On deciding transcendence of D-finite power series

(how to prove functional transcendence using a computer)

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Inria informatics mathematics

«Équations différentielles motiviques et au-delà»

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Goal, motivation, examples

In contrast with the "hard" theory of arithmetic transcendence, it is usually "easy" to establish transcendence of functions.

[Flajolet, Sedgewick, 2009]

▷ Definition: A power series f in $\mathbb{Q}[[t]]$ is called *algebraic* if it is a root of some algebraic equation $P(t, f(t)) = 0$, where $P(x, y) \in \mathbb{Z}[x, y] \setminus \{0\}$.

Otherwise, f is called *transcendental*.

▷ Goal: Given $f \in \mathbb{Q}[[t]]$, either in explicit form (by a formula), or in implicit form (by a functional equation), determine its *algebraicity* or *transcendence*.

Motivations

- Number theory: a first step towards proving the transcendence of a complex number is proving that some power series is transcendental
- Combinatorics: the nature of generating functions may reveal strong underlying structures
- Computer science: are algebraic power series (intrinsically) easier to manipulate?

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- $\sum_n \frac{(2n)!(5n)!^2}{(3n)!^4} t^n, \quad \sum_n \frac{(30n)!n!}{(15n)!(10n)!(6n)!} t^n$

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- $\sum_n (n\text{-th prime number})\, t^n, \quad \sum_n \left(n\text{-th decimal digit of } \sqrt{2}\right) t^n$

▷ Which ones are algebraic?

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Examples (II): power series given implicitly, as solutions of equations

- $f(t) = 1 + 3t + 18t^2 + 105t^3 + \cdots$, solution of

$$t^2(1+t)(1-2t)(1+4t)(1-8t)f'''(t) + t\left(576t^4 + 200t^3 - 252t^2 - 33t + 5\right)f''(t)$$
$$+4\left(288t^4 + 22t^3 - 117t^2 - 12t + 1\right)f'(t) + 12\left(32t^3 - 6t^2 - 12t - 1\right)f(t) = 0,$$

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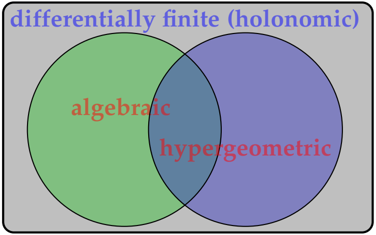
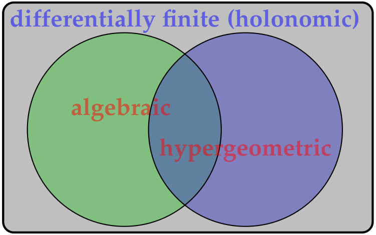
Classes of power series





$$f(t) = \sum_{n=0}^{\infty} a_n t^n \in \mathbb{Q}[[t]] \quad \text{is}$$

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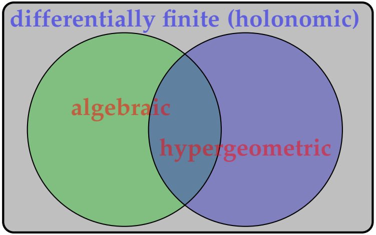

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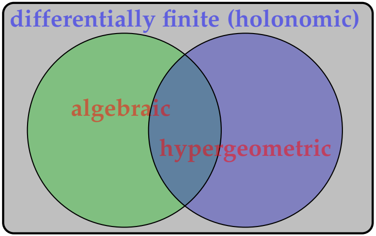

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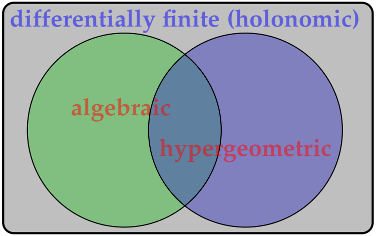
▷ *hypergeometric* if $\frac{a_{n+1}}{a_n} \in \mathbb{Q}(n)$. E.g., $\ln(1-t)$; $\frac{\arcsin(\sqrt{t})}{\sqrt{t}}$; $(1-t)^{\alpha}$, $\alpha \in \mathbb{Q}$



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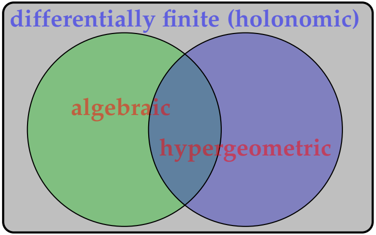

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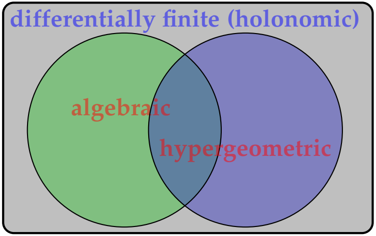
Theorem [Schwarz 1873; Landau 1911; Errera 1913; Beukers, Heckman 1989; Fürnsinn, Yurkevich 2023]

Full characterization of { *hypergeom* } ∩ { *algebraic* }



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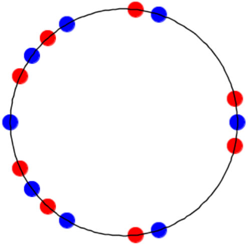
Algebraic hypergeometric series

Theorem [Beukers, Heckman, 1989] ("interlacing criterion")

Let $\{a_1, \ldots, a_k\}$ and $\{b_1, \ldots, b_{k-1}, b_k = 1\}$ be two sets of rational parameters, assumed disjoint modulo $\mathbb{Z}$. Let D be their common denominator. Then ${}_kF_{k-1}\left(\begin{matrix} a_1 & a_2 & \cdots & a_k \\ b_1 & \cdots & b_{k-1} & \end{matrix} \middle| \, t\right)$ is algebraic iff $\{e^{2i\pi r a_j}, j \leq k\}$ and $\{e^{2i\pi r b_j}, j < k\}$ interlace on the unit circle for all $1 \leq r < D$ with $\gcd(r, D) = 1$.



