Computer algebra for combinatorics

Alin Bostan

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Exercise from last lecture ("Binomial sums")

Give (and prove!) a simple formula for

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{2k}{n}$$
Guessing the answer using an automated guesser

Give (and prove!) a simple formula for

\[
\sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{2k}{n}
\]

\[
T := (-1)^k \binom{n}{k} \binom{2k}{n}:
\]

\[
\text{first} \_\text{terms} := [\text{seq}(\text{add}(T, k=0..n), n=0..6)]:
\]

\[
\text{guess} \_\text{rec} := \text{gfun}:-\text{listtorec}(\text{first} \_\text{terms, u(n)})[1];
\]

\[
\{ u(n+1) + 2u(n) = 0, \quad u(0) = 1 \}
\]

\[
\text{rsolve} (\text{guess} \_\text{rec, u(n)});
\]

\[
(-2)^n
\]

▷ Is this a proof?
▷ Can it be turned into a proof?
▷ Is this guessing procedure always guaranteed to work?
Guessing the answer using a human guesser

Give (and prove!) a simple formula for

\[ \sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{2k}{n} \]

**Combinatorial Identities**

A STANDARDIZED SET OF TABLES
LISTING 500 BINOMIAL COEFFICIENT SUMMATIONS

"Scientia non habet inimitum nisi ignorantiam"

Henry W. Gould
Professor of Mathematics
West Virginia University

Revised Edition
Morgantown, W. Va. 1972

H. W. Gould

Computer algebra for combinatorics
Guessing the answer using a human guesser

Give (and prove!) a simple formula for

\[
\sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{2k}{n}
\]

▷ Table look-up:

\[
(3.64) \sum_{k=0}^{n} (-1)^k \binom{n}{k}\binom{2k}{j} = (-1)^n \binom{n}{j-n} 2^{n-j}
\]

\[
3.150 \sum_{k=0}^{n} (-1)^k \binom{n}{k}\binom{x+kz}{j} = \begin{cases} 
0, & 0 \leq j < n, \\
(-1)^n z^n, & j = n.
\end{cases}
\]

▷ Is this a proof?

▷ Can it be turned into a proof?

▷ Is this guessing procedure always guaranteed to work?
Proving the answer using a built-in summation software

Give (and prove!) a simple formula for

\[ \sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{2k}{n} \]

\[ T := (-1)^k \binom{n}{k} \binom{2k}{n} : \]
\[ \text{sum}(T, k=0..n); \]

\[ (-2)^n \]

▷ Is this a proof?

▷ Is it always guaranteed to work?

▷ What is behind this proof?
Proving the answer using Zeilberger’s algorithm

Give (and prove!) a simple formula for \( \sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{2k}{n} \)

\[
> T:=(-1)^k*\text{binomial}(n,k)*\text{binomial}(2*k,n):
> \text{Zpair:=SumTools[\text{Hypergeometric}][\text{Zeilberger}]}(T, n, k, \text{Sn}):
> \text{tel:=Zpair[1];}
\]

\[
S_n + 2
\]

\[
> \text{cert:=Zpair[2];}
\]

\[
\frac{(2k - n - 1) (2k - n) (-1)^k \binom{n}{k} \binom{2k}{n}}{(-n + k - 1) (n + 1)}
\]

\[
> \text{is_zero:=(subs(n=n+1,T) + 2*T) - (subs(k=k+1,cert) - cert)}:
> \text{simplify(convert(is_zero,GAMMA));}
\]

\[
0
\]
Proving the answer using residues

Give (and prove!) a simple formula for

\[
\sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{2k}{n}
\]

\[
\text{with(BinomSums):}
\]
\[
T2:=(-1)^k \ast \text{Binomial2}(n,k)\ast\text{Binomial2}(2*k,n):
\]
\[
S := \text{Sum}(t^n\ast\text{Sum}(T2, k=0..\infty), n=0..\infty):
\]
\[
\text{series(BinomSums[computesum]}(S, 5), t);
\]

\[
1 - 2t + 4t^2 - 8t^3 + 16t^4 - 32t^5 + O(t^6)
\]

\[
\text{R, ord := BinomSums[sumtores]}(S, u);
\]

\[
\frac{1}{2t + 1} \quad [t]
\]

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Computer algebra for combinatorics
Proving the answer using residues

Give (and prove!) a simple formula for \( \sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{2k}{n} \)

▷ What really happened in the previous slide?
▷ The algorithm started from the pre-tabulated formulas

\[
\binom{n}{k} = \text{res}_u \frac{(1 + u)^n}{u^{k+1}}, \quad \binom{2k}{n} = \text{res}_v \frac{(1 + v)^{2k}}{v^{n+1}}
\]

▷ It then performed the summation \( \sum_{n=0}^{\infty} \sum_{k=0}^{n} (-1)^k \cdot \frac{(1 + u)^n}{u^{k+1}} \cdot \frac{(1 + v)^{2k}}{v^{n+1}} \cdot t^n \)

expressing the GF of the input binomial sum as the residue (w.r.t. \( u \) and \( v \)) of

\[
R := \frac{1}{v \left(1 - \frac{(1+u)t}{v}\right)} + \frac{(1+v)^2}{uv \left(1 + \frac{(1+v)^2(1+u)t}{uv}\right)}
\]

\[
v^2 + u + 2v + 1
\]

▷ It finally performed a successive pole/residue analysis, proving that

\[
\text{res}_{u,v} R = \text{res}_v \frac{v}{t v^3 + 2t v^2 + v^2} = \frac{1}{2t + 1}.
\]
Bonus: a “hypergeometric proof” using a special-function identity

Up to \( k \leftrightarrow n - k \), the identity is equivalent to

\[
\sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{2n - 2k}{n} = 2^n.
\]

▷ Let us re-write binomials as quotients of products of factorials

\[
u_n = \sum_{k=0}^{n} (-1)^k \cdot \frac{n!}{k!(n-k)!} \cdot \frac{(2n-2k)!}{n!(n-2k)!} = \sum_{k=0}^{n} (-1)^k \cdot \frac{1}{k!} \cdot \frac{1}{(n-k)!} \cdot \frac{(2n-2k)!}{(n-2k)!}
\]

and then in terms of “rising factorials” (or, “Pochhammer symbols”)

\((a)_n = a(a+1) \cdots (a+n-1)\), using the rewriting rules:

\[
(n-k)! = \frac{(-1)^k \cdot n!}{(-n)_k} \quad \text{and} \quad (a)_{2k} = 4^k \cdot \left(\frac{a}{2}\right)_k \cdot \left(\frac{a+1}{2}\right)_k
\]

▷ We get

\[
u_n = \binom{2n}{n} \cdot \sum_{k=0}^{n} \frac{(-n)_k \cdot (-n)_{2k}}{(1)_k \cdot (-2n)_{2k}} = \binom{2n}{n} \cdot \sum_{k=0}^{n} \frac{(-\frac{n}{2})_k \cdot (-\frac{n+1}{2})_k}{(1)_k \cdot \left(\frac{1}{2} - n\right)_k}
\]

▷ We conclude using the “Chu-Vandermonde” hypergeometric identity that

\[
u_n = \binom{2n}{n} \cdot _2F_1 \left( -\frac{n}{2}, \frac{1}{2} - \frac{n}{2} \mid 1 \right) = 2^n.
\]

▷ “We reduced an identity to another identity: what’s the point?”
Bonus: a direct algebraic proof using Legendre polynomials

Up to $k \leftrightarrow n - k$, the identity is equivalent to

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{2n - 2k}{n} = 2^n.$$ 

▷ Consider the polynomial

$$P_n(x) := \frac{1}{n!} \cdot \frac{\partial^n}{\partial x^n} (x^2 - 1)^n$$

▷ By the Leibniz differentiation rule,

$$P_n(x) = \sum_{k=0}^{n} \binom{n}{k}^2 (x + 1)^{n-k} (x - 1)^k,$$

hence $P_n(x) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{2n - 2k}{n} x^{n-2k}$, in particular $P_n(1) = 2^n$.

▷ By the binomial theorem, $(x^2 - 1)^n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} x^{2n-2k}$, hence

$$P_n(x) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{2n - 2k}{n} x^{n-2k}.$$ 

▷ In conclusion, $2^n = P_n(1) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{2n - 2k}{n}$. 

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Computer algebra for combinatorics
We will prove that

\[ \sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{2n-2k}{n} = 2^n \]

by counting subsets of \{1, \ldots, n\} in the following way:

(1) There is an obvious bijection between subsets of \{1, \ldots, n\} and subsets with \(n\) elements of the set \{1, \ldots, n, \bar{1}, \ldots, \bar{n}\} which contain either \(k\) or \(\bar{k}\)

(2) Now to count the latter we can do inclusion/exclusion:

- \(\binom{2n}{n}\) counts all \(n\)-element sets
- This counts too many, because it counts also subsets which contain both \(k\) and \(\bar{k}\)
- To delete those, we subtract \(\binom{n}{1} \cdot \binom{2n-2}{n-2}\)
- But this deletes too many, since it counts those who have \(k\) and \(\bar{k}\) and \(\ell\) and \(\bar{\ell}\) twice
- Hence one adds \(\binom{n}{2} \cdot \binom{2n-4}{n-4}\)
- And so on.
Summary: ingredients of an experimental mathematics approach

- **Trial and error phase**
  - Guess the answer
  - Look up for possible generalizations

- **Reasoning and proving phase – understand what’s “inside the box”**
  - Use built-in routines
  - Use a specific summation approach
  - Use an alternative / better one

- **Bonus phase – attacking from different angles**
  - Hypergeometric approach
  - Direct algebraic approach
  - Combinatorial approach

- What is your favorite proof?

- Why? (Criteria: length/beauty/trickiness/naturalness)

Hermite-Padé approximants
bibliographic searches

computer algebra software
Zeilberger’s algorithm
residues

Chu-Vandermonde identity
Legendre polynomials
Bijection (most human creativity demanding)
Bonus: a ChatGPT proof

You
The question is: Find the sum over k from 0 to n of \((-1)^k\) times \(\binom{n}{k}\) \(\binom{2k}{n}\), we can use the principle of inclusion-exclusion.

Let's break it down:

1. Start with the binomial expansion: 
   \[ (x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k. \]
2. Substitute \(x = -1\) and \(y = 1\) into the binomial expansion to get: 
   \[ (-1 + 1)^n = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k}. \]
3. Note that \((-1 + 1)^n = 0\) for \(n > 0\) because it alternates between positive and negative values, and since the sum is to 0, it must be zero.
4. Now, let's use the principle of inclusion-exclusion to get rid of the \(\binom{2k}{n}\) term. This is where things get a bit tricky.

The principle of inclusion-exclusion states that for two sets \(A\) and \(B\), the cardinality of their union is given by:

\[ |A \cup B| = |A| + |B| - |A \cap B|. \]
Extending this to three sets, we have:
\[ |A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C| \]

In our case, we have two sets: \( A = \{ k \text{ such that } k \leq n \} \) and \( B = \{ k \text{ such that } 2k \geq n \} \).

The set \( A \) corresponds to the sum \( \sum_{k=0}^{n} (-1)^k \binom{n}{k} \), and \( B \) corresponds to the sum \( \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \binom{2k}{n} \).

1. **Using the principle of inclusion-exclusion, the expression becomes:**
\[
\sum_{k=0}^{n} (-1)^k \binom{n}{k} - \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \binom{2k}{n}
\]

Now, we have to handle each term separately.

The first term is \( \sum_{k=0}^{n} (-1)^k \binom{n}{k} \), which we already know is zero.

The second term is \( \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \binom{2k}{n} \). We can compute this using the formula for the binomial coefficient:
\[
\binom{2k}{n} = \frac{(2k)!}{n!(2k-n)!}
\]

1. **Plug the value of \( \binom{2k}{n} \) into the sum, and simplify it:**
\[
\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \frac{(2k)!}{n!(2k-n)!}
\]
This expression is a bit complex and might not simplify nicely. It might be helpful to consider specific cases or further manipulation based on the value of $n$.

Let me know if you need further assistance!
You
Can you simplify further? Ideally, an expression with no sum

ChatGPT
To simplify the expression $\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \frac{(2k)!}{n!(2k-n)!}$ further, we can rewrite the factorial terms in terms of binomial coefficients and then manipulate the expression to remove the summation.

First, recall that $(2k)! = (2k)(2k-1)(2k-2) \ldots (2)(1)$.

We can express $(2k)!$ as $(2k)(2k-1)(2k-2) \ldots (2)(1) = (2k) \times (2k-1) \times \ldots \times (k+1) \times k!$.

Now, we can rewrite the expression:

$$(-1)^k \frac{(2k)!}{n!(2k-n)!} = (-1)^k \frac{(2k)(2k-1)\ldots(k+1)k!}{n!(2k-n)!}$$

Using the definition of binomial coefficients, we can express $(2k)!$ and $(2k - n)!$ in terms of binomial coefficients:

$$(2k)! = (2k)(2k-1) \ldots (k+1)k! = \binom{2k}{k}k!$$

$$(2k - n)! = (2k - n)(2k - n - 1) \ldots (2)(1) = \binom{2k-n}{0}(2k - n)!$$
Substituting these into the expression:

\[
(-1)^k \frac{\binom{2k}{k} k!}{n! \binom{2k-n}{0} (2k-n)!}
\]

\[= (-1)^k \frac{\binom{2k}{k}}{\binom{2k-n}{0} n!} \]

\[= (-1)^k \frac{\binom{2k}{k}}{\binom{2k-n}{2k-n} n!} \]

\[= (-1)^k \binom{2k}{k} \left( \frac{n}{n-k} \right)^{-1} \]

Now, \( \binom{2k-n}{2k-n}^{-1} \) is equivalent to \( \binom{2k-n}{n-2k} \), so we can rewrite the expression as:

\[(-1)^k \binom{2k}{k} \left( \frac{n}{n-k} \right)^{-1} \binom{2k-n}{n-2k} \]

This expression contains no summation. It gives the simplified form of the original summation expression.
Bonus: a ChatGPT proof

You
Can you find the sum over k from 0 to n of \((-1)^k \binom{n}{k} \binom{2k}{n}\)? The final result should be a simple expression of n.

