem is in [**10**, pp. 81–85].

6. AN ALTERNATING SUM OF SQUARES OF ALTERNATING SUMS. This example also converges extremely slowly, and even after a standard transformation to speed up the convergence, it is still too slow to be useful. We will add a heuristic trick to reduce the work to a manageable level.

MONTHLY problem 11682 [**16**] asks for a closed form for

$$\sum_{n=0}^{\infty}(-1)^n\left(\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{n+k}\right)^2. \tag{8}$$

This is an intimidating-looking problem, and even getting a numerical estimate is challenging. The inner series is the tail of $\ln 2 = \sum(-1)^{k-1}/k$, which converges extremely slowly. The tail is about $\pm 1/(2n)$, so the outer sum converges slowly, too.

We can speed up the convergence of the inner sum by Euler's transformation (see, for example, [**17**, p. 244]). Write $\Delta a_n = a_{n+1} - a_n$ for the forward difference operator and $\Delta^k a_n$ for the composition of this operator k times. Euler's transformation states

$$\sum_{k=0}^{\infty}(-1)^k a_k = \sum_{n=0}^{\infty}\frac{\Delta^n a_0}{2^{n+1}}.$$

Our particular example is worked out in [**17**, p. 246, Example 1], where we find

$$\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{n+k} = \sum_{k=0}^{\infty}\frac{1}{2^{k+1}(n+k+1)\binom{n+k}{k}}. \tag{9}$$

The right-hand side converges quickly, and to get 15 decimals, we only need about 50 terms.

The inner sum is about $1/(4n^2)$, and the outer sum is an alternating series, so we would need about 10^7 or 10^8 terms to get 15 decimals, and each of those has an inner sum of 50 terms. That is a lot of terms, and we need a better way.

We will attempt to get a good value with much less work by using the following observation. We know that the partial sums of an alternating series lie alternately above and below the series value (and that the error is less than the first omitted term). Empirically, it is further true that for series with slowly and smoothly decreasing terms, the series value is almost exactly halfway between two successive partial sums (or what is the same, the series value is almost exactly the partial sum plus half the first omitted term). To take a simple example, $\ln 2 = 0.693147$. The first 100 terms of the series $\ln 2 = \sum_{k=1}^{\infty}(-1)^{k-1}/k$ give a poor approximation of 0.688172, but adding half the next term gives the much better approximation 0.693123. (This heuristic observation has been worked out in more generality and detail as the method of "repeated averaging"; see, for example, [**11**, p. 72] and [**12**, p. 278].)

Our method is to truncate the outer sum of (8) after 100,000 terms, and estimate each term (and the first omitted term) using Euler's transformation (9) with 50 terms. That is, write

$$d_n = \sum_{k=0}^{50}\frac{1}{2^{k+1}(n+k+1)\binom{n+k}{k}},$$

and sum all the included terms and add half the next term, giving to 25 digits

$$
\begin{aligned}
(8) &\approx \sum_{n=0}^{10^5-1} (-1)^n d_n^2 + \frac{1}{2} d_{100000}^2 \\
&= 0.411233516699556597589303 \\
&+ 0.000000000012499875000313 \\
&= 0.411233516712056472589616.
\end{aligned}
$$

This takes about 40 seconds in *Mathematica*, which is reasonable. The ISC+ (with advanced lookup) identifies this as

$$\frac{\pi^2}{24} = 0.4112335167120566091181038,$$

which matches the calculated value to 15 decimals.

Where does $\pi^2/24$ come from? The π^2 makes us think of $\sum_{n=1}^{\infty} \frac{1}{n^2} = \zeta(2) = \frac{\pi^2}{6}$, especially because of the terms in the outer sum being very nearly $1/(4n^2)$. However, naively applying this estimate to the sum gives $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{4n^2} = \frac{1}{8}\zeta(2) = \frac{\pi^2}{48}$, only half the calculated value, and it is not clear how $\zeta(2)$ might be generated.