▷ $\displaystyle\sum_n \frac{(30n)!n!}{(15n)!(10n)!(6n)!} t^n = {}_8F_7\left(\begin{matrix} \frac{1}{30} & \frac{7}{30} & \frac{11}{30} & \frac{13}{30} & \frac{17}{30} & \frac{19}{30} & \frac{23}{30} & \frac{29}{30} \\ \frac{1}{5} & \frac{1}{3} & \frac{2}{5} & \frac{1}{2} & \frac{3}{5} & \frac{2}{3} & \frac{4}{5} & \end{matrix} \middle| \, 2^{14}\, 3^9\, 5^5\, t\right)$ is algebraic

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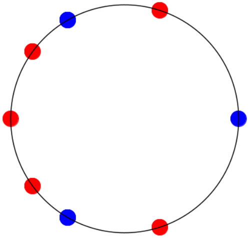

▷ $\sum_n \frac{(2n)!(5n)!^2}{(3n)!^4} t^n = {}_9F_8\left(\begin{matrix} \frac{1}{5} & \frac{1}{5} & \frac{2}{5} & \frac{2}{5} & \frac{1}{2} & \frac{3}{5} & \frac{3}{5} & \frac{4}{5} & \frac{4}{5} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{2}{3} & \frac{2}{3} & \frac{2}{3} & \frac{2}{3} & \end{matrix} \,\middle|\, \frac{2^2\, 5^{10}}{3^{12}}\, t\right)$ is transcendental

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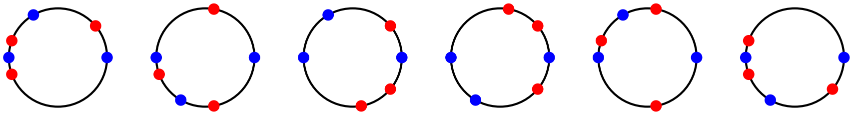

▷ $_3F_2\left(\begin{matrix} \frac{1}{9} & \frac{4}{9} & \frac{5}{9} \\ \frac{1}{2} & \frac{1}{3} & \end{matrix}\,\middle|\, 3^6\,t\right) = 1+120\,t+54600\,t^2+29995680\,t^3+17853428736\,t^4+11111241596928\,t^5+7114982545305600\,t^6+\cdots$ is transcendental

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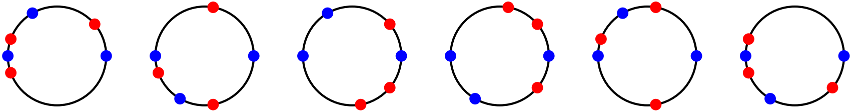

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▷ Zagier's problem: consider the P-recursive sequence [Bertola et. al, 2015]

$$c_{n-3} + 20\left(4500n^2 - 18900n + 19739\right) c_{n-2} + 80352000n(5n-1)(5n-2)(5n-4)c_n$$
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with initial terms $c_0 = 1, c_1 = -161/(2^{10} \cdot 3^5)$ and $c_2 = 26605753/(2^{23} \cdot 3^{12} \cdot 5^2)$.

Task: find $(u, v) \in \mathbb{Q}$ s. t. all $w^n \cdot (u)_n \cdot (v)_n \cdot c_n$ are in $\mathbb{Z}$ (for some $w \in \mathbb{Z}$)

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▷ [B., Weil, Yurkevich]: 7 more pairs $\longrightarrow$ all have algebraic GFs (!)

#	u	v	ODE order	alg. degree	#	u	v	ODE order	alg. degree
1	1/5	4/5	2	120	6	19/60	49/60	4	155520
2	3/5	4/5	2	120	7	19/60	59/60	4	46080
3	2/5	9/10	4	120	8	29/60	49/60	4	46080
4	7/30	9/10	4	155520	9	29/60	59/60	4	155520
5	9/10	17/30	4	155520					

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E.g.,

$$f = \ln(1-t) = -t - \frac{t^2}{2} - \frac{t^3}{3} - \frac{t^4}{4} - \frac{t^5}{5} - \frac{t^6}{6} - \cdots$$

is D-finite and can be represented by the second-order equation

$$\left((t-1)\partial_t^2 + \partial_t\right)(f) = 0, \quad f(0) = 0, f'(0) = -1.$$

▷ An algorithm should recognize (from this data) that f is transcendental.

Stanley's problem

Design an algorithm suitable for computer implementations which decides if a D-finite power series —given by a linear differential equation with polynomial coefficients and initial conditions— is algebraic, or not.

[Stanley, 1980]

▷ Notation: For a D-finite series f, we write $L_f^{\min}$ for the least-order, monic, linear differential operator in $\mathbb{Q}(t)\langle\partial_t\rangle$ that cancels f.

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▷ Warning: $L_f^{\min}$ is not known a priori; only some multiple L of it is given.

▷ Difficulty: $L_f^{\min}$ might not be irreducible. E.g., $L_{\ln(1-t)}^{\min} = \left(\partial_t + \frac{1}{t-1}\right)\partial_t$.

A few starting remarks on Stanley's problem

▷ Analogy between transcendence in $\mathbb{Q}[[t]]$ and irreducibility in $\mathbb{Q}[t]$:

- "generic" series are transcendent, "generic" polynomials are irreducible
- sufficient criteria exist (e.g., Eisenstein's), but none is also necessary
- irreducibility is decidable; what about transcendence?