ChatGPT
Certainly! We'll approach this problem using a combinatorial identity and some algebraic manipulation.

The given sum is:

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{2k}{n}$$

First, let's rewrite the binomial coefficients in terms of factorials:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

$$\binom{2k}{n} = \frac{(2k)!}{n!(2k-n)!}$$

Substitute these into the sum:

$$\sum_{k=0}^{n} (-1)^k \frac{n!}{k!(n-k)!} \frac{(2k)!}{n!(2k-n)!}$$
Now, let's simplify:

\[
\sum_{k=0}^{n} (-1)^k \frac{n!}{k!(n-k)!} \frac{(2k)!}{n!(2k-n)!} = \sum_{k=0}^{n} (-1)^k \frac{n!}{k!(n-k)!} \frac{(2k)(2k-1)(2k-2) \ldots (2)(1)}{n!(2k-n)(2k-n-1) \ldots (2)(1)}
\]

\[
= \sum_{k=0}^{n} (-1)^k \frac{n!}{k!(n-k)!} \frac{(2k)!}{n!(2k-n)!} \frac{1}{n!(2k-n)(2k-n-1) \ldots (2)(1)}
\]

\[
= \sum_{k=0}^{n} (-1)^k \frac{n!}{k!(n-k)!} \frac{(2k)!}{n!(2k-n)!} \frac{1}{n!(2k-n)(2k-n-1) \ldots (2)(1)}
\]

\[
= \sum_{k=0}^{n} (-1)^k \frac{n!}{k!(n-k)!} \frac{(2k)!}{n!(2k-n)!} \frac{(2k)(2k-1)(2k-2) \ldots (2)(1)}{n!(2k-n)(2k-n-1) \ldots (2)(1)}
\]

\[
= \sum_{k=0}^{n} (-1)^k \frac{n!}{k!(n-k)!} \frac{(2k)!}{n!(2k-n)!} \frac{(2k)(2k-1)(2k-2) \ldots (2)(1)}{n!(2k-n)(2k-n-1) \ldots (2)(1)}
\]

\[
= \sum_{k=0}^{n} (-1)^k \frac{n!}{k!(n-k)!} \frac{(2k)!}{n!(2k-n)!}
\]

\[
= \sum_{k=0}^{n} (-1)^k \frac{n!}{k!(n-k)!} \frac{(2k)!}{n!(2k-n)!} \frac{1}{n!(2k-n)(2k-n-1) \ldots (2)(1)}
\]

\[
= \sum_{k=0}^{n} (-1)^k \frac{n!}{k!(n-k)!} \frac{(2k)!}{n!(2k-n)!} \frac{(2k)(2k-1)(2k-2) \ldots (2)(1)}{n!(2k-n)(2k-n-1) \ldots (2)(1)}
\]

\[
= \sum_{k=0}^{n} (-1)^k \frac{n!}{k!(n-k)!} \frac{(2k)!}{n!(2k-n)!}
\]
$$= \sum_{k=0}^{n} (-1)^k \frac{n!}{k!(n-k)!} \frac{(2k)(2k-1)(2k-2)\ldots(2)(1)}{n!(2k-n)(2k-n-1)\ldots(2)(1)}$$

$$= \sum_{k=0}^{n} (-1)^k \frac{n!}{k!(n-k)!} \frac{(2k)!}{n!(2k-n)!}$$

$$= \sum_{k=0}^{n} (-1)^k \frac{n!}{k!(n-k)!} \frac{(2k)(2k-1)(2k-2)\ldots(2)(1)}{n!(2k-n)(2k-n-1)\ldots(2)(1)}$$

$$= \sum_{k=0}^{n} (-1)^k \frac{n!}{k!(n-k)!} \frac{(2k)!}{n!(2k-n)!}$$
Bonus: a ChatGPT proof

**You**
There is still a sum involved in the answer, and I would like the answer to contain no sum, just a simple formula in terms of \( n \)

**ChatGPT**
Apologies for the oversight. Let’s try to find a simplified expression without a sum.

We're looking at the expression:

\[
\sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{2k}{n}
\]

We know that:

\[
\binom{n}{k} = \frac{n!}{k!(n-k)!}
\]

\[
\binom{2k}{n} = \frac{(2k)!}{n!(2k-n)!}
\]

Let's simplify these expressions:

\[
\binom{n}{k} = \frac{n(n-1)(n-2)\ldots(n-k+1)}{k!}
\]

\[
\binom{2k}{n} = \frac{(2k)(2k-1)(2k-2)\ldots(2k-n+1)}{n!}
\]

Now, let's substitute these back into the original expression:
Bonus: a ChatGPT proof

\[
\sum_{k=0}^{n} (-1)^k \frac{n(n-1)(n-2)\ldots(n-k+1)}{k!(n-k)!} \frac{(2k)(2k-1)(2k-2)\ldots(2k-n+1)}{n!}
\]

We can simplify this further:

\[
\frac{n(n-1)(n-2)\ldots(n-k+1)}{k!} = \frac{n!}{k!(n-k)!}
\]

\[
\frac{(2k)(2k-1)(2k-2)\ldots(2k-n+1)}{n!} = \frac{(2k)!}{n!(2k-n)!}
\]

So, the expression becomes:

\[
\sum_{k=0}^{n} (-1)^k \frac{n!}{k!(n-k)!} \frac{(2k)!}{n!(2k-n)!}
\]

\[
= \sum_{k=0}^{n} (-1)^k \binom{n}{k} \left(\frac{2k}{n}\right)
\]

\[
= \sum_{k=0}^{n} (-1)^k \frac{n!}{k!(n-k)!} \frac{(2k)!}{n!(2k-n)!}
\]

\[
= \sum_{k=0}^{n} (-1)^k \frac{n!}{k!(n-k)!} \frac{(2k)!}{n!(2k-n)!}
\]

\[
= \sum_{k=0}^{n} (-1)^k \binom{n}{k} \left(\frac{2k}{n}\right)
\]
Bonus: a ChatGPT proof

\[= \sum_{k=0}^{n} (-1)^k \frac{n!}{k!(n-k)!} \frac{(n-k+1)(n-k+2)\ldots(n-2)(n-1)}{n!}\]

\[= \sum_{k=0}^{n} (-1)^k \frac{n!}{k!(n-k)!} \frac{(2k)!}{n!(2k-n)!}\]

\[= \sum_{k=0}^{n} (-1)^k \frac{n!(n-1)(n-2)\ldots(n-k+1)}{k!(n-k)!} \frac{(2k)!}{n!(2k-n)!}\]

\[= \sum_{k=0}^{n} (-1)^k \frac{n!(n-1)(n-2)\ldots(n-k+1)}{k!(n-k)!} \frac{(2k)!}{n!(2k-n)!}\]

\[= \sum_{k=0}^{n} (-1)^k \frac{n!(n-1)(n-2)\ldots(n-k+1)}{k!(n-k)!} \frac{(2k)!}{n!(2k-n)!}\]
Bonus: a new exercise

Enumeration of multi-rooted plane trees

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Abstract

We give closed form expressions for the numbers of multi-rooted plane trees with specified degrees of root vertices. This results in an infinite number of integer sequences some of which are known to have an alternative interpretation. We also propose recursion relations for numbers of such trees as well as for the corresponding generating functions. Explicit expressions for the generating functions corresponding to plane trees having two and three roots are derived. As a by-product, we obtain a new binomial identity and a conjecture relating hypergeometric functions.

MSC: 05A19, 05C05, 11Y55

Keywords: rooted maps; generating functions; ribbon graphs; integer sequences; plane trees; combinatorial identities.
Conjecture 2. Let $n, r, s$ be integers and $n \geq 0$. Then

$$
\sum_{k=0}^{n} \binom{2n-2k+s}{n-k} \binom{2k+r}{k} = \sum_{k=0}^{[n/2]} \binom{2n+2+r+s}{n-2k},
$$

where $[n/2]$ denotes the integer part of $n/2$.

Exercise: prove that both sides satisfy the recurrence

$$
4 (2n + r + s + 3) (2n + r + s + 2) a_n
- \left( 8n (n + r + s) + (r + s)^2 + 13(2n + r + s) + 22 \right) a_{n+1}
+ (n + 2) (n + s + r + 2) a_{n+2} = 0
$$

with

$$
a_0 = 1 \quad \text{and} \quad a_1 = r + s + 4.
$$
Enumerative Combinatorics: science of counting
Area of mathematics primarily concerned with counting discrete objects.
▷ Main outcome: theorems

Computer Algebra: effective mathematics
Area of computer science primarily concerned with the algorithmic manipulation of algebraic objects.
▷ Main outcome: algorithms

Computer Algebra for Enumerative Combinatorics
An (innocent looking) combinatorial question

Let $\mathcal{I} = \{\uparrow, \leftarrow, \downarrow\}$. An $\mathcal{I}$-walk is a path in $\mathbb{Z}^2$ using only steps from $\mathcal{I}$. Show that, for any integer $n$, the following quantities are equal:

(i) number $a_n$ of $n$-steps $\mathcal{I}$-walks confined to the upper half plane $\mathbb{Z} \times \mathbb{N}$ that start and finish at the origin $(0, 0)$ (excursions);

(ii) number $b_n$ of $n$-steps $\mathcal{I}$-walks confined to the quarter plane $\mathbb{N}^2$ that start at the origin $(0, 0)$ and finish on the diagonal of $\mathbb{N}^2$ (diagonal walks).
An (innocent looking) combinatorial question

Let $\mathcal{I} = \{\uparrow, \leftarrow, \searrow\}$. An $\mathcal{I}$-walk is a path in $\mathbb{Z}^2$ using only steps from $\mathcal{I}$. Show that, for any integer $n$, the following quantities are equal:

(i) number $a_n$ of $n$-steps $\mathcal{I}$-walks confined to the upper half plane $\mathbb{Z} \times \mathbb{N}$ that start and finish at the origin $(0,0)$ (excursions);

(ii) number $b_n$ of $n$-steps $\mathcal{I}$-walks confined to the quarter plane $\mathbb{N}^2$ that start at the origin $(0,0)$ and finish on the diagonal of $\mathbb{N}^2$ (diagonal walks).

For instance, for $n = 3$, this common value is $a_3 = b_3 = 3$:
Teaser 1: This “exercise” is non-trivial

Teaser 2: It can be solved using Experimental Math and Computer Algebra

Teaser 3: …by two robust and efficient algorithmic techniques, Guess-and-Prove and Creative Telescoping
Why count walks?

Many objects can be encoded by (confined) walks:

- probability theory (voting, games of chance, branching processes, …)
- discrete mathematics (permutations, trees, words, urns, …)
- statistical physics (Ising model, …)
- operations research (queueing theory, …)
Counting walks is an old topic: the ballot problem [Bertrand, 1887]

Suppose that candidates $A$ and $B$ are running in an election. If $a$ votes are cast for $A$ and $b$ votes are cast for $B$, where $a > b$, then the probability that $A$ stays ahead of $B$ throughout the counting of the ballots is $(a - b)/(a + b)$.

**Lattice path reformulation**: find the number of paths in $\mathbb{Z}^2$ with $a$ upsteps $\uparrow$ and $b$ downsteps $\downarrow$ that start at the origin and never touch the $x$-axis.

Without the constraint, the number of such paths is $\binom{a+b}{a}$ → a Guess-and-Prove proof in a few slides.
Counting walks is an old topic: the ballot problem \[\text{[Bertrand, 1887]}\]

Suppose that candidates A and B are running in an election. If \(a\) votes are cast for A and \(b\) votes are cast for B, where \(a > b\), then the probability that A stays ahead of B throughout the counting of the ballots is \((a - b)/(a + b)\).

**Lattice path reformulation:** find the number of paths in \(\mathbb{Z}^2\) with \(a - 1\) upsteps \(\uparrow\) and \(b\) downsteps \(\downarrow\) that start at \((1, 1)\) and never touch the \(x\)-axis.

**Reflection principle** \[\text{[Aebly, 1923]}\]: paths in \(\mathbb{Z}^2\) from \((1, 1)\) to \(T(a + b, a - b)\) that do touch the \(x\)-axis are in bijection with paths in \(\mathbb{Z}^2\) from \((1, -1)\) to \(T\).

**Answer:**

\[
\begin{align*}
(\text{paths in } \mathbb{Z}^2 \text{ from } (1, 1) \text{ to } T) &= \binom{a + b - 1}{a - 1} \\
(\text{paths in } \mathbb{Z}^2 \text{ from } (1, -1) \text{ to } T) &= \binom{a + b - 1}{b - 1}
\end{align*}
\]
Counting walks is an old topic: the ballot problem [Bertrand, 1887]

Suppose that candidates $A$ and $B$ are running in an election. If $a$ votes are cast for $A$ and $b$ votes are cast for $B$, where $a > b$, then the probability that $A$ stays ahead of $B$ throughout the counting of the ballots is $(a - b)/(a + b)$.

Lattice path reformulation: find the number of paths in $\mathbb{Z}^2$ with $a - 1$ upsteps $\uparrow$ and $b$ downsteps $\downarrow$ that start at $(1, 1)$ and never touch the $x$-axis.

Reflection principle [Aebly, 1923]: paths in $\mathbb{Z}^2$ from $(1, 1)$ to $T(a + b, a - b)$ that do touch the $x$-axis are in bijection with paths in $\mathbb{Z}^2$ from $(1, -1)$ to $T$.

Answer: \[
\binom{a+b-1}{a-1} - \binom{a+b-1}{b-1} = \frac{a-b}{a+b} \binom{a+b}{a}
\]
...but it is still a very hot topic

Lot of recent activity; many recent contributors:

...but it is still a very hot topic

Lot of recent activity; many recent contributors:

...but it is still a very hot topic
Our approach: Experimental Mathematics using Computer Algebra
Our approach: Experimental Mathematics using Computer Algebra
Lattice walks with small steps in the quarter plane

Walks in $\mathbb{N}^2$ starting at $(0, 0)$ and using steps in a fixed subset $\mathcal{S}$ of 
\{↙, ←, ↖, ↑, ↗, →, ↘, ↓\}.

