Computer algebra for combinatorics

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Exercise from Jan-20 lecture (“Binomial sums”)

Give (and prove!) a simple formula for

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{2k}{n}$$

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```
> T:=(-1)^k*binomial(n,k)*binomial(2*k,n):
> first_terms:=[seq(add(T, k=0..n), n=0..6)]:
> guess_rec:=gfun:-listtorec(first_terms, u(n))[1];
```

$$\{u(n+1) + 2u(n) = 0, \quad u(0) = 1\}$$

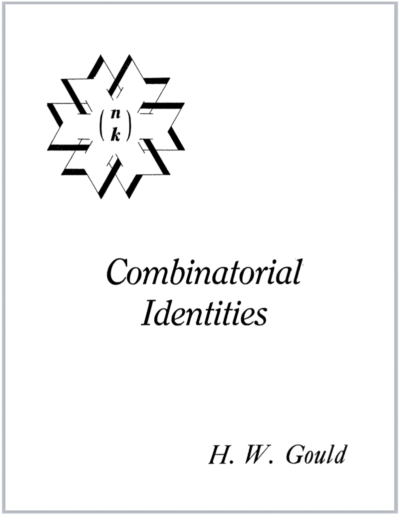
```
> rsolve(guess_rec,u(n));
```

$$(-2)^n$$

- ▷ Is this a proof?
- ▷ Can it be turned into a proof?
- ▷ Is this guessing procedure always guaranteed to work?

Give (and prove!) a simple formula for

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{2k}{n}$$



Combinatorial Identities

A STANDARDIZED SET OF TABLES

LISTING 500 BINOMIAL COEFFICIENT SUMMATIONS

"*Scientia non habet inimicum nisi ignorantiam*"

HENRY W. GOULD

Professor of Mathematics

West Virginia University

Revised Edition

Morgantown, W. Va. 1972

Give (and prove!) a simple formula for

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{2k}{n}$$

▷ Table look-up:

(3.64) $\displaystyle\sum_{k=0}^{n} (-1)^k \binom{n}{k}\binom{2k}{j} = (-1)^n \binom{n}{j-n} 2^{2n-j}$

(3.150) $\displaystyle\sum_{k=0}^{n} (-1)^k \binom{n}{k}\binom{x+kz}{j} = \begin{cases} 0, & 0 \leq j < n, \\ (-1)^n z^n, & j = n, \end{cases}$

▷ Is this a proof?

▷ Can it be turned into a proof?

▷ Is this guessing procedure always guaranteed to work?

Give (and prove!) a simple formula for

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{2k}{n}$$

```
> T:=(-1)^k*binomial(n,k)*binomial(2*k,n):
> sum(T, k=0..n);
```

$$(-2)^n$$

▷ Is this a proof?

▷ Is it always guaranteed to work?

▷ What is behind this proof?

Give (and prove!) a simple formula for $\sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{2k}{n}$

```
> T:=(-1)^k*binomial(n,k)*binomial(2*k,n):
> Zpair:=SumTools[Hypergeometric][Zeilberger](T, n, k, Sn):
> tel:=Zpair[1];
```

$$S_n + 2$$

```
> cert:=Zpair[2];
```

$$\frac{(2k-n-1)(2k-n)(-1)^k \binom{n}{k}\binom{2k}{n}}{(-n+k-1)(n+1)}$$

```
> is_zero:=(subs(n=n+1,T) + 2*T) - (subs(k=k+1,cert) - cert):
> simplify(convert(is_zero,GAMMA));
```

$$0$$

Give (and prove!) a simple formula for $\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{2k}{n}$

```
> with(BinomSums):
> T2:=(-1)^k * Binomial2(n,k)*Binomial2(2*k,n):
> S := Sum(t^n*Sum(T2, k=0..infinity), n=0..infinity):
> series(BinomSums[computesum](S, 5), t);
```

$$1-2t+4t^2-8t^3+16t^4-32t^5+O(t^6)$$