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▷ The minimal polynomial can have arbitrarily large size (degrees) w.r.t. the size (order/degree) of the differential equation:

solution of $N(t-1)f'(t) - f(t) = 0$, $f(0) = 1$ satisfies $f^N = 1 - t$

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▷ No characterization for coefficient sequences of algebraic power series

- smaller class: *rational functions* $\iff$ *C-recursive sequences*
- larger class: *D-finite functions* $\iff$ *P-recursive sequences*
- *diagonals* $\overset{\text{Christol's}}{\underset{\text{conjecture}}{\Longleftrightarrow}}$ *P-recursive, almost integer, seq. with geometric growth*

  (NB: in positive characteristic p, *algebraic functions* $\iff$ *p-automatic sequences*)

Related problems

(F) *Fuchs' problem*: Decide if all solutions of L are algebraic

(L) *Liouville's problem*: Decide if L has at least one algebraic solution ($\neq 0$)

(S) *Stanley's problem*: Decide if a given solution f of L is algebraic

▷ When L is irreducible, problems **(F)**, **(L)** and **(S)** are equivalent

- ▷ [Liouville, 1833]: algorithm for (basis of) *rational solutions* of linear ODEs
  $\longrightarrow$ solves the rational versions $\mathbf{(F_{rat})}$, $\mathbf{(L_{rat})}$ and $\mathbf{(S_{rat})}$ of **(F)**, **(L)** and **(S)**
- ▷ [Fuchs, 1866]: characterization of ODEs having *only rational solutions*
  $\longrightarrow$ alternative solution to $\mathbf{(F_{rat})}$

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(Gauss hypergeometric equation $t(t-1)y'' + ((a+b+1)t-c)y' + aby = 0$)

▷ [Baldassarri & Dwork 1979]: *solution to* **(F)** *for arbitrary second order ODEs*, building on works by [Klein, 1878] and [Fuchs, 1878]

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▷ Many tools: geometry (Schwarz, Klein), invariant theory (Fuchs, Gordan), group theory (Jordan), diff. Galois theory (Vessiot, Singer, Hrushovski), number theory and algebraic geometry (Grothendieck, Katz, André)

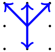
Three examples

(A) Apéry's power series [Apéry, 1978] (used in his proof of $\zeta(3) \notin \mathbb{Q}$)

$$\sum_n \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 t^n = 1 + 5\,t + 73\,t^2 + 1445\,t^3 + 33001\,t^4 + \cdots$$

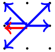
(B) GF of trident walks in the quarter plane

$$\sum_n a_n t^n = 1 + 2\,t + 7\,t^2 + 23\,t^3 + 84\,t^4 + 301\,t^5 + 1127\,t^6 + \cdots,$$

where $a_n = \#\Big\{$  $-$ walks of length n in $\mathbb{N}^2$ starting at $(0,0)\Big\}$

(C) GF of a quadrant model with repeated steps

$$\sum_n a_n t^n = 1 + t + 4\,t^2 + 8\,t^3 + 39\,t^4 + 98\,t^5 + 520\,t^6 + \cdots,$$

where $a_n = \#\Big\{$  $-$ walks of length n in $\mathbb{N}^2$ from $(0,0)$ to $(\star,0)\Big\}$

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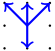
where $a_n = \#\Big\{$  – walks of length n in $\mathbb{N}^2$ starting at $(0,0)\Big\}$

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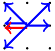
where $a_n = \#\Big\{$  – walks of length n in $\mathbb{N}^2$ from $(0,0)$ to $(\star,0)\Big\}$

Question: *How to prove that these three power series are transcendental?*

Main properties of algebraic series

If $f = \sum_n a_n t^n \in \mathbb{Q}[[t]]$ is algebraic, then

- Algebraic properties

  f is D-finite; $L_f^{\min}$ has a basis of algebraic solutions [Abel, 1827; Tannery, 1875]

- Arithmetic properties

  f is globally bounded [Eisenstein, 1852]

  $$\exists C \in \mathbb{N}^* \text{ with } a_n C^n \in \mathbb{Z} \text{ for } n \geq 1$$

- Analytic properties(⋆)

  $(a_n)_n$ has "nice" asymptotics [Puiseux, 1850; Darboux, 1878; Flajolet, 1987]

  Typically, $a_n \sim \kappa \, \rho^n \, n^\alpha$ with $\alpha \in \mathbb{Q} \setminus \mathbb{Z}_{<0}$ and $\rho \in \overline{\mathbb{Q}}$ and $\kappa \cdot \Gamma(\alpha+1) \in \overline{\mathbb{Q}}$

(⋆) "It is usually 'easy' to establish transcendence of functions, by exhibiting a local expansion that contradicts the Newton–Puiseux Theorem" [Flajolet, Sedgewick, 2009]

…and resulting transcendence criteria

For $f = \sum_n a_n t^n \in \mathbb{Q}[[t]]$, if one of the following holds

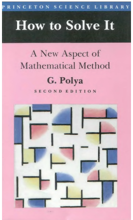
- f is not D-finite $\qquad \prod_{n \geq 1} \frac{1}{1-t^n}$
- f is not globally bounded $\qquad \sum_{n \geq 1} \frac{1}{n} t^n$
- $(a_n)_n$ has incompatible asymptotics $\qquad \sum_{n \geq 0} \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 t^n$ (†)

then f is transcendental

(†) $a_n \sim \frac{(1+\sqrt{2})^{4n+2}}{2^{9/4}\pi^{3/2}n^{3/2}}$ and $\frac{\Gamma(-1/2)}{\pi^{3/2}} = -\frac{2}{\pi} \notin \overline{\mathbb{Q}}$

Guess-and-Prove

Guess-and-Prove



Guessing and Proving

George Pólya

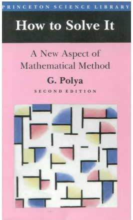
What is "scientific method"? Philosophers and non-philosophers have discussed this question and have not yet finished discussing it. Yet as a first introduction it can be described in three syllables:

Guess and test.

Mathematicians too follow this advice in their research although they sometimes refuse to confess it. They have, however, something which the other scientists cannot really have. For mathematicians the advice is

First guess, then prove.