Counting sequence: $q_{\mathcal{S}}(n) = \text{number of } \mathcal{S}\text{-walks of length } n$

Length generating function:

$$Q_{\mathcal{S}}(t) = \sum_{n=0}^{\infty} q_{\mathcal{S}}(n) t^n \in \mathbb{Z}[[t]]$$
Lattice walks with small steps in the quarter plane

Walks in \( \mathbb{N}^2 \) starting at \((0, 0)\) and using steps in a fixed subset \( \mathcal{S} \) of \{↙, ←, ↘, ↑, →, ↗, ↓\}.

Refinement: \( q_{\mathcal{S}}(i, j; n) = \) number of \( \mathcal{S} \)-walks of length \( n \) ending at \((i, j)\)

Full generating function (with “catalytic” variables \( x, y \)):

\[
Q_{\mathcal{S}}(x, y; t) = \sum_{i,j,n=0}^{\infty} q_{\mathcal{S}}(i, j; n) x^i y^j t^n \in \mathbb{Z}[[x, y, t]]
\]

Actually: \( Q_{\mathcal{S}}(x, y; t) \in \mathbb{Z}[x, y][[t]] \) and \( Q_{\mathcal{S}}(1, 1; t) = Q_{\mathcal{S}}(t) \)
Entire books dedicated to small-steps walks in the quarter plane!
Small-steps models of interest

Among the $2^8$ step sets $\mathcal{S} \subseteq \{-1, 0, 1\}^2 \setminus \{(0, 0)\}$, some are:

- Trivial
- Simple
- Intrinsic to the half plane
- Symmetrical

One is left with 79 interesting distinct models.
Small-steps models of interest

Among the $2^8$ step sets $\mathcal{S} \subseteq \{-1,0,1\}^2 \setminus \{(0,0)\}$, some are:

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- simple,
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Small-steps models of interest

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Among the $2^8$ step sets $\mathcal{I} \subseteq \{-1,0,1\}^2 \setminus \{(0,0)\}$, some are:

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Small-steps models of interest

Among the $2^8$ step sets $\mathcal{S} \subseteq \{-1,0,1\}^2 \setminus \{(0,0)\}$, some are:

- trivial,
- simple,
- intrinsic to the half plane,
- symmetrical.

One is left with 79 interesting distinct models.
The 79 small-steps quadrant models

Non-singular

Singular
Task: classify their generating functions!
Classification criterion: properties of generating functions

- differentially finite (holonomic)
- algebraic
- hypergeometric

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Classification criterion: properties of generating functions

- algebraic
- hypergeometric
- differentially finite (holonomic)
- differentially algebraic

\[(1 - t)^\alpha\]
Classification criterion: properties of generating functions

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\[ \sqrt{1 - t} + \sqrt[3]{1 - 2t} \]

\[ (1 - t)^\alpha \]
Classification criterion: properties of generating functions

- algebraic
- hypergeometric
- differentially finite (holonomic)
- differentially algebraic

\[
\sqrt{1 - t} + \sqrt[3]{1 - 2t} + \ln(1 - t)
\]
Classification criterion: properties of generating functions

\[ \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{t^n}{n!} = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{t^n}{n!}, \text{ where } (a)_n = a(a+1) \cdots (a+n-1). \]
Classification criterion: properties of generating functions

differentially algebraic

algebraic

\( (1-t)^\alpha \)

\( \frac{1}{\sqrt{1-t}} + \frac{3}{\sqrt{1-2t}} \)

hypergeometric

\( \ln(1-t) \)

\( \binom{2}{1} F_1 \left( \begin{array}{c} a \ b \ c \\ 1 \ | \ t \end{array} \right) \)

E.g., \( (1-t)^\alpha = 2F_1 \left( \begin{array}{c} -\alpha \ 1 \\ 1 \ | \ t \end{array} \right) \), \( \ln(1-t) = -t \cdot 2F_1 \left( \begin{array}{c} 1 \ 1 \\ 2 \ | \ t \end{array} \right) = - \sum_{n=1}^{\infty} \frac{t^n}{n} \)
Classification criterion: properties of generating functions

- Algebraic
- Hypergeometric
- Differentially finite (holonomic)
- Differentially algebraic

\[ \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{t^n}{n!}, \quad \text{where} \quad (a)_n = a(a+1) \cdots (a+n-1). \]
Classification criterion: properties of generating functions

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\[ 2F_1 \left( \begin{array}{c} a \\ b \\ c \end{array} \left| t \right. \right) := \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{t^n}{n!} \], where \( (a)_n = a(a+1) \cdots (a+n-1) \).
Classification criterion: properties of generating functions

D-algebraic (solutions of polynomial differential equations)

D-finite (solutions of linear differential equations)

algebraic

(roots of polynomials)

\[ \sqrt{1 - t} \]

\[ \frac{\ln(1 - t)}{t} \]

D-transcendental

\[ y' = 1 + y^2 \]

\[ y'' + (3t - 2)y' + y = 0 \]

\[ \tan(t) \]

\[ \Gamma(t) = \int_0^\infty x^{t-1}e^{-x} \, dx \]

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Classification criterion: properties of generating functions

D-algebraic (solutions of polynomial differential equations)

D-finite (solutions of linear differential equations)

\[ \sqrt{1 - t} \]

algebraic (roots of polynomials)

\[ y' = 1 + y^2 \]

\[ \tan(t) \]

\[ _2F_1 \left( \begin{array}{c} a & b \\ c \end{array} \bigg| t \right) := \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{t^n}{n!} \]

where \( (a)_n = a(a+1) \cdots (a+n-1) \).
Generating function: \( Q(x,y) \equiv Q(x,y;t) = \sum_{i,j,n=0}^{\infty} q(i,j;n)x^i y^j t^n \in \mathbb{Z}[[x,y,t]] \)

Recursive construction yields the kernel equation

\[
Q(x,y) = 1 + t \left( y + \frac{1}{x} + x \frac{1}{y} \right) Q(x,y) - t \frac{1}{x} Q(0,y) - tx \frac{1}{y} Q(x,0)
\]
Algebraic reformulation of main task: solving a functional equation

Generating function: $Q(x, y) \equiv Q(x, y; t) = \sum_{i,j,n=0}^{\infty} q(i, j; n)x^i y^j t^n \in \mathbb{Z}[[x, y, t]]$

Recursive construction yields the kernel equation

$$\left(1 - t \left(y + \frac{1}{x} + x \frac{1}{y}\right)\right) xyQ(x, y) = xy - tyQ(0, y) - tx^2 Q(x, 0)$$
Algebraic reformulation of main task: solving a functional equation

Generating function: \( Q(x, y) \equiv Q(x, y; t) = \sum_{i,j,n=0}^{\infty} q(i, j; n) x^i y^j t^n \in \mathbb{Z}[[x, y, t]] \)

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\]

New task: Solve this functional equation!
Algebraic reformulation of main task: solving a functional equation

Generating function: \( Q(x, y) \equiv Q(x, y; t) = \sum_{i,j,n=0}^{\infty} q(i, j; n) x^i y^j t^n \in \mathbb{Z}[[x, y, t]] \)

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\]

New task: For the other models – solve 78 similar equations!
“Special” models of walks in the quarter plane

- Dyck
- Motzkin
- Pólya
- Kreweras
- Gessel
- Gouyou-Beauchamps
- King walks
- Tandem walks
Kernel equation:

\[(y - tx(1 + y^2)) \cdot Q(x, y) = y - tx \cdot Q(x, 0)\]

Kernel method [Knuth, 1968]:

- let \(y_0 \in \mathbb{Q}[x][[t]]\) be the power series root of \(K = y - tx(1 + y^2)\)
  
  \[y_0 = \frac{1 - \sqrt{1 - 4t^2x^2}}{2tx} = tx + t^3x^3 + 2t^5x^5 + \cdots \in \mathbb{Q}[x][[t]]\]

- plug \(y = y_0\) in the kernel equation \(\implies\) \(Q(x, 0) = \frac{y_0}{tx}\)

- conclude algebraicity:
  
  \[Q(x, y) = \frac{y - y_0}{K} = \frac{\sqrt{1 - 4t^2x^2} + 2txy - 1}{2tx(y - tx(1 + y^2))}\]

Same method proves algebraicity for all models intrinsic to the half plane.
• $g(n)$ = number of $n$-steps $\{↗, ↘, ←, →\}$-walks in $\mathbb{N}^2$
  
1, 2, 7, 21, 78, 260, 988, 3458, 13300, 47880, …

**Question**: What is the nature of the generating function

$$G(t) = \sum_{n=0}^{\infty} g(n) t^n$$
• $g(i, j; n) =$ number of $n$-steps $\{\uparrow, \downarrow, \leftarrow, \rightarrow\}$-walks in $\mathbb{N}^2$ from $(0, 0)$ to $(i, j)$

**Question:** What is the nature of the generating function

$$G(x, y; t) = \sum_{i,j,n=0}^{\infty} g(i, j; n) x^i y^j t^n$$
• $g(i, j; n)$ = number of $n$-steps $\{\nearrow, \searrow, \leftarrow, \rightarrow\}$-walks in $\mathbb{N}^2$ from $(0, 0)$ to $(i, j)$

**Question**: What is the nature of the generating function

$$G(x, y; t) = \sum_{i, j, n=0}^{\infty} g(i, j; n) x^i y^j t^n$$

**Theorem [B., Kauers, 2010]**

$G(x, y; t)$ is an algebraic function$^\dagger$.

$\triangleright$ computer-driven discovery/proof via *algorithmic Guess-and-Prove*

$
^\dagger$ Minimal polynomial $P(G(x, y; t); x, y, t) = 0$ has $> 10^{11}$ terms; $\approx 30$ Gb (6 DVDs!)
• \( g(n) \) = number of \( n \)-steps \( \{ \nearrow, \swarrow, \leftarrow, \rightarrow \} \)-walks in \( \mathbb{N}^2 \)

**Question**: What is the nature of the generating function

\[
G(t) = \sum_{n=0}^{\infty} g(n) t^n
\]

**Corollary [B., Kauers, 2010]** (former conjecture of Gessel’s)

\[(3n + 1) g(2n) = (12n + 2) g(2n - 1) \text{ and } (n + 1) g(2n + 1) = (4n + 2) g(2n)\]

▷ computer-driven discovery/proof via algorithmic Guess-and-Prove
Guessing and Proving

George Pólya

What is “scientific method”? Philosophers and non-philosophers have discussed this question and have not yet finished discussing it. Yet as a first introduction it can be described in three syllables:

Guess and test.

Mathematicians too follow this advice in their research although they sometimes refuse to confess it. They have, however, something which the other scientists cannot really have. For mathematicians the advice is

First guess, then prove.
Guess-and-Prove

Guessing and Proving

George Pólya

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**Guess and test.**

Mathematicians too follow this advice in their research although they sometimes refuse to confess it. They have, however, something which the other scientists cannot really have. For mathematicians the advice is

- generate data
- make conjectures
- prove them
Question: Find $B_{i,j} :=$ the number of $\{\rightarrow, \uparrow\}$-walks in $\mathbb{N}^2$ from $(0,0)$ to $(i,j)$.
**Guess-and-Prove: a toy example**

**Question:** Find $B_{i,j} :=$ the number of $\{\rightarrow, \uparrow\}$-walks in $\mathbb{N}^2$ from $(0,0)$ to $(i,j)$

1. There are 2 ways to get to $(i,j)$, either from $(i-1,j)$, or from $(i,j-1)$:
   \[
   B_{i,j} = B_{i-1,j} + B_{i,j-1}
   \]

2. There is only one way to get to a point on an axis: $B_{i,0} = B_{0,j} = 1$

▷ These two rules completely determine all the numbers $B_{i,j}$
Guess-and-Prove: a toy example

Question: Find $B_{i,j}$ := the number of $\{\rightarrow, \uparrow\}$-walks in $\mathbb{N}^2$ from $(0,0)$ to $(i,j)$

1. There are 2 ways to get to $(i,j)$, either from $(i-1,j)$, or from $(i,j-1)$:

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$\triangleright$ These two rules completely determine all the numbers $B_{i,j}$

\begin{align*}
\text{(I) Generate data:} \\
1 & 7 & 28 & 84 & 210 & 462 & 924 \\
1 & 6 & 21 & 56 & 126 & 252 & 462 \\
1 & 5 & 15 & 35 & 70 & 126 & 210 \\
1 & 4 & 10 & 20 & 35 & 56 & 84 \\
1 & 3 & 6 & 10 & 15 & 21 & 28 \\
1 & 2 & 3 & 4 & 5 & 6 & 7 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\end{align*}
### Guess-and-Prove: a toy example

**Question:** Find $B_{i,j} :=$ the number of $\{\rightarrow, \uparrow\}$-walks in $\mathbb{N}^2$ from $(0,0)$ to $(i,j)$

1. There are 2 ways to get to $(i,j)$, either from $(i-1,j)$, or from $(i,j-1)$:
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(II) Guess:

\[
\begin{align*}
1 & \quad 2 & \quad 3 & \quad 4 & \quad 5 & \quad 6 & \quad 7 \\
1 & \quad 1 & \quad 1 & \quad 1 & \quad 1 & \quad 1 & \\
1 & \quad 6 & \quad 21 & \quad 56 & \quad 126 & \quad 252 & \quad 462 \\
1 & \quad 5 & \quad 15 & \quad 35 & \quad 70 & \quad 126 & \quad 210 \\
1 & \quad 4 & \quad 10 & \quad 20 & \quad 35 & \quad 56 & \quad 84 \\
(\text{II}) \quad \text{Guess:} & \quad (i+1)(i+2) \\
& \quad \begin{cases} 
1 & \quad \text{for } i = 0 \\
1 & \quad \text{for } i = 1 \\
\end{cases}
\end{align*}
\]
Question: Find $B_{i,j} :=$ the number of $\{\rightarrow, \uparrow\}$-walks in $\mathbb{N}^2$ from $(0,0)$ to $(i,j)$

1. There are 2 ways to get to $(i,j)$, either from $(i-1,j)$, or from $(i,j-1)$:
   \[ B_{i,j} = B_{i-1,j} + B_{i,j-1} \]

2. There is only one way to get to a point on an axis: $B_{i,0} = B_{0,j} = 1$

$\blacktriangle$ These two rules completely determine all the numbers $B_{i,j}$

```
: (I) Generate data:
1| 7   28  84  210  462  924
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1| 3   6   10  15   21   28
1| 2   3   4   5   6    7
1| 1   1   1   1   1    1
```

(II) Guess:
\[ B_{i,j} = \frac{(i+j)!}{i!j!} \]
Guess-and-Prove: a toy example

Question: Find $B_{i,j}$ := the number of $\{\rightarrow, \uparrow\}$-walks in $\mathbb{N}^2$ from $(0,0)$ to $(i,j)$

1. There are 2 ways to get to $(i,j)$, either from $(i-1,j)$, or from $(i,j-1)$:

   $$B_{i,j} = B_{i-1,j} + B_{i,j-1}$$

2. There is only one way to get to a point on an axis: $B_{i,0} = B_{0,j} = 1$

> These two rules completely determine all the numbers $B_{i,j}$

(1) Generate data:


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(III) Prove: If $C_{i,j} \overset{\text{def}}{=} \frac{(i+j)!}{i!j!}$, then

$$\frac{C_{i-1,j}}{C_{i,j}} + \frac{C_{i,j-1}}{C_{i,j}} = \frac{i}{i+j} + \frac{j}{i+j} = 1$$

and $C_{i,0} = C_{0,j} = 1$.