However, thinking about the double (or triple) series and rummaging through zeta function lore might make us think of Tom Apostol's evaluation [**1**] of $\zeta(2)$ using the double integral

$$\zeta(2) = \int_0^1 \int_0^1 \frac{1}{1-xy}\, dx\, dy.$$

(This method appeared earlier as an exercise in LeVeque [**20**, Section 6-10, exercise 6, p. 122], and later Apostol independently rediscovered it and popularized it.) Apostol then used an extremely clever change of variables to evaluate the integral. It is easy to turn our sum into a double integral, too, and it looks a little like Apostol's:

$$
\begin{aligned}
\sum_{n=0}^{\infty} (-1)^n \left(\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{n+k} \right)^2 &= \sum_{n=0}^{\infty} (-1)^n \left(\int_0^1 \sum_{k=1}^{\infty} (-1)^{k-1} x^{n+k-1}\, dx \right)^2 \\
&= \sum_{n=0}^{\infty} (-1)^n \left(\int_0^1 \frac{x^n}{1+x}\, dx \right)^2 \\
&= \sum_{n=0}^{\infty} \int_0^1 \int_0^1 \frac{(-1)^n x^n y^n}{(1+x)(1+y)}\, dx\, dy \\
&= \int_0^1 \int_0^1 \frac{1}{(1+x)(1+y)(1+xy)}\, dx. \qquad (10)
\end{aligned}
$$

Somewhat miraculously, *Mathematica* knows the value of this integral: $\pi^2/24$, which confirms our guess. If we trust *Mathematica*, our job is done!

If we do not trust *Mathematica* that much, we can work the integral by hand. *Mathematica* can help with this too because it also knows the value of the indefinite integral:

$$\int\int \frac{1}{(1+x)(1+y)(1+xy)}\,dx$$
$$= \frac{1}{2}\Bigg(-\mathrm{Li}_2\left(\frac{xy+1}{1-y}\right) + \mathrm{Li}_2\left(\frac{xy+1}{y+1}\right) - \ln\left(\frac{(x+1)y}{y-1}\right)\ln(xy+1)$$
$$+ \ln\left(-\frac{(x-1)y}{y+1}\right)\ln(xy+1) + 2\tanh^{-1}(x)\ln(y+1)\Bigg),$$

where we need the dilogarithm function

$$\mathrm{Li}_2(x) = \sum_{n=1}^{\infty} x^n/n^2 = -\int_0^x \frac{\ln(1-t)}{t}\,dt. \tag{11}$$

We can verify the indefinite integral by hand by differentiating, but it is easier to work forward now that we have the hint of using Li_2. We expand the integrand of (10) in partial fractions twice to get

$$\int_0^1\int_0^1 \frac{1}{(1+x)(1+y)(1+xy)}\,dx$$
$$= \int_0^1\int_0^1 \frac{1}{1-x^2}\left(\frac{1}{1+y} - \frac{x}{1+xy}\right) dy\,dx$$
$$= \int_0^1 \frac{1}{1-x^2}(\ln 2 - \ln(1+x))\,dx$$
$$= \frac{1}{2}\int_0^1 \frac{1}{1+x}(\ln 2 - \ln(1+x))\,dx + \frac{1}{2}\int_0^1 \frac{1}{1-x}(\ln 2 - \ln(1+x))\,dx.$$

The first integral is easily evaluated as $\frac{1}{2}\ln^2 2$. To evaluate the second integral, we make the change of variables $x = 1 - 2t$ to get

$$\int_0^1 \frac{1}{1-x}(\ln 2 - \ln(1+x))\,dx = \int_{1/2}^0 \frac{1}{2t}(-\ln(1-t))\,(-2\,dt)$$
$$= \mathrm{Li}_2(\tfrac{1}{2}) - \mathrm{Li}_2(0) = \frac{\pi^2}{12} - \frac{1}{2}\ln^2 2,$$

where we have used the value $\mathrm{Li}_2(0) = 0$ from the definition (11), and the value $\mathrm{Li}_2(\frac{1}{2}) = \pi^2/12 - \frac{1}{2}\ln^2 2$ that comes from setting $x = \frac{1}{2}$ in the functional equation (see [**24**, p. 611, formula 25.12.6]):

$$\mathrm{Li}_2(x) + \mathrm{Li}_2(1-x) = \frac{1}{6}\pi^2 - (\ln x)(\ln(1-x)) \quad \text{for } 0 < x < 1.$$

Combining this with the first integral, the $\ln^2 2$ terms cancel and we are left with $(10) = \pi^2/24$.