Guess-and-Prove



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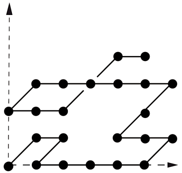
generate data → make conjectures → prove them

- $g(i,j,n)$ = number of n-steps $\{\nearrow, \swarrow, \leftarrow, \rightarrow\}$-walks in $\mathbb{N}^2$ from $(0,0)$ to (i,j)

▷ **Question**: What is the nature of the generating function

$$G(x,y,t) = \sum_{i,j,n=0}^{\infty} g(i,j,n)\, x^i y^j t^n \ ?$$



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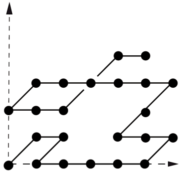

▷ **Algebraic reformulation**: Solve the "kernel equation"

$$\begin{aligned} G(x,y,t) = & 1 + t\left(xy + x + \frac{1}{xy} + \frac{1}{x}\right) G(x,y,t) \\ & - t\left(\frac{1}{x} + \frac{1}{x}\frac{1}{y}\right) G(0,y,t) - t\,\frac{1}{xy}\Big(G(x,0,t) - G(0,0,t)\Big) \end{aligned}$$

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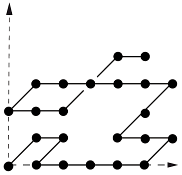

Answer: [B., Kauers, 2010] $G(x,y,t)$ is an algebraic function†.

▷ **Approach**:

1. Generate data: compute $G(x,y,t)$ to precision t^{1200} ($\approx$ 1.5 billion coeffs!)
2. Guess: conjecture polynomial equations for $G(x,0,t)$ and $G(0,y,t)$ (degree 24 each, coeffs. of degree $(46,56)$, with 80-bit digits coeffs.)
3. Prove: multivariate resultants of (very big) polynomials (30 pages each)

† Minimal polynomial $P(G(x,y,t);x,y,t) = 0$ has $> 10^{11}$ terms; $\approx$ 30 Gb (6 DVDs!)

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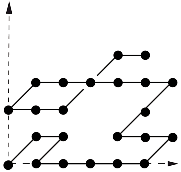

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An easier, but typical Guess-and-Prove algorithmic proof

Theorem ["Gessel excursions are algebraic"]

$$g(t) := G(0,0,\sqrt{t}) = \sum_{n=0}^{\infty} \frac{(5/6)_n (1/2)_n}{(5/3)_n (2)_n} (16t)^n \quad \text{is algebraic.}$$

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Proof: First guess a polynomial $P(t,T)$ in $\mathbb{Q}[t,T]$, then prove that P admits the power series $g(t) = \sum_{n=0}^{\infty} g_n t^n$ as a root.

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$$(n+2)(3n+5)r_{n+1} - 4(6n+5)(2n+1)r_n = 0, \qquad r_0 = 1$$

$\Rightarrow$ solution $r_n = \frac{(5/6)_n (1/2)_n}{(5/3)_n (2)_n} 16^n = g_n$, thus $g(t) = r(t)$ is algebraic.

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```
> P:=gfun:-listtoalgeq([seq(pochhammer(5/6,n)*pochhammer(1/2,n)/
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```

▷ The approach applies (in principle) to any instance of Stanley's problem.

The arithmetic and topology of differential equations

Don Zagier

Example 3. Hypergeometric algebraic units

The last example is of a somewhat different nature. In Example 4 of Section 3 we discussed hypergeometric functions $F(t)$ of the form (3.9) that are algebraic, giving Villegas's criterion for this and also the examples (3.10) and (3.11). Here Golyshev predicted, based on an argument about extensions of motives that I will not reproduce, that the power series $Q(t) = \exp\left(\int \frac{F(t)}{t}\,dt\right) = t\exp\left(\sum_{n>0} a_n \frac{t^n}{n}\right)$, where a_n denotes the coefficient of t^n in $F(t)$, must always be an algebraic function in the field $\mathbb{Q}(t, F(t))$, and in fact always an algebraic unit over $\mathbb{Z}[1/t]$. (This implies in particular that the value of $Q(t)$ if one substitutes for t the reciprocal of any integer bigger than the inverse of the radius of convergence is an algebraic unit in $\overline{\mathbb{Q}}$.)

Yan Soibelman.) I also checked Golyshev's prediction for the first two power series in (3.11) (Proposition 4 below), but in view of the huge degree I was not able to do the same for the third example. Spencer Bloch sketched to me a proof of the algebraicity of $Q(t)$ whenever the curve defined by the algebraic hypergeometric function $F(t)$ is rational (as happens for $B_{M,2}(t)$ for all M and also for $F_{(6,1),(3,2,2)}(t)$; see below), but as far as I know there is no proof yet for the general case.

$$\sum_{n=0}^{\infty} \frac{(6n)!\,n!}{(3n)!\,(2n)!^2}\,t^n, \quad \sum_{n=0}^{\infty} \frac{(10n)!\,n!}{(5n)!\,(4n)!\,(2n)!}\,t^n, \quad \sum_{n=0}^{\infty} \frac{(30n)!\,n!}{(15n)!\,(10n)!\,(6n)!}\,t^n. \quad (3.11)$$

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The last example is of a somewhat different nature. In Example 4 of Section 3 we discussed hypergeometric functions $F(t)$ of the form (3.9) that are algebraic, giving Villegas's criterion for this and also the examples (3.10) and (3.11). Here Golyshev predicted, based on an argument about extensions of motives that I will not reproduce, that the power series $Q(t) = \exp\left(\int \frac{F(t)}{t}\,dt\right) = t\exp\left(\sum_{n>0} a_n \frac{t^n}{n}\right)$, where a_n denotes the coefficient of t^n in $F(t)$, must always be an algebraic function in the field $\mathbb{Q}(t, F(t))$, and in fact always an algebraic unit over $\mathbb{Z}[1/t]$. (This implies in particular that the value of $Q(t)$ if one substitutes for t the reciprocal of any integer bigger than the inverse of the radius of convergence is an algebraic unit in $\overline{\mathbb{Q}}$.)