Thus $B_{i,j} = C_{i,j}$
• $g(i, j; n)$ = number of $n$-steps $\{\uparrow, \downarrow, \leftarrow, \rightarrow\}$-walks in $\mathbb{N}^2$ from $(0, 0)$ to $(i, j)$

**Question:** What is the nature of the generating function

$$G(x, y; t) = \sum_{i,j,n=0}^{\infty} g(i, j; n) x^i y^j t^n$$

**Answer:** [B., Kauers, 2010] $G(x, y; t)$ is an algebraic function†.

**Approach:**

1. **Generate data:** compute $G$ to precision $t^{1200}$ ($\approx 1.5$ billion coeffs!)
2. **Guess:** conjecture polynomial equations for $G(x, 0; t)$ and $G(0, y; t)$ (degree 24 each, coeffs. of degree $(46, 56)$, with 80-bits digits coeffs.)
3. **Prove:** multivariate resultants of (very big) polynomials (30 pages each)

† Minimal polynomial $P(G(x, y; t); x, y, t) = 0$ has $> 10^{11}$ terms; $\approx 30$ Gb (6 DVDs!)
A typical Guess-and-Prove algorithmic proof

**Theorem** [“Gessel excursions are algebraic”]

\[ g(t) := G(0, 0; \sqrt{t}) = \sum_{n=0}^{\infty} \frac{(5/6)_n (1/2)_n}{(5/3)_n (2)_n} (16t)^n \]

is algebraic.
A typical Guess-and-Prove algorithmic proof

**Theorem** [“Gessel excursions are algebraic”]

\[ g(t) := G(0, 0; \sqrt{t}) = \sum_{n=0}^{\infty} \frac{(5/6)_n (1/2)_n}{(5/3)_n (2)_n} (16t)^n \]

is algebraic.

**Proof**: First guess a polynomial \( P(t, T) \) in \( \mathbb{Q}[t, T] \), then prove that \( P \) admits the power series \( g(t) = \sum_{n=0}^{\infty} g_n t^n \) as a root.
A typical Guess-and-Prove algorithmic proof

**Theorem** ["Gessel excursions are algebraic"]

\[ g(t) := G(0, 0; \sqrt{t}) = \sum_{n=0}^{\infty} \frac{(5/6)^n (1/2)^n}{(5/3)^n (2)^n} (16t)^n \] is algebraic.

**Proof:** First guess a polynomial \( P(t, T) \) in \( \mathbb{Q}[t, T] \), then prove that \( P \) admits the power series \( g(t) = \sum_{n=0}^{\infty} g_n t^n \) as a root.

1. Find \( P \) such that \( P(t, g(t)) = 0 \mod t^{100} \) by (structured) linear algebra.
A typical Guess-and-Prove algorithmic proof

**Theorem [“Gessel excursions are algebraic”]**

\[
g(t) := G(0, 0; \sqrt{t}) = \sum_{n=0}^{\infty} \frac{(5/6)_n (1/2)_n}{(5/3)_n (2)_n} (16t)^n
\]

is algebraic.

**Proof:** First guess a polynomial \( P(t, T) \) in \( \mathbb{Q}[t, T] \), then prove that \( P \) admits the power series \( g(t) = \sum_{n=0}^{\infty} g_n t^n \) as a root.

1. **Find** \( P \) such that \( P(t, g(t)) = 0 \mod t^{100} \) by (structured) linear algebra.

2. **Implicit function theorem:** \( \exists! \) root \( r(t) \in \mathbb{Q}[[t]] \) of \( P \).
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3. \( r(t) = \sum_{n=0}^{\infty} r_n t^n \) being algebraic, it is D-finite, and so \( (r_n) \) is P-recursive:

\[
(n + 2)(3n + 5)r_{n+1} - 4(6n + 5)(2n + 1)r_n = 0, \quad r_0 = 1
\]