7. A RAPIDLY CONVERGING PRODUCT. MONTHLY problem 11677 [**25**] asks for an evaluation of

$$P = \prod_{n=1}^{\infty} \left(1 + 2e^{-n\pi\sqrt{3}} \cosh\left(n\pi/\sqrt{3}\right)\right).$$

Just as in Section 2, this expression converges very rapidly, and we only need a few terms to get a good approximation. If we write $a_n = e^{-n\pi\sqrt{3}} \cosh\left(n\pi/\sqrt{3}\right)$ (so that we seek $P = \prod_1^{\infty}(1 + 2a_n)$), then

$$\ln a_n \approx -n\pi\left(\sqrt{3} - 1/\sqrt{3}\right) \approx -3.6276n \approx -1.57545n \ln 10,$$

so we get about 1.5 significant digits for each term we take in the product.

Taking the first 15 terms and calculating to 25 digits, we get

$$P \approx 1.028032541689576770462884.$$

But now we hit a snag: We ask ISC+ about this, and it says it found nothing, both in the standard lookup and the advanced lookup. (We asked on March 18, 2016; the database is updated continually, and ISC+ may someday be able to identify this constant.)

Because the item we seek is a product, we wonder if we would have better luck working with its logarithm, $\ln P = \sum_{n=1}^{\infty} \ln(1 + 2a_n)$. Taking the first 15 terms of this and calculating to 25 digits, we get

$$\ln P \approx 0.02764682187200888558353500.$$

This has the same problem, though: ISC+ cannot find it.

The ISC+ lookups almost always work for MONTHLY problems, perhaps because those usually have neat answers, but this is an exception, and we look at other methods to identify the number. Testing whether it is an algebraic integer using `RootApproximant` does not produce any useful answers. It does misidentify the 25-digit version of $\ln P$ as

$$\frac{5657351 - \sqrt{29079344023205}}{9578834},$$

which agrees to 24 digits but is not correct. We will try integer relation detection.

There are two challenges to using integer relation detection. The first is that often the desired number must be calculated to a very high precision, sometimes to hundreds of digits. For our example, this is not much of a problem because the product converges so rapidly. The other problem is guessing which constants should go into a linear combination to get the desired number. These guesses are based on experience and similar expressions for which we know the constants. In MONTHLY problems we are not given the context, and we may not have any experience with the particular expressions, so guessing the constants may be especially challenging.

We are going to work with $\ln P$ again. We do not have much idea what constants to use, but we will guess that we should include the constants that appear explicitly in the product, namely π, $\sqrt{3}$, and $\pi\sqrt{3}$ and their logarithms, $\ln \pi$ and $\ln 3$. (Do not use both $\pi\sqrt{3}$ and $\pi/\sqrt{3}$ because one is a rational multiple of the other, and do not use both $\ln 3$ and $\ln \sqrt{3}$, for the same reason.) A good rule of thumb when looking for a logarithm is to throw in the logs of small primes because expressions often have small integer factors in addition to the transcendental factors. We add $\ln 2$, $\ln 5$, $\ln 7$, and $\ln 11$ to the mix. We will also bump up the precision of our approximation by calculating $\ln P$ with 100 terms and 100 digits of precision.

Somewhat miraculously, this very loose procedure produces a neat answer when `FindIntegerNullVector` tells us that (within the precision of the calculations)

$$36 \cdot \ln P - 2 \cdot \pi\sqrt{3} + 9 \cdot \ln 3 = 0,$$

in other words

$$P = e^{\pi\sqrt{3}/18} \big/ \sqrt[4]{3}. \tag{12}$$

We test this against the product with 200 terms and find they agree to about 317 digits, so we conjecture that this is the correct value of the product.

Unfortunately, the explicit answer does not seem to point to any method of proof. One oddity that might catch our eye is the 18th root; that is, in the product and the final answer, we have a term with $\exp(\pi\sqrt{3})$, but in the final answer, it appears to the $1/18$ power. If we know a lot about special functions, this might remind us of the modular functions and in particular of the Dedekind eta function, which includes a $1/12$ power and that appears in a discriminant formula to the 24th power:

$$\eta(\tau) = e^{\pi i\tau/12} \prod_{n=1}^{\infty} \left(1 - e^{2\pi i n\tau}\right), \quad \operatorname{Im}\tau > 0.$$

This turns out to be the key observation, as it is possible to express the given product in terms of a ratio of eta function values, and a functional equation allows us to express the ratio as $1/\sqrt[4]{3}$. The complete solution is in [**25**].

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