Yan Soibelman.) I also checked Golyshev's prediction for the first two power series in (3.11) (Proposition 4 below), but in view of the huge degree I was not able to do the same for the third example. Spencer Bloch sketched to me a proof of the algebraicity of $Q(t)$ whenever the curve defined by the algebraic hypergeometric function $F(t)$ is rational (as happens for $B_{M,2}(t)$ for all M and also for $F_{(6,1),(3,2,2)}(t)$; see below), but as far as I know there is no proof yet for the general case.

$$\sum_{n=0}^{\infty} \frac{(6n)!\,n!}{(3n)!\,(2n)!^2}\,t^n, \quad \sum_{n=0}^{\infty} \frac{(10n)!\,n!}{(5n)!\,(4n)!\,(2n)!}\,t^n, \quad \sum_{n=0}^{\infty} \frac{(30n)!\,n!}{(15n)!\,(10n)!\,(6n)!}\,t^n. \tag{3.11}$$

Finally, we verify Golyshev's prediction for the first two series in (3.11).

Proposition 7.4 *Each of the two power series*

$$t\exp\left(\sum_{n=1}^{\infty} \frac{(6n)!\,n!}{(3n)!\,(2n)!^2}\,\frac{t^n}{n}\right), \quad t\exp\left(\sum_{n=1}^{\infty} \frac{(10n)!\,n!}{(5n)!\,(4n)!\,(2n)!}\,\frac{t^n}{n}\right) \tag{7.6}$$

is algebraic, and is a unit over the ring $\mathbb{Z}[1/t]$.

Proof. The proof is purely computational, using the first terms of each power series to guess the algebraic equation and then verifying that it satisfies the correct differential equation, so we content ourselves with describing the structure of the equations of the hypergeometric series $F(t) = F_{\mathbf{c},\mathbf{d}}(t)$ and the corresponding unit $Q(t)$

The arithmetic and topology of differential equations

Don Zagier

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▷ [Delaygue, Rivoal, 2022]: proof of the 3rd prediction (suspected algebraicity degree 483 840)

Singer's algorithm and Stanley's problem

Singer's algorithm

Problem **(F)**: Decide if *all* solutions of a given ODE L of order n are algebraic

• Starting point [Jordan, 1878]: If so, then for some solution y of L, $u = y'/y$ has alg. degree at most $(49n)^{n^2}$ and satisfies a Riccati equation of order $n-1$

Algorithm (L irreducible) [Painlevé, 1887], [Boulanger, 1898], [Singer, 1979]

1. Decide if the Riccati equation has an algebraic solution u of degree at most $(49n)^{n^2}$ — degree bounds + algebraic elimination
2. (Abel's problem) Given an algebraic u, decide whether $y'/y = u$ has an algebraic solution y — [Risch 1970], [Baldassarri & Dwork 1979]

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▷ [Singer, 2014; B., Salvy, Singer, 2023]: compute L^{alg}, factor of L with solution space spanned by alg. solutions of L $\longrightarrow$ requires ODE factoring

Problem **(S)**: Decide if a D-finite power series $f \in \mathbb{Q}[[t]]$, given by an ODE $L(f) = 0$ and sufficiently many initial terms, is transcendental.

1. Compute L^{alg}
2. Decide if L^{alg} annihilates f

▷ Benefit: Solves (in principle) Stanley's problem **(S)**: *algebraicity is decidable*

▷ Drawbacks: Step 1 involves *impractical bounds* & *requires ODE factorization*

▷ ODE factorization is effective
[Schlesinger, 1897], [Singer, 1979], [Grigoriev, 1990], [van Hoeij, 1997]

▷ ...but possibly extremely costly:
$(N\mathcal{L})^{O(n^4)}$, with $\mathcal{L} = \mathrm{bitsize}(L)$ and $N \leq e^{(\mathcal{L}\cdot 2^n)^{o(2^n)}}$ [Grigoriev, 1990]

A practical method, based on Minimization

Problem **(S)**: Decide if a D-finite power series $f \in \mathbb{Q}[[t]]$, given by an ODE $L(f) = 0$ and sufficiently many initial terms, is transcendental.

Key property: If $L_f^{\min}$ has a logarithmic singularity, then f is transcendental.

▷ Pros and cons: Avoids factorization of L, but requires to compute $L_f^{\min}$.

Ex. (A): Apéry's power series

Theorem (Apéry's power series is transcendental)

$$f(t) = \sum_n A_n t^n, \quad \text{where } A_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2, \quad \text{is transcendental.}$$

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1. Creative telescoping: [Zagier, 1979], [Zeilberger, 1990]

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2. Conversion from recurrence to differential equation $L(f) = 0$, where

   $$L = (t^4 - 34t^3 + t^2)\partial_t^3 + (6t^3 - 153t^2 + 3t)\partial_t^2 + (7t^2 - 112t + 1)\partial_t + t - 5$$

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4. Local solutions of $L_f^{\min}$: [Frobenius, 1873], [Chudnovsky[2], 1987]

   $$\left\{1 + 5t + O(t^2),\ \ln(t) + (5\ln(t) + 12)t + O(t^2),\ \ln(t)^2 + (5\ln(t)^2 + 24\ln(t))t + O(t^2)\ \right\}$$

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5. Conclusion: f is transcendental†

† f algebraic would imply a full basis of algebraic solutions for $L_f^{\min}$ [Tannery, 1875].