\[ \implies \text{solution } r_n = \frac{(5/6)_n (1/2)_n}{(5/3)_n (2)_n} 16^n = g_n, \text{ thus } g(t) = r(t) \text{ is algebraic.} \]
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```plaintext
> P:=gfun:-listtoalgeq([seq(pochhammer(5/6,n)*pochhammer(1/2,n)/pochhammer(5/3,n)/pochhammer(2,n)*16^n, n=0..100)], g(t));
> gfun:-diffeqtorec(gfun:-algeqtodiffeq(P[1], g(t)), g(t), r(n));
```

Alin Bostan
Computer algebra for combinatorics
A typical Guess-and-Prove algorithmic proof

**Theorem** ["Gessel excursions are algebraic"]

\[ g(t) := G(0, 0; \sqrt{t}) = \sum_{n=0}^{\infty} \frac{(5/6)_n (1/2)_n}{(5/3)_n (2)_n} \frac{(16t)^n}{n!} \text{ is algebraic.} \]

**Proof:** First **guess** a polynomial \( P(t, T) \) in \( \mathbb{Q}[t, T] \), then **prove** that \( P \) admits the power series \( g(t) = \sum_{n=0}^{\infty} g_n t^n \) as a root.

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\( > \) \( \text{gfun:-diffeqtorec(gfun:-algeqtodiffeq(P[1], g(t)), g(t), r(n));} \)

\( \triangleright \) **Steps 1 & 3** rely on **polynomial linear algebra** (*Hermite-Padé approximants*).
Sur la généralisation des fractions continues algébriques.

(Par M. Ch. Hermite, membre de l'Institut, à Paris.)

[Extrait d'une lettre à M. Pincherle (*).]

...... Le problème que j'ai en vue est le suivant: Etant donné \( n \) séries \( S_1, S_2, \ldots, S_n \) procédant suivant les puissances d'une variable \( x \), déterminer les polynômes \( X_1, X_2, \ldots, X_n \) des degrés \( \mu_1, \mu_2, \ldots, \mu_n \) de manière à avoir

\[
S_1 X_1 + S_2 X_2 + \cdots + S_n X_n = S x^{\mu_1 + \mu_2 + \cdots + \mu_n + n - 1},
\]

où \( S \) est une série de même nature que \( S_1, S_2, \) etc. La question ainsi posée est entièrement déterminée, et une remarque de calcul intégral en donne la complète solution dans le cas particulier où les séries sont de simples exponentielles. C'est ce que je vais montrer, je me proposerai ensuite de faire sortir, en vue du cas général, les enseignements que contient cette solution.
Hermite-Padé approximants

[**Hermite, 1893**], [**Padé, 1894**]

*Sur la généralisation des fractions continues algébriques;*

**PAR M. H. PADÉ,**
Docteur ès Sciences mathématiques,
Professeur au lycée de Lille.

---

**INTRODUCTION.**

M. Hermite s’est, dans un travail récemment paru (*†*), occupé de la généralisation des fractions continues algébriques. La question est de déterminer les polynômes $X_1, X_2, \ldots, X_n$, de degrés $\nu_1, \nu_2, \ldots, \nu_n$, qui satisfont à l’équation

$$S_1 X_1 + S_2 X_2 + \ldots + S_n X_n = S_x^{\nu_1 + \nu_2 + \ldots + \nu_n + n - 1},$$

$S_1, S_2, \ldots, S_n$ étant des séries entières données, et $S$ une série égale-
ment entière. Ou plutôt, il s’agit d’*obtenir un algorithme* qui permette le calcul de proche en proche de ces systèmes de $n$ polynômes, et qui
Definition: A Hermite-Padé approximant of type $\mathbf{d} = (d_1, \ldots, d_n) \in \mathbb{N}^n$ for $\mathbf{F} = (f_1, \ldots, f_n) \in \mathbb{K}[x]^n$ is a $\mathbf{P} = (P_1, \ldots, P_n) \in \mathbb{K}[x]^n \setminus \{0\}$ such that:

1. $P_1 f_1 + \cdots + P_n f_n = O(x^\sigma)$ with $\sigma = \sum_i (d_i + 1) - 1$,
2. $\deg(P_i) \leq d_i$ for all $i$. 
**Hermite-Padé approximants**

**Definition:** A Hermite-Padé approximant of type \( \mathbf{d} = (d_1, \ldots, d_n) \in \mathbb{N}^n \) for \( \mathbf{F} = (f_1, \ldots, f_n) \in \mathbb{K}[[x]]^n \) is a \( \mathbf{P} = (P_1, \ldots, P_n) \in \mathbb{K}[x]^n \setminus \{0\} \) such that:

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2. \( \deg(P_i) \leq d_i \) for all \( i \).

▷ Very useful concept in number theory (irrationality/transcendence):
- [Hermite, 1873]: \( e \) is transcendent; [Lindemann, 1882]: \( \pi \) is transc.
- [Apéry, 1978; Beukers, 1981]: \( \zeta(3) = \sum_{n \geq 1} \frac{1}{n^3} \) is irrational;
  [Rivoal, 2000]: there are infinitely many \( k \) such that \( \zeta(2k + 1) \notin \mathbb{Q} \).

▷ Very useful tool in computer algebra
- **algebraic approximants** when \( f_\ell = A^{\ell-1} \) for a given \( A \in \mathbb{K}[[x]] \)
- **differential approximants** when \( f_\ell = A^{(\ell-1)} \) for a given \( A \in \mathbb{K}[[x]] \)
Hermite-Padé approximants

**Definition:** A Hermite-Padé approximant of type \( \mathbf{d} = (d_1, \ldots, d_n) \in \mathbb{N}^n \) for \( \mathbf{F} = (f_1, \ldots, f_n) \in \mathbb{K}[[x]]^n \) is a \( \mathbf{P} = (P_1, \ldots, P_n) \in \mathbb{K}[x]^n \setminus \{0\} \) such that:

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▷ **gfun** (Maple), **Guess.m** (Mathematica), **ore_algebra** (SageMath), etc.
### Algorithmic classification of models with D-finite $Q_\mathcal{S}(t) := Q_\mathcal{S}(1, 1; t)$

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Equation sizes = (order, degree)

▷ Computerized discovery: enumeration + guessing [B., Kauers, 2009]
Algorithmic classification of models with D-finite $Q_S(t) := Q_S(1, 1; t)$$\quad$

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Equation sizes = (order, degree)

- Computerized discovery: enumeration + guessing [B., Kauers, 2009]
- 23: DF confirmed by a human proof in [B., Kurkova, Raschel, 2017]
- All: explicit eqs. proved via CA [B., Chyzak, van Hoeij, Kauers, Pech, 2017]
Algorithmic classification of models with D-finite $Q(1, 1; t)$:

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$\lambda = 7 + 3\sqrt{6}, \mu = \sqrt{\frac{4\sqrt{6}-1}{19}}$

$\mu = \sqrt{\frac{3}{\pi} \frac{\sqrt{2\lambda^2/2} (2A)^n}{n^{3/2}}}$

$A = 1 + \sqrt{2}, B = 1 + \sqrt{3}, C = 1 + \sqrt{6}$

$\Gamma(1/4)$

▷ Computerized discovery: convergence acceleration + LLL [B., Kauers, ’09]

Alin Bostan
Computer algebra for combinatorics
### Algorithmic classification of models with D-finite $Q_\mathcal{F} (t) := Q_\mathcal{F} (1, 1; t)$

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- Computerized discovery: convergence acceleration + LLL [B., Kauers, ’09]
- Asympt. confirmed by human proofs via ACSV in [Melczer, Wilson, 2016]
- Transcendence proofs via CA [B., Chyzak, van Hoeij, Kauers, Pech, 2017]
Let $\mathcal{S}$ be one of the models 1–19. Then

- $Q_{\mathcal{S}}(x, y; t)$ is expressible using (integrals of) $2F_1$ expressions.
- $Q_{\mathcal{S}}(x, y; t)$ is transcendental.
Theorem [B., Chyzak, van Hoeij, Kauers, Pech, 2017]

Let $\mathcal{S}$ be one of the models 1–19. Then

- $Q_\mathcal{S}(t)$ is expressible using (integrals of) $\,_{2}F_{1}$ expressions.
- $Q_\mathcal{S}(t)$ is transcendental, except for $\mathcal{S} = \quad$ and $\mathcal{S} = \quad.$

Example (King walks in the quarter plane, A151331)

$$Q_\mathcal{S}(t) = \frac{1}{t} \int_0^t \frac{1}{(1 + 4x)^3} \cdot \,_{2}F_{1}\left(\begin{array}{c}
\frac{3}{2} \\
2
\end{array} \mid \frac{16x(1 + x)}{(1 + 4x)^2}\right) \, dx$$

$$= 1 + 3t + 18t^2 + 105t^3 + 684t^4 + 4550t^5 + 31340t^6 + 219555t^7 + \cdots$$

Computer-driven discovery and proof; no human proof yet.

Proof uses: (1) kernel method and (2) creative telescoping \[ \quad \quad \quad + \quad (3) \text{ODE factoring} \quad \quad \quad + \quad (4) \text{ODE solving}. \]
Theorem [B., Chyzak, van Hoeij, Kauers, Pech, 2017]

Let $\mathcal{S}$ be one of the models 1–19. Then
- $Q_{\mathcal{S}}(t)$ is expressible using (integrals of) $\,_{2}F_{1}$ expressions.
- $Q_{\mathcal{S}}(t)$ is transcendental, except for $\mathcal{S} = \bigtriangleup$ and $\mathcal{S} = \bigtriangledown$.

Example (King walks in the quarter plane, A151331)

\[
Q_{\bigtriangleup}(t) = \frac{1}{t} \int_{0}^{t} \frac{1}{(1 + 4x)^3} \cdot \,_{2}F_{1}\left(\frac{3}{2}, \frac{3}{2} \mid \frac{16x(1 + x)}{(1 + 4x)^2}\right) \, dx
\]

\[
= 1 + 3t + 18t^2 + 105t^3 + 684t^4 + 4550t^5 + 31340t^6 + 219555t^7 + \cdots
\]

▷ Computer-driven discovery and proof; no human proof yet.
▷ Proof uses: (1) kernel method and (2) creative telescoping
   + (3) ODE factoring and (4) ODE solving.
The kernel $K(x, y; t) := 1 - t \cdot \sum_{(i,j) \in \mathcal{S}} x^i y^j = 1 - t \left(x + \frac{1}{x} + y + \frac{1}{y}\right)$ is left invariant under the change of $(x, y)$ into the elements of $G_{\mathcal{S}} := \left\{ (x, y), \left(\frac{1}{x}, y\right), \left(\frac{1}{x}, \frac{1}{y}\right), (x, \frac{1}{y}) \right\}$
The kernel $K(x, y; t) := 1 - t \cdot \sum_{(i,j) \in \mathcal{S}} x^i y^j = 1 - t \left( x + \frac{1}{x} + y + \frac{1}{y} \right)$ is left invariant under the change of $(x, y)$ into the elements of

$$
\mathcal{G}_{\mathcal{S}} := \left\{ (x, y), \left( \frac{1}{x}, y \right), \left( \frac{1}{x}, \frac{1}{y} \right), (x, \frac{1}{y}) \right\}
$$

Kernel equation:

$$
K(x, y; t)xyQ(x, y; t) = xy - txQ(x, 0; t) - tyQ(0, y; t)
$$
(1) Kernel method [Bousquet-Mélou, Mishna, 2010]

The kernel $K(x, y; t) := 1 - t \cdot \sum_{(i,j) \in \mathcal{G}} x^i y^j = 1 - t \left( x + \frac{1}{x} + y + \frac{1}{y} \right)$

is left invariant under the change of $(x, y)$ into the elements of

$$\mathcal{G} := \left\{ (x, y), \left( \frac{1}{x}, y \right), \left( x, \frac{1}{y} \right) \right\}$$

Kernel equation:

$$K(x, y; t) xy Q(x, y; t) = xy - txQ(x, 0; t) - tyQ(0, y; t)$$

$$- K(x, y; t) \frac{1}{x} y Q \left( \frac{1}{x}, y; t \right) = - \frac{1}{x} y + t \frac{1}{x} Q \left( \frac{1}{x}, 0; t \right) + tyQ(0, y; t)$$
The kernel $K(x, y; t) := 1 - t \cdot \sum_{(i,j) \in \mathcal{S}} x^i y^j = 1 - t \left( x + \frac{1}{x} + y + \frac{1}{y} \right)$
is left invariant under the change of $(x, y)$ into the elements of

$$\mathcal{G}_\mathcal{S} := \left\{ (x, y), \left( \frac{1}{x}, y \right), \left( \frac{1}{x}, \frac{1}{y} \right), (x, \frac{1}{y}) \right\}$$

**Kernel equation:**

\[
K(x, y; t) x y Q(x, y; t) = xy - t x Q(x, 0; t) - t y Q(0, y; t)
\]

\[
- K(x, y; t) \frac{1}{x} y Q\left( \frac{1}{x}, y \right; t) = -\frac{1}{x} y + t \frac{1}{x} Q\left( \frac{1}{x}, 0 \right; t) + t y Q(0, y; t)
\]

\[
K(x, y; t) \frac{1}{x} \frac{1}{y} Q\left( \frac{1}{x}, \frac{1}{y} \right; t) = \frac{1}{x} \frac{1}{y} - t \frac{1}{x} Q\left( \frac{1}{x}, 0 \right; t) - t \frac{1}{y} Q(0, \frac{1}{y}; t)
\]
The kernel $K(x, y; t) := 1 - t \cdot \sum_{(i,j) \in \mathcal{S}} x^i y^j = 1 - t \left(x + \frac{1}{x} + y + \frac{1}{y}\right)$ is left invariant under the change of $(x, y)$ into the elements of $\mathcal{G}_\mathcal{S} := \left\{(x, y), \left(\frac{1}{x}, y\right), \left(\frac{1}{x}, \frac{1}{y}\right), (x, \frac{1}{y})\right\}$.

Kernel equation:

$$K(x, y; t)xyQ(x, y; t) = xy - txQ(x, 0; t) - tyQ(0, y; t)$$

$$- K(x, y; t)\frac{1}{x}yQ\left(\frac{1}{x}, y; t\right) = -\frac{1}{x}y + t\frac{1}{x}Q\left(\frac{1}{x}, 0; t\right) + tyQ(0, y; t)$$

$$K(x, y; t)\frac{1}{x}\frac{1}{y}Q\left(\frac{1}{x}, \frac{1}{y}; t\right) = \frac{1}{x} \frac{1}{y} - t\frac{1}{x}Q\left(\frac{1}{x}, 0; t\right) - t\frac{1}{y}Q(0, \frac{1}{y}; t)$$

$$- K(x, y; t)x\frac{1}{y}Q(x, \frac{1}{y}; t) = -x\frac{1}{y} + txQ(x, 0; t) + t\frac{1}{y}Q(0, \frac{1}{y}; t)$$
The kernel $K(x, y; t) := 1 - t \cdot \sum_{(i,j) \in \mathcal{S}} x^i y^j = 1 - t \left( x + \frac{1}{x} + y + \frac{1}{y} \right)$ is left invariant under the change of $(x, y)$ into the elements of

$$G_\mathcal{S} := \left\{ (x, y), \left( \frac{1}{x}, y \right), \left( \frac{1}{x}, \frac{1}{y} \right), (x, \frac{1}{y}) \right\}$$

Kernel equation:

$$K(x, y; t) x y Q(x, y; t) = x y - t x Q(x, 0; t) - t y Q(0, y; t)$$

$$- K(x, y; t) \frac{1}{x} y Q\left( \frac{1}{x}, y; t \right) = - \frac{1}{x} y + t \frac{1}{x} Q\left( \frac{1}{x}, 0; t \right) + t y Q(0, y; t)$$

$$K(x, y; t) \frac{1}{y} x Q\left( \frac{1}{x}, \frac{1}{y}; t \right) = \frac{1}{x} \frac{1}{y} - t \frac{1}{x} Q\left( \frac{1}{x}, 0; t \right) - t \frac{1}{y} Q(0, \frac{1}{y}; t)$$

$$- K(x, y; t) x \frac{1}{y} Q(x, \frac{1}{y}; t) = - x \frac{1}{y} + t x Q(x, 0; t) + t \frac{1}{y} Q(0, \frac{1}{y}; t)$$

Summing up yields the orbit equation:

$$\sum_{\theta \in G} (-1)^\theta \theta (x y Q(x, y; t)) = \frac{xy - \frac{1}{x} y + \frac{1}{x} \frac{1}{y} - x \frac{1}{y}}{K(x, y; t)}$$
The kernel $K(x, y; t) := 1 - t \cdot \sum_{(i,j) \in \mathcal{S}} x^i y^j = 1 - t \left( x + \frac{1}{x} + y + \frac{1}{y} \right)$
is left invariant under the change of $(x, y)$ into the elements of

$G_{\mathcal{S}} := \{(x, y), (\frac{1}{x}, y), (\frac{1}{x}, \frac{1}{y}), (x, \frac{1}{y})\}$

**Kernel equation:**

$$K(x, y; t)xyQ(x, y; t) = xy - txQ(x, 0; t) - tyQ(0, y; t)$$

$$- K(x, y; t)\frac{1}{x}yQ(\frac{1}{x}, y; t) = -\frac{1}{x}y + t\frac{1}{x}Q(\frac{1}{x}, 0; t) + tyQ(0, y; t)$$

$$K(x, y; t)\frac{1}{y}xQ(\frac{1}{x}, \frac{1}{y}; t) = \frac{1}{x}y - t\frac{1}{x}Q(\frac{1}{x}, 0; t) - t\frac{1}{y}Q(0, \frac{1}{y}; t)$$

$$- K(x, y; t)x\frac{1}{y}Q(x, \frac{1}{y}; t) = -x\frac{1}{y} + txQ(x, 0; t) + t\frac{1}{y}Q(0, \frac{1}{y}; t)$$

**Taking positive parts yields:**

$$[x^> y^>] \sum_{\theta \in G} (-1)^\theta \theta(xyQ(x, y; t)) = [x^> y^>] \frac{xy - \frac{1}{x}y + \frac{1}{x}y - x\frac{1}{y}}{K(x, y; t)}$$
The kernel $K(x,y;t) := 1 - t \cdot \sum_{(i,j) \in \mathcal{S}} x^i y^j = 1 - t \left( x + \frac{1}{x} + y + \frac{1}{y} \right)$
is left invariant under the change of $(x,y)$ into the elements of $G_{\mathcal{S}} := \left\{ (x,y), \left( \frac{1}{x}, y \right), \left( \frac{1}{x}, \frac{1}{y} \right), (x, \frac{1}{y}) \right\}$

Kernel equation:

\[
K(x,y;t)xyQ(x,y;t) = xy - txQ(x,0;t) - tyQ(0,y;t) \\
- K(x,y;t)\frac{1}{x} y Q\left( \frac{1}{x}, y; t \right) = -\frac{1}{x} y + t \frac{1}{x} Q\left( \frac{1}{x}, 0; t \right) + t y Q(0, y; t) \\
K(x,y;t)\frac{1}{y} x Q\left( x, \frac{1}{y}; t \right) = \frac{1}{y} x - t \frac{1}{y} Q\left( 1, \frac{1}{y}; t \right) - t \frac{1}{y} Q(0, \frac{1}{y}; t) \\
- K(x,y;t)x\frac{1}{y} Q(x, \frac{1}{y}; t) = -x \frac{1}{y} + txQ(x,0;t) + t \frac{1}{y} Q(0, \frac{1}{y}; t)
\]

Summing up and taking positive parts yields:

\[
xy Q(x,y;t) = [x^> y^>] \frac{xy - \frac{1}{x} y + \frac{1}{x} \frac{1}{y} - x \frac{1}{y}}{K(x,y;t)}
\]
The kernel $K(x,y;t) := 1 - t \cdot \sum_{(i,j) \in \mathcal{S}} x^i y^j = 1 - t \left( x + \frac{1}{x} + y + \frac{1}{y} \right)$
is left invariant under the change of $(x,y)$ into the elements of

$$\mathcal{G}_\mathcal{S} := \left\{ (x,y), \left( \frac{1}{x}, y \right), \left( \frac{1}{x}, \frac{1}{y} \right), (x, \frac{1}{y}) \right\}$$

Kernel equation:

$$K(x,y;t) x y Q(x,y;t) = x y - t x Q(x,0;t) - t y Q(0,y;t)$$

$$- K(x,y;t) \frac{1}{x} y Q\left( \frac{1}{x}, y \right; t) = - \frac{1}{x} y + t \frac{1}{x} Q\left( \frac{1}{x}, 0 \right; t) + t y Q(0,y;t)$$

$$- K(x,y;t) x \frac{1}{y} Q(x, \frac{1}{y};t) = - x \frac{1}{y} + t x Q(x,0;t) + t \frac{1}{y} Q(0, \frac{1}{y};t)$$

Summing up and taking positive parts yields:

$$G\text{Fun} = \text{PosPart} \left( \frac{\text{OrbitSum}}{\text{Kernel}} \right) = \iint \text{RatFrac}$$
The kernel $K(x, y; t) := 1 - t \cdot \sum_{(i,j) \in \mathcal{J}} x^i y^j = 1 - t \left( x + \frac{1}{x} + y + \frac{1}{y} \right)$
is left invariant under the change of $(x, y)$ into the elements of

$$\mathcal{G}_\mathcal{J} := \left\{ (x, y), \left( \frac{1}{x}, y \right), \left( \frac{1}{x}, \frac{1}{y} \right), (x, \frac{1}{y}) \right\}$$

Kernel equation:

$$K(x, y; t) xy Q(x, y; t) = xy - tx Q(x, 0; t) - ty Q(0, y; t)$$

$$- K(x, y; t) \frac{1}{x} y Q\left( \frac{1}{x}, y; t \right) = -\frac{1}{x} y + t \frac{1}{x} Q\left( \frac{1}{x}, 0; t \right) + ty Q(0, y; t)$$

$$K(x, y; t) \frac{1}{y} x Q\left( x, \frac{1}{y}; t \right) = \frac{1}{1} \frac{1}{y} x Q\left( \frac{1}{x}, \frac{1}{y}; t \right) - t \frac{1}{y} x Q(0, \frac{1}{y}; t)$$

$$- K(x, y; t) x \frac{1}{y} Q(x, \frac{1}{y}; t) = -x \frac{1}{y} + tx Q(x, 0; t) + t \frac{1}{y} Q(0, \frac{1}{y}; t)$$

$$\text{GF} = \text{PosPart} \left( \frac{\text{OS}}{\text{Ker}} \right) \text{ is D-finite \cite{Lipshitz, 1988}}$$
The kernel \( K(x, y; t) := 1 - t \cdot \sum_{(i,j) \in \mathcal{S}} x^i y^j = 1 - t \left( x + \frac{1}{x} + y + \frac{1}{y} \right) \) is left invariant under the change of \((x, y)\) into the elements of \( \mathcal{G}_\mathcal{S} := \{ (x, y), (\frac{1}{x}, y), (\frac{1}{x}, \frac{1}{y}), (x, \frac{1}{y}) \} \)

Kernel equation:

\[
\begin{align*}
K(x, y; t) xy Q(x, y; t) &= xy - tx Q(x, 0; t) - ty Q(0, y; t) \\
- K(x, y; t) \frac{1}{x} y Q\left(\frac{1}{x}, y; t\right) &= -\frac{1}{x} y + t \frac{1}{x} Q\left(\frac{1}{x}, 0; t\right) + ty Q(0, y; t) \\
K(x, y; t) \frac{1}{y} x Q\left(x, \frac{1}{y}; t\right) &= \frac{1}{y} - t \frac{1}{y} Q\left(0, \frac{1}{y}; t\right) - t \frac{1}{y} Q\left(0, \frac{1}{y}; t\right) \\
- K(x, y; t) x \frac{1}{y} Q(x, \frac{1}{y}; t) &= -\frac{1}{y} + tx Q(x, 0; t) + t \frac{1}{y} Q(0, \frac{1}{y}; t)
\end{align*}
\]

\( \text{GF} = \text{PosPart} \left( \frac{\text{OS}}{\text{Ker}} \right) \) is D-finite [Lipshitz, 1988]

▷ Argument works if \( \text{OS} \neq 0 \): algebraic version of the reflection principle
The kernel \( K(x, y; t) := 1 - t \cdot \sum_{(i,j) \in \mathcal{S}} x^i y^j = 1 - t \left( x + \frac{1}{x} + y + \frac{1}{y} \right) \) is left invariant under the change of \((x, y)\) into the elements of

\[
\mathcal{G}_\mathcal{S} := \left\{ (x, y), \left( \frac{1}{x}, y \right), \left( \frac{1}{x}, \frac{1}{y} \right), (x, \frac{1}{y}) \right\}
\]

Kernel equation:

\[
\begin{align*}
K(x, y; t)xyQ(x, y; t) &= xy - txQ(x, 0; t) - tyQ(0, y; t) \\
-K(x, y; t)\frac{1}{x}yQ\left( \frac{1}{x}, y; t \right) &= -\frac{1}{x}y + t\frac{1}{x}Q\left( \frac{1}{x}, 0; t \right) + tyQ(0, y; t) \\
K(x, y; t)\frac{1}{y}Q\left( \frac{1}{x}, \frac{1}{y}; t \right) &= \frac{1}{x}y - t\frac{1}{x}Q\left( \frac{1}{x}, 0; t \right) - t\frac{1}{y}Q(0, \frac{1}{y}; t) \\
-K(x, y; t)x\frac{1}{y}Q(x, \frac{1}{y}; t) &= -x\frac{1}{y} + txQ(x, 0; t) + t\frac{1}{y}Q(0, \frac{1}{y}; t)
\end{align*}
\]

\( \text{GF} = \text{PosPart} \left( \frac{\text{OS}}{\text{Ker}} \right) \) is D-finite [Lipshitz, 1988]

▷ Creative Telescoping finds a differential equation for \( \text{GF} = \text{RatFrac} \)
(2) Creative Telescoping

“An algorithmic toolbox for multiple sums and integrals with parameters”

Combinatorial Identities

H. W. Gould
(2) Creative Telescoping

"An algorithmic toolbox for multiple sums and integrals with parameters"

6. \[ \int \frac{1}{x^2 + y^2 + z^2 + x^2 - y^2 - 4z^2} \, dx = \frac{(2x^2 + 4y^2 + 4z^2)^2}{16y^4 + 16z^2 + 16} \left( \frac{a^2}{4y^2 + 16z^2} \right) \] 

7. \[ \int \frac{1}{x^2 + y^2 + z^2 + y^2 - 4z^2} \, dx = \frac{(2x^2 + 4y^2 + 4z^2)^2}{16y^4 + 16z^2 + 16} \left( \frac{a^2}{4y^2 + 16z^2} \right) \]

8. \[ \int \frac{1}{x^2 + y^2 + z^2 + y^2 - 4z^2} \, dx = \frac{(2x^2 + 4y^2 + 4z^2)^2}{16y^4 + 16z^2 + 16} \left( \frac{a^2}{4y^2 + 16z^2} \right) \]

9. \[ \int \frac{1}{x^2 + y^2 + z^2 + y^2 - 4z^2} \, dx = \frac{(2x^2 + 4y^2 + 4z^2)^2}{16y^4 + 16z^2 + 16} \left( \frac{a^2}{4y^2 + 16z^2} \right) \]

10. \[ \int \frac{1}{x^2 + y^2 + z^2 + y^2 - 4z^2} \, dx = \frac{(2x^2 + 4y^2 + 4z^2)^2}{16y^4 + 16z^2 + 16} \left( \frac{a^2}{4y^2 + 16z^2} \right) \]

11. \[ \int \frac{1}{x^2 + y^2 + z^2 + y^2 - 4z^2} \, dx = \frac{(2x^2 + 4y^2 + 4z^2)^2}{16y^4 + 16z^2 + 16} \left( \frac{a^2}{4y^2 + 16z^2} \right) \]

12. \[ \int \frac{1}{x^2 + y^2 + z^2 + y^2 - 4z^2} \, dx = \frac{(2x^2 + 4y^2 + 4z^2)^2}{16y^4 + 16z^2 + 16} \left( \frac{a^2}{4y^2 + 16z^2} \right) \]

13. \[ \int \frac{1}{x^2 + y^2 + z^2 + y^2 - 4z^2} \, dx = \frac{(2x^2 + 4y^2 + 4z^2)^2}{16y^4 + 16z^2 + 16} \left( \frac{a^2}{4y^2 + 16z^2} \right) \]

3.2.12. Integrals containing orthogonal polynomials.

1. \[ \int \frac{1}{(1 - \mu) a + x} \, dx = \frac{\mu a}{1 - \mu} + \frac{\mu a}{1 - \mu} \ln \left| \frac{1 - \mu}{1 - \mu} \right| \]

2. \[ \int \frac{1}{(1 - \mu) a^2 + x^2} \, dx = \frac{\mu a^2}{1 - \mu} + \frac{\mu a^2}{1 - \mu} \ln \left| \frac{1 - \mu}{1 - \mu} \right| \]

3. \[ \int \frac{1}{(1 - \mu) a^2 + x^2} \, dx = \frac{\mu a^2}{1 - \mu} + \frac{\mu a^2}{1 - \mu} \ln \left| \frac{1 - \mu}{1 - \mu} \right| \]

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(2) Creative Telescoping

“An algorithmic toolbox for multiple sums and integrals with parameters”

Example [Apéry 1978]: \( A_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 \) satisfies the recurrence

\[
(n + 1)^3 A_{n+1} + n^3 A_{n-1} = (2n + 1) (17n^2 + 17n + 5) A_n.
\]

▷ Key fact used to prove that \( \zeta(3) := \sum_{n \geq 1} \frac{1}{n^3} \approx 1.202056903 \ldots \) is irrational.

1. Journées Arithmétiques de Marseille-Luminy, June 1978

The board of programme changes informed us that R. Apéry (Caen) would speak Thursday, 14.00 “Sur l’irrationalité de \( \zeta(3) \).” Though there had been earlier rumours of his claiming a proof, scepticism was general. The lecture tended to strengthen this view to rank disbelief. Those who listened casually, or who were afflicted with being non-Francophone, appeared to hear only a sequence of unlikely assertions.

[Van der Poorten, 1979: “A proof that Euler missed”]

7. ICM ’78, Helsinki, August 1978

Neither Cohen nor I had been able to prove (5) or (5) in the intervening 2 months. After a few days of fruitless effort the specific problem was mentioned to Don Zagier (Bonn), and with irritating speed he showed that indeed the sequence \( \{b'_n\} \) satisfies the recurrence (4). This more or less broke the dam and (5) and (5) were quickly conquered. Henri Cohen addressed a very well-attended meeting at 17.00 on Friday, August 18 in the language of the majority, proving (5) and explaining how this implied the...
(2) Creative Telescoping

“The algorithmic toolbox for multiple sums and integrals with parameters”

Example [Apéry 1978]: \( A_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 \) satisfies the recurrence

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(2) Creative Telescoping

“An algorithmic toolbox for multiple sums and integrals with parameters”

Example [Euler, 1733]: Perimeter of an ellipse of eccentricity $e$, semi-major axis 1

$$p(e) = 4 \int_0^1 \sqrt{\frac{1 - e^2 u^2}{1 - u^2}} \, du = 4 \iint \frac{dxdy}{1 - \frac{1-e^2x^2}{(1-x^2)y^2}}$$

Principle: Find algorithmically
(2) Creative Telescoping

“An algorithmic toolbox for multiple sums and integrals with parameters”

Example [Euler, 1733]: Perimeter of an ellipse of eccentricity $e$, semi-major axis 1

$$p(e) = 4 \int_0^1 \sqrt{\frac{1 - e^2 u^2}{1 - u^2}} \, du = 4 \iint \frac{dxdy}{1 - \frac{1-e^2x^2}{(1-x^2)y^2}}$$

**Principle:** Find algorithmically

$$\left( (e - e^3) \frac{\partial^2}{\partial e^2} + (1 - e^2) \frac{\partial}{\partial e} + e \right) \left( \frac{1}{1 - \frac{1-e^2x^2}{(1-x^2)y^2}} \right) =$$

$$\partial_x \left( \frac{e(1+x-x^2-x^3) y^2 (2x-3+y^2+x^2(3e^2-y^2-2))}{(y^2+x^2(e^2-y^2)-1)^2} \right) + \partial_y \left( \frac{2e(e^2-1)x(1+x^3)y^3}{(y^2+x^2(e^2-y^2)-1)^2} \right)$$

▷ **Conclusion:**

$$\left( (e - e^3) \cdot p''(e) + (1 - e^2) \cdot p'(e) + e \cdot p(e) \right) = 0.$$
(2) Creative Telescoping