Ex. (B): D-Finite quadrant models [B., Chyzak, van Hoeij, Kauers & Pech, 2017]

	OEIS	$\mathscr{S}$	nature	ODE (ord, deg)		OEIS	$\mathscr{S}$	nature	ODE (ord, deg)
1	A005566		T	(3, 4)	13	A151275		T	(5, 24)
2	A018224		T	(3, 5)	14	A151314		T	(5, 24)
3	A151312		T	(3, 8)	15	A151255		T	(4, 16)
4	A151331		T	(3, 6)	16	A151287		T	(5, 19)
5	A151266		T	(5, 16)	17	A001006		A	(2, 3)
6	A151307		T	(5, 20)	18	A129400		A	(2, 3)
7	A151291		T	(5, 15)	19	A005558		T	(3, 5)
8	A151326		T	(5, 18)					
9	A151302		T	(5, 24)	20	A151265		A	(4, 9)
10	A151329		T	(5, 24)	21	A151278		A	(4, 12)
11	A151261		T	(4, 15)	22	A151323		A	(2, 3)
12	A151297		T	(5, 18)	23	A060900		A	(3, 5)

- ▷ Computer-driven discovery and proof; no human proof yet
- ▷ For models 5–10, asymptotics do not conclude. E.g.  $a_n \sim \frac{4}{3\sqrt{\pi}} \frac{4^n}{n^{1/2}}$

	OEIS	$\mathscr{S}$	nature	asympt
1	A005566		T	$\frac{4}{\pi}\frac{4^n}{n}$
2	A018224		T	$\frac{2}{\pi}\frac{4^n}{n}$
3	A151312		T	$\frac{\sqrt{6}}{\pi}\frac{6^n}{n}$
4	A151331		T	$\frac{8}{3\pi}\frac{8^n}{n}$
5	A151266		T	$\frac{1}{2}\sqrt{\frac{3}{\pi}}\frac{3^n}{n^{1/2}}$
6	A151307		T	$\frac{1}{2}\sqrt{\frac{5}{2\pi}}\frac{5^n}{n^{1/2}}$
7	A151291		T	$\frac{4}{3\sqrt{\pi}}\frac{4^n}{n^{1/2}}$
8	A151326		T	$\frac{2}{\sqrt{3\pi}}\frac{6^n}{n^{1/2}}$
9	A151302		T	$\frac{1}{3}\sqrt{\frac{5}{2\pi}}\frac{5^n}{n^{1/2}}$
10	A151329		T	$\frac{1}{3}\sqrt{\frac{7}{3\pi}}\frac{7^n}{n^{1/2}}$
11	A151261		T	$\frac{12\sqrt{3}}{\pi}\frac{(2\sqrt{3})^n}{n^2}$
12	A151297		T	$\frac{\sqrt{3}B^{7/2}}{2\pi}\frac{(2B)^n}{n^2}$
13	A151275		T	$\frac{12\sqrt{30}}{\pi}\frac{(2\sqrt{6})^n}{n^2}$
14	A151314		T	$\frac{\sqrt{6}\lambda\mu C^{5/2}}{5\pi}\frac{(2C)^n}{n^2}$
15	A151255		T	$\frac{24\sqrt{2}}{\pi}\frac{(2\sqrt{2})^n}{n^2}$
16	A151287		T	$\frac{2\sqrt{2}A^{7/2}}{\pi}\frac{(2A)^n}{n^2}$
17	A001006		A	$\frac{3}{2}\sqrt{\frac{3}{\pi}}\frac{3^n}{n^{3/2}}$
18	A129400		A	$\frac{3}{2}\sqrt{\frac{3}{\pi}}\frac{6^n}{n^{3/2}}$
19	A005558		T	$\frac{8}{\pi}\frac{4^n}{n^2}$
20	A151265		A	$\frac{2\sqrt{2}}{\Gamma(1/4)}\frac{3^n}{n^{3/4}}$
21	A151278		A	$\frac{3\sqrt{3}}{\sqrt{2}\Gamma(1/4)}\frac{3^n}{n^{3/4}}$
22	A151323		A	$\frac{\sqrt{2}3^{3/4}}{\Gamma(1/4)}\frac{6^n}{n^{3/4}}$
23	A060900		A	$\frac{4\sqrt{3}}{3\Gamma(1/3)}\frac{4^n}{n^{2/3}}$

$$A = 1+\sqrt{2},\ B = 1+\sqrt{3},\ C = 1+\sqrt{6},\ \lambda = 7+3\sqrt{6},\ \mu = \sqrt{\tfrac{4\sqrt{6}-1}{19}}$$

▷ Asymptotics conjectured by [B., Kauers, 2009], proved by [Melczer, Wilson, 2016]

Ex. (B): Models 1–19, explicit expressions and transcendence

Theorem [B., Chyzak, van Hoeij, Kauers, Pech, 2017]

Let $\mathscr{S}$ be one of the models 1–19. Then

- $Q_{\mathscr{S}}(0,0,t)$ is expressible using (integrals of) ${}_2F_1$ expressions.
- $Q_{\mathscr{S}}(0,0,t)$ is transcendental.

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- $Q_{\mathscr{S}}(1,1,t)$ is transcendental, except for $\mathscr{S} =$  and $\mathscr{S} =$ .

Example (King walks in the quarter plane, A151331)

Q

$$(t) = \frac{1}{t}\int_0^t \frac{1}{(1+4x)^3} \cdot {}_2F_1\left(\begin{matrix}\frac{3}{2} & \frac{3}{2}\\ & 2\end{matrix}\,\middle|\, \frac{16x(1+x)}{(1+4x)^2}\right) dx$$

$$= 1 + 3t + 18t^2 + 105t^3 + 684t^4 + 4550t^5 + 31340t^6 + 219555t^7 + \cdots$$

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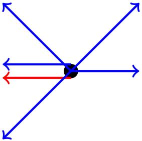
$$= 1 + 3t + 18t^2 + 105t^3 + 684t^4 + 4550t^5 + 31340t^6 + 219555t^7 + \cdots$$

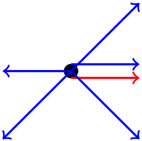
▷ Computer-driven discovery and proof; no human proof yet.

▷ Original proof uses creative telescoping, ODE factorization, ODE solving

▷ Alternative (easier) proof uses minimization



Case A



Case B

Theorem [B., Bousquet-Mélou, Kauers, Melczer, 2016]

- GF is D-finite and transcendental in Case A.
- GF is algebraic in Case B.