“An algorithmic toolbox for multiple sums and integrals with parameters”

Example [Euler, 1733]: Perimeter of an ellipse of eccentricity \( e \), semi-major axis 1

\[
p(e) = 4 \int_0^1 \sqrt{\frac{1 - e^2 u^2}{1 - u^2}} \, du = 4 \iint \frac{dx \, dy}{1 - \frac{1-e^2 x^2}{(1-x^2)y^2}}
\]

**Principle:** Find algorithmically

\[
\left( (e - e^3) \partial_e^2 + (1 - e^2) \partial_e + e \right) \left( \frac{1}{1 - \frac{1-e^2 x^2}{(1-x^2)y^2}} \right) =
\]

\[
\partial_x \left( \frac{e (1+x-x^2-x^3)}{(y^2+x^2(e^2-y^2)-1)^2} \frac{(2x-3+y^2+x^2 (3e^2-y^2-2))}{(y^2+x^2(e^2-y^2)-1)^2} \right)
\]

\[
+ \partial_y \left( \frac{2e (e^2-1) x (1+x^3)}{(y^2+x^2(e^2-y^2)-1)^2} \frac{y^3}{(y^2+x^2(e^2-y^2)-1)^2} \right)
\]

▷ Conclusion: \( p(e) = \frac{\pi}{2} \cdot 2 F_1 \left( -\frac{1}{2}, \frac{1}{2} \left| e^2 \right. \right) = 2\pi - \frac{\pi}{2} e^2 - \frac{3\pi}{32} e^4 - \cdots \).
(2) Creative Telescoping

“An algorithmic toolbox for multiple sums and integrals with parameters”

Example [Euler, 1733]: Perimeter of an ellipse of eccentricity $e$, semi-major axis 1

$$p(e) = 4 \int_0^1 \sqrt{1 - e^2u^2} \, du = 4 \iint \frac{dx \, dy}{1 - \frac{1-e^2x^2}{(1-x^2)y^2}}$$

**Principle:** Find algorithmically

$$\left((e - e^3)\partial_e^2 + (1 - e^2)\partial_e + e\right) \left(\frac{1}{1 - \frac{1-e^2x^2}{(1-x^2)y^2}}\right) =$$

$$\partial_x \left(\frac{e(1+x-x^2-x^3)y^2(2x-3+y^2+x^2(3e^2-y^2-2))}{(y^2+x^2(e^2-y^2)-1)^2}\right)$$

$$+ \partial_y \left(\frac{2e(e^2-1)x(1+x^3)y^3}{(y^2+x^2(e^2-y^2)-1)^2}\right)$$

▷ **Drawback:** Size(Indicator) $\gg$ Size(telescoper).
(2) Creative Telescoping: several generations of algorithms

- **3G, combines 1G + 2G + linear algebra**: [Apagodu, Zeilberger, 2005], [Koutschan 2010], [Chen, Kauers 2012], [Chen, Kauers, Koutschan 2014]

▷ Advantages:
- 1G–3G: very general algorithms;
- 2G/3G algorithms are able to solve non-trivial applications.

▷ Drawbacks:
- 1G: slow;
- 2G: bad or unknown complexity;
- 1G and 3G: non-minimality of telescopers;
- 1G–3G: all compute (big) certificates.
(2) Creative Telescoping: several generations of algorithms

4G: roots in [Ostrogradsky, 1845], [Hermite, 1872] and [Picard, 1902]

Univariate:
- rational ∫: [B., Chen, Chyzak, Li, 2010]
- hyperexponential ∫: [B., Chen, Chyzak, Li, Xin, 2013]
- hypergeometric ∑: [Chen, Huang, Kauers, Li, 2015], [Huang, 2016]
- mixed ∫ + ∑: [B., Dumont, Salvy, 2016]
- algebraic ∫: [Chen, Kauers, Koutschan, 2016]
- D-finite Fuchsian ∫: [Chen, van Hoeij, Kauers, Koutschan, 2018]
- D-finite ∫: [B., Chyzak, Lairez, Salvy, 2018], [van der Hoeven, 2018]

Multiple:
- rational bivariate ∫∫: [Chen, Kauers, Singer, 2012], [Chen, Du, Kauers, 2021]
- rational: [B., Lairez, Salvy, 2013], [Lairez 2016]
- binomial sums: [B., Lairez, Salvy, 2017]

▷ Advantages:
- good complexity;
- minimality of telescopers;
- do not need to compute certificates;
- fast in practice.

▷ Drawback: not (yet) as general as 1G–3G algorithms.
Algorithm for the integration of rational functions [B., Lairez, Salvy, 2013]

- **Input:** $R(e, x)$ a rational function in $e$ and $x = x_1, \ldots, x_n$.
- **Output:** A linear ODE $T(e, \partial_e)y = 0$ satisfied by $y(e) = \int R(e, x)dx$.
- **Complexity:** $O(D^{8n+2})$, where $D = \deg R$.
- **Output size:** $T$ has order $\leq D^n$ in $\partial_e$ and degree $\leq D^{3n+2}$ in $e$.

▷ Roots in [B., Chen, Chyzak, Li, 2010] ($n = 1$).
▷ Relies on generalized Hermite reduction and polynomial linear algebra.
▷ Avoids the (costly) computation of certificates, of size $\Omega(D^{n^2/2})$.
▷ Previous algorithms: complexity (at least) doubly exponential in $n$.
▷ Highly non-trivial extension by [Lairez, 2016]: very efficient in practice.
**Theorem** [B., Chyzak, van Hoeij, Kauers, Pech, 2017]

Let $S$ be one of the models 1–19. Then

- $Q_{S}(t)$ is expressible using (integrals of) $\mathbf{2}_1$ expressions.
- $Q_{S}(t)$ is transcendental, except for $S = \begin{array}{c|c|c}
\vline & \vline & \\
\vline & ( & \\
\vline & \end{array}$ and $S = \begin{array}{c|c|c}
\vline & \vline & \\
\vline & ( & \\
\vline & \end{array}$.

**Example** (King walks in the quarter plane, A151331)

\[
Q_{\begin{array}{c|c|c}
\vline & \vline & \\
\vline & ( & \\
\vline & \end{array}}(t) = \frac{1}{t} \int_{0}^{t} \frac{1}{(1 + 4x)^3} \cdot \mathbf{2}_1 \left( \begin{array}{c|c|c}
\frac{3}{2} \vline & \frac{3}{2} \vline & \\
\vline & ( & \\
\vline & \end{array} \right) \frac{16x(1 + x)}{(1 + 4x)^2} \, dx
\]

\[
= 1 + 3t + 18t^2 + 105t^3 + 684t^4 + 4550t^5 + 31340t^6 + 219555t^7 + \cdots
\]

▷ Computer-driven discovery and proof; no human proof yet.
▷ Proof uses: (1) kernel method and (2) creative telescoping
  + (3) ODE factoring and (4) ODE solving.
Algorithmic transcendence proofs: blending Guess-and-Prove and CT

Theorem (Apéry’s power series is transcendental)

\[ f(t) = \sum_{n} A_n t^n, \quad \text{where} \quad A_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2, \quad \text{is transcendental.} \]
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**Proof:**

1. **Creative telescoping:**

   \[
   (n + 1)^3 A_{n+1} + n^3 A_{n-1} = (2n + 1) (17n^2 + 17n + 5) A_n, \quad A_0 = 1, \ A_1 = 5
   \]
**Algorithmic transcendence proofs**: blending Guess-and-Prove and CT

Theorem (Apéry’s power series is transcendental)

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Proof:

1. **Creative telescoping:**

   \[(n + 1)^3 A_{n+1} + n^3 A_{n-1} = (2n + 1) (17n^2 + 17n + 5) A_n, \quad A_0 = 1, \ A_1 = 5\]

2. **Conversion from recurrence to differential equation** \(L(f) = 0\), where

   \[ L = (t^4 - 34t^3 + t^2) \partial_t^3 + (6t^3 - 153t^2 + 3t) \partial_t^2 + (7t^2 - 112t + 1) \partial_t + t - 5 \]
Algorithmic transcendence proofs: blending Guess-and-Prove and CT

Theorem \((\text{Apéry’s power series is transcendental})\)

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f(t) = \sum_{n} A_n t^n, \quad \text{where } A_n = \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k}^2, \quad \text{is transcendental.}
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Proof:

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\]

3. **Guess-and-Prove:**

compute least-order \(L_f^\text{\text{min}}\) in \(\mathbb{Q}(t) \langle \partial_t \rangle\) such that \(L_f^\text{\text{min}}(f) = 0\)
Algorithmic transcendence proofs: blending Guess-and-Prove and CT

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\]

3. Guess-and-Prove:

compute least-order \( L_f^{\min} \) in \( \mathbb{Q}(t)\{\partial_t\} \) such that \( L_f^{\min}(f) = 0 \)

4. Basis of formal solutions of \( L_f^{\min} \) at \( t = 0 \):

\[
\left\{ 1 + 5t + O(t^2), \ \ln(t) + (5\ln(t) + 12)t + O(t^2), \ \ln(t)^2 + (5\ln(t)^2 + 24\ln(t))t + O(t^2) \right\}
\]

Conclusion: \( f \) is transcendental

† algebraic would imply a full basis of algebraic solutions for \( L_f^{\min} \). [Tannery, 1875]
Algorithmic transcendence proofs: blending Guess-and-Prove and CT

**Theorem** (Apéry’s power series is transcendental)

\[ f(t) = \sum_n A_n t^n, \quad \text{where } A_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2, \quad \text{is transcendental.} \]

Proof:

1. **Creative telescoping:**
   \[(n + 1)^3 A_{n+1} + n^3 A_{n-1} = (2n + 1) (17 n^2 + 17 n + 5) A_n, \quad A_0 = 1, \; A_1 = 5 \]

2. **Conversion from recurrence to differential equation** \( L(f) = 0 \), where
   \[ L = (t^4 - 34t^3 + t^2) \partial_t^3 + (6t^3 - 153t^2 + 3t) \partial_t^2 + (7t^2 - 112t + 1) \partial_t + t - 5 \]

3. **Guess-and-Prove:**
   compute least-order \( L^\text{min}_f \) in \( \mathbb{Q}(t) \langle \partial_t \rangle \) such that \( L^\text{min}_f(f) = 0 \)

4. **Basis of formal solutions** of \( L^\text{min}_f \) at \( t = 0 \):
   \[ \{ 1 + 5t + O(t^2), \; \ln(t) + (5 \ln(t) + 12)t + O(t^2), \; \ln(t)^2 + (5 \ln(t)^2 + 24 \ln(t))t + O(t^2) \} \]

5. **Conclusion:** \( f \) is transcendental

\[ f \text{ algebraic would imply a full basis of algebraic solutions for } L^\text{min}_f \quad \text{[Tannery, 1875].} \]
Summary: classification of walks with small steps in $\mathbb{N}^2$

$Q_{\mathcal{G}}$ is D-finite $\iff$ a certain group $G_{\mathcal{G}}$ is finite (!)

quadrant models $\mathcal{I}$: 79

$|G_{\mathcal{G}}| < \infty$: 23

orbits sum $= 0$: 4

Guess-and-Prove algebraic

orbits sum $\neq 0$: 19

Creative Telescoping

$|G_{\mathcal{G}}| = \infty$: 56

asymptotics + Gröbner Bases

D-finite

non-D-finite
Summary: classification of walks with small steps in $\mathbb{N}^2$

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    - algebraic
  - $\text{OS} \neq 0$: 19
    - kernel + CT
    - D-finite transc.
- $|G_{\mathcal{G}}| = \infty$: 56
  - $\exists$ decoupling*: 9
    - Tutte’s invariants
    - D-algebraic
  - $\nexists$ decoupling: 47
    - diff. Galois
    - D-transcendental

$\exists U \in \mathbb{Q}(x,t), V \in \mathbb{Q}(y,t)$ s.t. $U(x) + V(y) = xy$ on the kernel $K(x,y;t) = 0$. 

Alin Bostan  
Computer algebra for combinatorics
Summary: classification of walks with small steps in \( \mathbb{N}^2 \)

\[ Q \mathcal{I} \text{ is D-finite} \iff \text{a certain group } \mathcal{G}_\mathcal{I} \text{ is finite (!!)} \]

- Quadrant models \( \mathcal{I} \): 79
  - \( |\mathcal{G}_\mathcal{I}| < \infty \): 23
    - \( \text{OS} = 0: 4 \)
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Alin Bostan | Computer algebra for combinatorics
Summary: classification of walks with small steps in $\mathbb{N}^2$

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  - diff. Galois
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▷ Proofs use various tools: algebra, complex analysis, probability theory, differential Galois theory, computer algebra, etc.
Enumerative Combinatorics and Computer Algebra enrich one another

Classification of $Q(x,y;t)$ fully completed for 2D small-steps walks

Robust algorithmic methods, based on efficient algorithms:
- Guess-and-Prove
- Creative Telescoping

Brute-force and/or use of naive algorithms = hopeless.
E.g. size of algebraic equations for $G(x,y;t) \approx 30$Gb.
Enumerative Combinatorics and Computer Algebra enrich one another

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Lack of “purely human” proofs for some results.

Many beautiful open questions for 2D walk models with repeated or large steps, and in different cones, and in dimension $> 2$. 
Thanks for your attention!


Bonus
Beyond dimension 2: walks with small-steps in $\mathbb{N}^3$

$2^{3^3-1} \approx 67$ million models, of which $\approx 11$ million inherently 3D

3D octant models $\mathcal{I}$ with $\leq 6$ steps: 20804

- $|\mathcal{G}_{\mathcal{I}}| < \infty$: 170
  - orbit sum $\neq 0$: 108
    - Creative Telescoping: D-finite
  - orbit sum $= 0$: 62
    - 2D-reducible: 43
    - not 2D-reducible: 19
- $|\mathcal{G}_{\mathcal{I}}| = \infty$: 20634
  - non-D-finite?

Question: differential finiteness $\iff$ finiteness of the group?
Answer: probably no

References:
[B., Bousquet-Mélou, Kauers, Melczer, 2016] + [Du, Hou, Wang, 2017]; completed by [Bacher, Kauers, Yatchak, 2016]
Beyond dimension 2: walks with small-steps in $\mathbb{N}^3$

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Question: differential finiteness $\iff$ finiteness of the group?

Answer: probably no
19 mysterious 3D-models: finite $\mathcal{G}$ and possibly non-D-finite $\mathcal{Q}$.
Open question: 3D Kreweras excursions

Numerical computations [Dahne, Salvy, 2020] suggest:

\[ k_{4n} = C \cdot 256^n / n^\alpha, \text{ for } \alpha = \frac{3.3257570041744 \ldots}{\notin \mathbb{Q}}, \]

so excursions are very probably non-D-finite
Beyond small steps: Walks in $\mathbb{N}^2$ with large steps

quadrant models with steps in $\{-2, -1, 0, 1\}^2$: 13 110

$|G_\mathcal{F}| < \infty$: 240

$|G_\mathcal{F}| = \infty$: 12 870

OS $\neq 0$: 431

OS = 0: 9

$\alpha$ rational: 16

$\alpha$ irrational: 12 854

D-finite

D-finite?

non-D-finite

non-D-finite

[B., Bousquet-Mélou, Melczer, 2021]

Question: differential finiteness $\iff$ finiteness of the group?

Answer: ?
Two challenging models with large steps

**Conjecture 1 [B., Bousquet-Mélou, Melczer, 2021]**

For the model the excursions generating function $Q(0, 0; t^{1/2})$ equals

$$\frac{1}{3t} - \frac{1}{6t} \cdot \left( \frac{1 - 12t}{(1 + 36t)^{1/3}} \right) \cdot 2F_1 \left( \frac{1}{6}, \frac{2}{3} \bigg| \frac{108t(1 + 4t)^2}{(1 + 36t)^2} \right) + \sqrt{1 - 12t} \cdot 2F_1 \left( -\frac{1}{6}, \frac{2}{3} \bigg| \frac{108t(1 + 4t)^2}{(1 - 12t)^2} \right).$$

**Conjecture 2 [B., Bousquet-Mélou, Melczer, 2021]**

For the model the excursions generating function $Q(0, 0; t)$ equals

$$\frac{(1 - 24U + 120U^2 - 144U^3)(1 - 4U)}{(1 - 3U)(1 - 2U)^{3/2}(1 - 6U)^{9/2}},$$

where $U = t^4 + 53t^8 + 4363t^{12} + \cdots$ is the unique series in $Q[[t]]$ satisfying

$$U (1 - 2U)^3 (1 - 3U)^3 (1 - 6U)^9 = t^4 (1 - 4U)^4.$$
Beyond the first quadrant: three-quadrant walks with small steps

three-quadrant models $\mathcal{I}$: 74

| $|G_\mathcal{I}| < \infty$: 23 |
|----------------------------------|
| D-finite?                        |
| 3 (4 ?)                          |
| algebraic                        |

| $|G_\mathcal{I}| = \infty$: 51 |
|---------------------------------|
| non-D-finite                    |
| 1 (9 ?)                         |
| D-algebraic                     |
| 3 (42 ?)                        |
| D-transcendental                |

▷ Partial classification due to
[Bousquet-Mélou, 2016], [Raschel, Trotignon, 2019],
[Mustapha, 2019], [Dreyfus, Trotignon, 2020],
[Bousquet-Mélou, Wallner, 2021], [Bousquet-Mélou, 2021]
A difficult quadrant model with repeated steps

Theorem [B., Bousquet-Mélou, Kauers, Melczer, 2016]

Let $a_n = \# \left\{ \text{walks of length } n \text{ in } \mathbb{N}^2 \text{ from } (0,0) \text{ to } (\star,0) \right\}$. Then $f(t) = \sum_n a_n t^n = 1 + t + 4 t^2 + 8 t^3 + 39 t^4 + 98 t^5 + \cdots$ is transcendental.
A difficult quadrant model with repeated steps

**Theorem** [B., Bousquet-Mélou, Kauers, Melczer, 2016]

Let \( a_n = \# \left\{ \begin{array}{c} \mathcal{X} \end{array} \right\} \) — walks of length \( n \) in \( \mathbb{N}^2 \) from \((0,0)\) to \((\star,0)\). Then

\[
f(t) = \sum_n a_n t^n = 1 + t + 4 t^2 + 8 t^3 + 39 t^4 + 98 t^5 + \cdots \text{ is transcendental.}
\]

**Proof:**

1. Discover and certify a differential equation \( L \) for \( f(t) \) of order 11 and degree 73
2. If \( \text{ord}(L_{f\min}) \leq 10 \), then \( \text{deg}_t(L_{f\min}) \leq 580 \)
3. Rule out this possibility
4. Thus, \( L_{f\min} = L \)
5. \( L \) has a log singularity at \( t = 0 \), and so \( f \) is transcendental \( \square \)
A difficult quadrant model with repeated steps

Theorem [B., Bousquet-Mélou, Kauers, Melczer, 2016]

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is transcendental.

Proof:

\begin{enumerate}
\item Discover and certify a differential equation \( L \) for \( f(t) \) of order 11 and degree 73 \hspace{1cm} \text{high-tech Guess-and-Prove apparent singularities}
\item If \( \text{ord}(L_f^{\text{min}}) \leq 10 \), then \( \text{deg}_t(L_f^{\text{min}}) \leq 580 \)
\item Rule out this possibility
\item Thus, \( L_f^{\text{min}} = L \)
\item \( L \) has a log singularity at \( t = 0 \), and so \( f \) is transcendental \hspace{0.5cm} \square
\end{enumerate}

▷ General minimization algorithm and application to transcendence

[B., Rivoal, Salvy, 2021]
Solution of the “exercise”
Excursions in $\mathbb{Z} \times \mathbb{N}$

• The kernel equation reads (with $K(x,y) = 1 - t(y + \bar{x} + x\bar{y})$):

$$K(x,y)yH(x,y) = y - txH(x,0)$$
The kernel equation reads (with \( K(x, y) = 1 - t(y + \bar{x} + x\bar{y}) \)):

\[
K(x, y) y H(x, y) = y - tx H(x, 0)
\]

Let

\[
y_0 = \frac{x - t - \sqrt{(t-x)^2 - 4t^2x^3}}{2tx}
\]

be the (unique) root in \( Q[x, \bar{x}][[t]] \) of \( K(x, y_0) = 0 \).
Excursions in $\mathbb{Z} \times \mathbb{N}$

• The kernel equation reads (with $K(x, y) = 1 - t(y + \bar{x} + x\bar{y})$):

$$K(x, y)yH(x, y) = y - txH(x, 0)$$

• Let

$$y_0 = \frac{x - t - \sqrt{(t - x)^2 - 4t^2x^3}}{2tx} = xt + t^2 + (x^2 + \bar{x})t^3 + (3x + \bar{x}^2)t^4 + \cdots$$

be the (unique) root in $\mathbb{Q}[x, \bar{x}][[t]]$ of $K(x, y_0) = 0$.

• Then

$$0 = K(x, y_0)yH(x, y_0) = y_0 - txH(x, 0),$$

thus

$$H(x, 0) = \frac{y_0}{tx} \quad \text{and} \quad A(t) = \left[ x^0 \right] \frac{y_0}{tx}.$$
Excursions in $\mathbb{Z} \times \mathbb{N}$

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$$0 = K(x, y_0)yH(x, y_0) = y_0 - txH(x,0),$$

thus

$$H(x, 0) = \frac{y_0}{tx} \quad \text{and} \quad A(t) = \left[ x^0 \right] \frac{y_0}{tx}.$$ 

- Creative telescoping then proves:

$$(27t^4 - t)A''(t) + (108t^3 - 4)A'(t) + 54t^2A(t) = 0.$$ 

> Zeilberger(1/x * sqrt((t-x)^2 - 4*t^2*x^3)/(2*t^2*x^2), t, x, Dt);
The group of the model \{↑, ←, ↘\}

Step set \( \mathcal{S} = \{(-1,0), (0,1), (1,-1)\} \), with characteristic polynomial

\[
\chi(x,y) = \frac{1}{x} + y + x \cdot \frac{1}{y} = \bar{x} + y + x\bar{y}
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\( \chi(x, y) \) is left unchanged by the rational transformations

\[
\Phi : (x, y) \mapsto (\bar{y}x, y) \quad \text{and} \quad \Psi : (x, y) \mapsto (x, x\bar{y}).
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The group of the model \( \{\uparrow, \leftarrow, \downarrow\} \)

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\]

\( \Phi \) and \( \Psi \) are involutions, and generate a finite dihedral group \( D_3 \) of order 6:

\[
\begin{align*}
\Phi & : (x, y) \mapsto (\bar{x}y, y) \\
\Psi & : (x, y) \mapsto (x, x\bar{y}) \\
\Phi & : (\bar{x}y, y) \mapsto (\bar{x}y, \bar{x}) \\
\Phi & : (\bar{x}y, \bar{x}) \mapsto (\bar{y}, \bar{x}) \\
\Psi & : (x, x\bar{y}) \mapsto (\bar{y}, x\bar{y}) \\
\Psi & : (\bar{y}, x\bar{y}) \mapsto (\bar{x})
\end{align*}
\]
Diagonal walks in $\mathbb{N}^2$

- Orbit equation:

$$xyQ(x, y) - \bar{x}y^2Q(\bar{x}y, y) + \bar{x}^2yQ(\bar{x}y, \bar{x})$$

$$- \bar{y}\bar{x}Q(\bar{y}, \bar{x}) + x\bar{y}^2Q(x, x\bar{y}) - x^2\bar{y}Q(x, x\bar{y}) =$$

$$\frac{xy - \bar{x}y^2 + \bar{x}^2y - \bar{x}\bar{y} + x\bar{y}^2 - x^2\bar{y}}{1 - t(y + \bar{x} + x\bar{y})}$$
Diagonal walks in $\mathbb{N}^2$

- Orbit equation:

$$xyQ(x, y) - \bar{x}y^2Q(\bar{y}, y) + \bar{x}^2yQ(\bar{y}, \bar{x})$$

$$- \bar{x}\bar{y}Q(\bar{y}, \bar{x}) + x\bar{y}^2Q(y, x\bar{y}) - x^2\bar{y}Q(x, x\bar{y}) = \frac{xy - \bar{x}y^2 + \bar{x}^2y - \bar{x}\bar{y} + x\bar{y}^2 - x^2\bar{y}}{1 - t(y + \bar{x} + x\bar{y})}$$

- Corollary [Bousquet-Mélou & Mishna, 2010]:

$$xyQ(x, y) = [x>0, y>0] \frac{xy - \bar{x}y^2 + \bar{x}^2y - \bar{x}\bar{y} + x\bar{y}^2 - x^2\bar{y}}{1 - t(y + \bar{x} + x\bar{y})}$$
Diagonal walks in $\mathbb{N}^2$

- **Orbit equation:**

\[
\begin{align*}
xyQ(x, y) - \bar{x}y^2Q(\bar{x}y, y) + \bar{x}^2yQ(\bar{x}y, \bar{x}) \\
- \bar{x}\bar{y}Q(\bar{y}, \bar{x}) + x\bar{y}^2Q(\bar{y}, x\bar{y}) - x^2\bar{y}Q(x, x\bar{y}) = \\
\frac{xy - \bar{x}y^2 + \bar{x}^2y - \bar{x}\bar{y} + x\bar{y}^2 - x^2\bar{y}}{1 - t(y + \bar{x} + x\bar{y})}
\end{align*}
\]

- **Corollary [Bousquet-Mélou & Mishna, 2010]:**

\[
xyQ(x, y) = [x^{>0}y^{>0}] \frac{xy - \bar{x}y^2 + \bar{x}^2y - \bar{x}\bar{y} + x\bar{y}^2 - x^2\bar{y}}{1 - t(y + \bar{x} + x\bar{y})}
\]

- **Corollary [B.-Chyzak-van Hoeij-Kauers-Pech, 2017]:**

\[
B(t) = [z^0]Q(z, \bar{z}) = [u^{-1}v^{-1}z^{-1}] \frac{\bar{u}\bar{v} - u\bar{v}^2 + u^2\bar{v} - uv + \bar{u}v^2 - \bar{u}^2v}{z(1 - zu)(1 - v\bar{z})(1 - t(\bar{v} + u + \bar{u}v))}
\]
Diagonal walks in $\mathbb{N}^2$

- Orbit equation:
  \[
  xyQ(x, y) - \bar{x}y^2Q(\bar{x}y, y) + \bar{x}^2yQ(\bar{x}y, \bar{x}) \\
  - \bar{y}yQ(\bar{y}, \bar{x}) + x\bar{y}^2Q(\bar{y}, x\bar{y}) - x^2\bar{y}Q(x, x\bar{y}) = \\
  \frac{xy - \bar{x}y^2 + \bar{x}^2y - \bar{x}\bar{y} + x\bar{y}^2 - x^2\bar{y}}{1 - t(y + \bar{x} + x\bar{y})}
  \]

- Corollary [Bousquet-Mélou & Mishna, 2010]:
  \[
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  \]

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  \]

- Creative Telescoping gives a differential equation for $B(t)$:
  \[
  (27t^4 - t)B''(t) + (108t^3 - 4)B'(t) + 54t^2B(t) = 0.
  \]
Conclusion

We have proved that $A(t)$ and $B(t)$ are both solutions of

$$(27t^4 - t)y''(t) + (108t^3 - 4)y'(t) + 54t^2y(t) = 0.$$  

Solving this equation proves:

$$A(t) = B(t) = 2F_1\left(\frac{1}{3}, \frac{2}{3} \mid 27t^3\right) = \sum_{n=0}^{\infty} \frac{(3n)!}{n!^3} \frac{t^{3n}}{n+1}. $$

Thus the two sequences are equal to

$$a_{3n} = b_{3n} = \frac{(3n)!}{n!^2 \cdot (n+1)!}, \quad \text{and} \quad a_m = b_m = 0 \quad \text{if 3 does not divide } m.$$
Example with infinite group: the scarecrows

[B., Raschel, Salvy, 2014]: $Q_{\mathcal{G}}(0,0;t)$ is not D-finite for the models

For the 1st and the 3rd, the excursions sequence $[t^n] Q_{\mathcal{G}}(0,0;t)$

$$1, 0, 0, 2, 4, 8, 28, 108, 372, \ldots$$

is $\sim K \cdot 5^n \cdot n^{-\alpha}$, with $\alpha = 1 + \pi/ \arccos(1/4) = 3.383396 \ldots$

[Denisov, Wachtel, 2015]

The irrationality of $\alpha$ prevents $Q_{\mathcal{G}}(0,0;t)$ from being D-finite.

[Katz, 1970; Chudnovsky, 1985; André, 1989]
The characteristic polynomial \( \chi_S := x + \frac{1}{x} + y + \frac{1}{y} \)
The characteristic polynomial \( \chi_{\mathcal{S}} := x + \frac{1}{x} + y + \frac{1}{y} \) is left invariant under

\[
\psi(x,y) = \left( x, \frac{1}{y} \right), \quad \phi(x,y) = \left( \frac{1}{x}, y \right),
\]
The group of a model: the simple walk case

The characteristic polynomial \( \chi_S := x + \frac{1}{x} + y + \frac{1}{y} \) is left invariant under

\[
\psi(x, y) = \left( x, \frac{1}{y} \right), \quad \phi(x, y) = \left( \frac{1}{x}, y \right),
\]

and thus under any element of the group

\[
\langle \psi, \phi \rangle = \left\{ (x, y), \left( x, \frac{1}{y} \right), \left( \frac{1}{x}, \frac{1}{y} \right), \left( \frac{1}{x}, y \right) \right\}.
\]
The group of a model

The generating polynomial $\chi_{\mathcal{S}} := \sum_{(i,j) \in \mathcal{S}} x^i y^j = \sum_{i=-1}^{1} B_i(y) x^i = \sum_{j=-1}^{1} A_j(x) y^j$
The group of a model

The generating polynomial $\chi_{S} := \sum_{(i,j) \in S} x^i y^j = \sum_{i=-1}^{1} B_i(y)x^i = \sum_{j=-1}^{1} A_j(x)y^j$

is left invariant under the birational involutions

$$\psi(x, y) = \left(x, \frac{A_{-1}(x)}{A_1(x)} \frac{1}{y} \right), \quad \phi(x, y) = \left(\frac{B_{-1}(y)}{B_1(y)} \frac{1}{x}, y \right),$$

and thus under any element of the (dihedral) group

$$\mathcal{G}_S := \langle \psi, \phi \rangle.$$
Examples of groups

Order 4,
Examples of groups

Order 4,

order 6,
Examples of groups

Order 4,

order 6,

order 8,
Examples of groups

Order 4,

Order 6,

Order 8,

Order $\infty$. 
Examples of groups

Order 4,

\[
x, y
\]

\[
x, \frac{x}{y}
\]

\[
\frac{y}{x}, y
\]

\[
\frac{y}{x}, \frac{1}{x}
\]

\[
\frac{1}{y}, \frac{1}{x}
\]

order 6,

order 8,

order \( \infty \).