▷ Computer-driven discovery and proof; no human proof yet.
▷ Proof uses minimization.
▷ All other criteria and algorithms fail or do not terminate.

Input: A D-finite $f(t) \in \mathbb{Q}[[t]]$, given by an ODE $L(f) = 0$ plus initial terms
Output: T if $f(t)$ is transcendental, A if $f(t)$ is algebraic

▷ Principle: **(S)** reduced to **(F)** via minimization

1. Compute $L_f^{\min}$ [Adamczewski, Rivoal, 2018], [B., Rivoal, Salvy, 2022]
2. Decide if $L_f^{\min}$ has only algebraic solutions; if so return A, else return T. [Singer, 1979]

▷ Benefit: Solves (in principle) Stanley's problem: *algebraicity is decidable*

▷ Drawback: Step 2 can be very costly in practice.

Input: A D-finite $f(t) \in \mathbb{Q}[[t]]$, given by an ODE $L(f) = 0$ plus initial terms
Output: T if $f(t)$ is transcendental, A if $f(t)$ is algebraic

1. Compute $L_f^{\min}$ [Adamczewski, Rivoal, 2018], [B., Rivoal, Salvy, 2022]
2. If $L_f^{\min}$ has a logarithmic singularity, return T; otherwise return A

▷ This algorithm is always correct when it returns T

▷ *Conjecturally*, under the additional assumption that *f is globally bounded*$^\diamondsuit$, it is also always correct$^\clubsuit$ when it returns A [Christol, 1986], [André, 1997]

$^\diamondsuit$ E.g. if f is given as GF of a binomial sum, or as the diagonal of a rational function

$^\clubsuit$ NB: not true without the global boundedness assumption, e.g. $f(t) = {}_2F_1\left(\begin{matrix}\frac{1}{6} & \frac{5}{6}\\ \frac{7}{6} & \end{matrix} \,\middle|\, t\right)$

Central sub-task: Minimization

Problem: Given a D-finite power series $f \in \mathbb{Q}[[t]]$ by a differential equation $L(f) = 0$ and sufficiently many initial terms, compute $L_f^{\min}$.

▷ Why isn't this easy? After all, it is just a differential analogue of:

Given an algebraic power series $f \in \mathbb{Q}[[t]]$
by an algebraic equation $P(t, f) = 0$ and sufficiently many initial terms,
compute its minimal polynomial P_f^{min}.

▷ $L_f^{\min}$ is a (right) factor of L, but contrary to the commutative case:

- $L_f^{\min}$ might not be irreducible. E.g., $L_{\ln(1-t)}^{\min} = \left(\partial_t + \frac{1}{t-1}\right)\partial_t$.
- factorization of diff. operators is not unique $\quad \partial_t^2 = (\partial_t + \frac{1}{t-c})(\partial_t - \frac{1}{t-c})$
- ...and it is difficult to compute
- $\deg_t L_f^{\min} > \deg_t L$, due to apparent singularities $\quad (t\partial_t - N) \mid \partial_t^{N+1}$

▷ $\deg_t L_f^{\min}$ can be bounded w.r.t. n and local data of L via Fuchs' relation

Central sub-task: Minimization [B., Rivoal, Salvy, 2022]

Input: $L \in \mathbb{Q}(t)\langle\partial_t\rangle$ such that $L(f) = 0$ (+ initial conditions) **Output**: $L_f^{\min}$

▷ Strategy (inspired by the approach in [van Hoeij, 1997], itself based on ideas from [Chudnovsky, 1980], [Bertrand & Beukers, 1982], [Ohtsuki, 1982])

1. If $L_f^{\min}$ is Fuchsian (e.g., if f is a diagonal), then it can be written

$$L_f^{\min} = \partial_t^n + \frac{a_{n-1}(t)}{A(t)}\partial_t^{n-1} + \cdots + \frac{a_0(t)}{A(t)^n}, \qquad n \leq \mathrm{ord}(L)$$

   with $A(t)$ squarefree and $\deg(a_{n-i}) \leq \deg(A^i) - i$.

2. $\deg(A)$ can be bounded in terms of n and (local) data of L (via *apparent singularities* and *Fuchs' relation*)

3. Guess and Prove: For $n = 1, 2, \ldots,$
   1. Guess differential equation of order n for f (use bounds and linear algebra)
   2. Once found a nontrivial candidate, certify it using L, or go to previous step.

Central sub-task: Minimization [B., Rivoal, Salvy, 2022]

Input: $L \in \mathbb{Q}(t)\langle\partial_t\rangle$ such that $L(f) = 0$ (+ initial conditions) **Output**: $L_f^{\min}$

▷ Strategy (inspired by the approach in [van Hoeij, 1997], itself based on ideas from [Chudnovsky, 1980], [Bertrand & Beukers, 1982], [Ohtsuki, 1982])

1. If $L_f^{\min}$ is Fuchsian (e.g., if f is a diagonal), then it can be written

$$L_f^{\min} = \partial_t^n + \frac{a_{n-1}(t)}{A(t)}\partial_t^{n-1} + \cdots + \frac{a_0(t)}{A(t)^n}, \qquad n \le \operatorname{ord}(L)$$

   with $A(t)$ squarefree and $\deg(a_{n-i}) \le \deg(A^i) - i$.

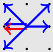
2. $\deg(A)$ can be bounded in terms of n and (local) data of L (via *apparent singularities* and *Fuchs' relation*)

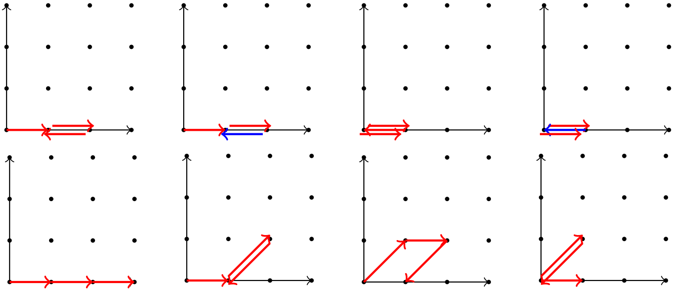
3. Guess and Prove: For $n = 1, 2, \ldots,$
   1. Guess differential equation of order n for f (use bounds and linear algebra)
   2. Once found a nontrivial candidate, certify it using L, or go to previous step.

▷ If $L_f^{\min}$ is not Fuchsian: Newton polygons, generalized Fuchs relation, various optimizations

Ex. (C): a difficult quadrant model with repeated steps

Theorem [B., Bousquet-Mélou, Kauers, Melczer, 2016]

Let $a_n = \#\Big\{$  $-$ walks of length n in $\mathbb{N}^2$ from $(0,0)$ to $(\star,0)\Big\}$. Then $f(t) = \sum_n a_n t^n = 1 + t + 4\,t^2 + 8\,t^3 + 39\,t^4 + 98\,t^5 + \cdots$ is transcendental.



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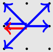
Let $a_n = \#\Big\{$  $-$ walks of length n in $\mathbb{N}^2$ from $(0,0)$ to $(\star,0)\Big\}$. Then $f(t) = \sum_n a_n t^n = 1 + t + 4\,t^2 + 8\,t^3 + 39\,t^4 + 98\,t^5 + \cdots$ is transcendental.

Proof:

1. Discover and certify a differential equation L for $f(t)$ of order 11 and degree 73 — high-tech Guess-and-Prove
2. If $\operatorname{ord}(L_f^{\min}) \leq 10$, then $\deg_t(L_f^{\min}) \leq 580$ — apparent singularities
3. Rule out this possibility — differential Hermite-Padé approximants
4. Thus, $L_f^{\min} = L$
5. L has a log singularity at $t = 0$, and so f is transcendental $\square$

Ex. (C): a difficult quadrant model with repeated steps

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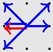
Let $a_n = \# \Big\{$  $-$ walks of length n in $\mathbb{N}^2$ from $(0,0)$ to $(\star, 0)\Big\}$. Then $f(t) = \sum_n a_n t^n = 1 + t + 4\,t^2 + 8\,t^3 + 39\,t^4 + 98\,t^5 + \cdots$ is transcendental.

Proof:

1. Discover and certify a differential equation L for $f(t)$ of order 11 and degree 73 — high-tech Guess-and-Prove
2. If $\operatorname{ord}(L_f^{\min}) \leq 10$, then $\deg_t(L_f^{\min}) \leq 580$ — apparent singularities
3. Rule out this possibility — [Beckermann, Labahn, 1994]
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- Problems **(F)**, **(L)**, **(S)** of algebraicity of solutions of ODEs are decidable
- In practice, proving *transcendence is easier* than proving algebraicity (!)
- ODE minimization is a practical alternative for proving transcendence
  - ☺ $\longrightarrow$ allows to solve difficult problems from applications
  - ☺ $\longrightarrow$ also useful in other contexts (effective Siegel-Shidlovskii)
- Guess-and-Prove is a powerful method for proving algebraicity
  - ☺ $\longrightarrow$ robust: adapts to other functional equations
  - ☹ $\longrightarrow$ main limitation: output size!
- Brute-force / naive algorithms $\longrightarrow$ hopeless on "real-life" applications

Thanks for your attention!

Bonus

Bounds for $L_f^{\min} = \partial_t^n + \frac{a_{n-1}(t)}{A(t)}\partial_t^{n-1} + \cdots + \frac{a_0(t)}{A(t)^n}, \qquad n \le \mathrm{ord}(L)$

Task: get a bound on $\deg(A)$ in terms of n and (local) data of L

- $A(t) = A_{\mathsf{sing}}(t) A_{\mathsf{app}}(t)$, where the roots of A_{sing}, resp. of A_{app}, are the finite *true* singular points, resp. the finite *apparent* singular points, of $L_f^{\min}$.

- Trivial: $\deg(A_{\mathsf{sing}}) \le \#\{\text{finite true singularities of } L\}$

- Fuchs' relation

$$\sum_{z \in \mathbb{C} \cup \{\infty\}} S_z(L_f^{\min}) = \sum_{z \text{ singularity of } L_f^{\min}} S_z(L_f^{\min}) = -n(n-1),$$

with $S_z(L_f^{\min}) = (\text{sum of local exponents of } L_f^{\min} \text{ at } z) - (0+1+\cdots+(n-1))$

- Main point: If z is an apparent singularity of $L_f^{\min}$ then $S_z(L_f^{\min}) \ge 1$, thus:

$$\deg(A_{\mathsf{app}}) \le -n(n-1) - \sum_{z \text{ true singularity of } L} \min(0, S_z^{(n)}(L)),$$

where $S_z^{(n)}(L) := (\text{sum of the smallest } n \text{ exponents of } L \text{ at } z) - \binom{n}{2}$

Conjecture

Let $f \in \mathbb{Q}[[z]]$ be a globally bounded and D-finite power series. Then:

- [Christol, 1990] f is the diagonal of a rational function;
- [Christol-André, 1997, 2004] If $z = 0$ is an ordinary point for $L_f^{\min}$, then f is algebraic;
- [André] If the monodromy of $L_f^{\min}$ at $z = 0$ is semisimple (i.e., $z = 0$ is not a logarithmic singularity of $L_f^{\min}$), then f is algebraic.

▷ Concrete subproblem: is

$${}_3F_2\left(\left.\begin{matrix}\frac{1}{9} & \frac{4}{9} & \frac{5}{9}\\ & \frac{1}{3} & 1\end{matrix}\,\right|\, 729\, t\right) = 1 + 36t + 10530t^2 + 4401540t^3 + \cdots \quad \text{a diagonal?}$$