```
> R, ord := BinomSums[sumtores](S, u);
```

$$\frac{1}{2t+1}, \quad [t]$$

Give (and prove!) a simple formula for $\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{2k}{n}$

▷ What really happened in the previous slide?

▷ The algorithm started from the pre-tabulated formulas

$$\binom{n}{k} = \operatorname{res}_u \frac{(1+u)^n}{u^{k+1}}, \quad \binom{2k}{n} = \operatorname{res}_v \frac{(1+v)^{2k}}{v^{n+1}}$$

▷ It then performed the summation $\sum_{n=0}^{\infty}\sum_{k=0}^{n}(-1)^k \cdot \frac{(1+u)^n}{u^{k+1}} \cdot \frac{(1+v)^{2k}}{v^{n+1}} \cdot t^n$

expressing the GF of the input binomial sum as the residue (w.r.t. u and v) of

$$R := \frac{\frac{1}{v\left(1-\frac{(1+u)t}{v}\right)} + \frac{(1+v)^2}{uv\left(1+\frac{(1+v)^2(1+u)t}{uv}\right)}}{v^2+u+2v+1}$$

▷ It finally performed a successive pole/residue analysis, proving that

$$\operatorname{res}_{u,v} R = \operatorname{res}_v \frac{v}{t\,v^3+2t\,v^2+v^2} = \frac{1}{2t+1}.$$

Up to $k \leftrightarrow n-k$, the identity is equivalent to $\sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{2n-2k}{n} = 2^n.$

▷ Let us re-write binomials as quotients of products of factorials

$$u_n = \sum_{k=0}^{n} (-1)^k \cdot \frac{n!}{k!(n-k)!} \cdot \frac{(2n-2k)!}{n!(n-2k)!} = \sum_{k=0}^{n} (-1)^k \cdot \frac{1}{k!} \cdot \frac{1}{(n-k)!} \cdot \frac{(2n-2k)!}{(n-2k)!}$$

and then in terms of "rising factorials" (or, "Pochhammer symbols") $(a)_n = a(a+1)\cdots(a+n-1)$, using the rewriting rules:

$$(n-k)! = \frac{(-1)^k \cdot n!}{(-n)_k} \quad \text{and} \quad (a)_{2k} = 4^k \cdot \left(\frac{a}{2}\right)_k \cdot \left(\frac{a+1}{2}\right)_k$$

▷ We get $u_n = \binom{2n}{n} \cdot \sum_{k=0}^{n} \frac{(-n)_k \cdot (-n)_{2k}}{(1)_k \cdot (-2n)_{2k}} = \binom{2n}{n} \cdot \sum_{k=0}^{n} \frac{\left(-\frac{n}{2}\right)_k \cdot \left(\frac{1-n}{2}\right)_k}{(1)_k \cdot \left(\frac{1-2n}{2}\right)_k}$

▷ We conclude using the "Chu-Vandermonde" hypergeometric identity that

$$u_n = \binom{2n}{n} \cdot {}_2F_1\left(\begin{matrix} -\frac{n}{2} \;\; \frac{1}{2} - \frac{n}{2} \\ \frac{1}{2} - n \end{matrix} \;\middle|\; 1\right) = 2^n.$$

▷ *"We reduced an identity to another identity: what's the point?"*

Up to $k \leftrightarrow n-k$, the identity is equivalent to $\sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{2n-2k}{n} = 2^n$.

▷ Consider the polynomial

$$P_n(x) := \frac{1}{n!} \cdot \frac{\partial^n}{\partial x^n} (x^2-1)^n$$

▷ By the Leibniz differentiation rule,

$$P_n(x) = \frac{1}{n!} \cdot \sum_{k=0}^{n} \binom{n}{k} \cdot \frac{\partial^k}{\partial x^k} (x+1)^n \cdot \frac{\partial^{n-k}}{\partial x^{n-k}} (x-1)^n,$$

hence $P_n(x) = \sum_{k=0}^{n} \binom{n}{k}^2 (x+1)^{n-k} (x-1)^k$, in particular $P_n(1) = 2^n$.

▷ By the binomial theorem, $(x^2-1)^n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} x^{2n-2k}$, hence

$$P_n(x) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{2n-2k}{n} x^{n-2k}.$$

▷ In conclusion, $\sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{2n-2k}{n} = P_n(1) = 2^n$.

We will prove that

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{2n-2k}{n} = 2^n$$

by counting subsets of $\{1, \ldots, n\}$ in the following way:

(1) There is an obvious bijection between subsets of $\{1, \ldots, n\}$ and subsets with n elements of the set $\{1, \ldots, n, \overline{1}, \ldots, \overline{n}\}$ which contain either k or $\overline{k}$

(2) Now to count the latter we can do inclusion/exclusion:

- $\binom{2n}{n}$ counts all n-element sets
- This counts too many, because it counts also subsets which contain both k and $\overline{k}$
- To delete those, we subtract $\binom{n}{1} \cdot \binom{2n-2}{n-2}$
- But this deletes too many, since it counts those who have k and $\overline{k}$ and ℓ and $\overline{\ell}$ twice
- Hence one adds $\binom{n}{2} \cdot \binom{2n-4}{n-4}$
- And so on.

Summary: ingredients of an experimental mathematics approach

- Trial and error phase
  - Guess the answer — Hermite-Padé approximants
  - Look up for possible generalizations — bibliographic searches
- Reasoning and proving phase – understand what's "inside the box"
  - Use built-in routines — computer algebra software
  - Use a specific summation approach — Zeilberger's algorithm
  - Use an alternative / better one — residues
- Bonus phase – attacking from different angles
  - Hypergeometric approach — Chu-Vandermonde identity
  - Direct algebraic approach — Legendre polynomials
  - Combinatorial approach — Bijection (most human creativity demanding)

▷ What is your favorite proof?

▷ Why? (Criteria: length/beauty/trickiness/naturalness)

Bonus: a ChatGPT "proof" (2024)

You

The question is: Find the sum over k from 0 to n of (-1)^k times n choose k times 2k choose n

ChatGPT

Sure, I can help you with that.

To find the sum $\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{2k}{n}$, we can use the principle of inclusion-exclusion.

Let's break it down:

1. Start with the binomial expansion: $(x+y)^n=\sum_{k=0}^{n}\binom{n}{k}x^{n-k}y^k$.
2. Substitute $x=-1$ and $y=1$ into the binomial expansion to get: $(-1+1)^n=\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}$.
3. Note that $(-1+1)^n=0$ for $n>0$ because it alternates between positive and negative values, and since the sum is to 0, it must be zero.
4. Now, let's use the principle of inclusion-exclusion to get rid of the $\binom{2k}{n}$ term. This is where things get a bit tricky.

The principle of inclusion-exclusion states that for two sets A and B, the cardinality of their union is given by:

$|A\cup B|=|A|+|B|-|A\cap B|$

Extending this to three sets, we have:

$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$

In our case, we have two sets: $A = \{k \text{ such that } k \leq n\}$ and $B = \{k \text{ such that } 2k \geq n\}$.

The set A corresponds to the sum $\sum_{k=0}^{n}(-1)^k \binom{n}{k}$, and B corresponds to the sum $\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor}(-1)^k \binom{2k}{n}$.

1. Using the principle of inclusion-exclusion, the expression becomes: $\sum_{k=0}^{n}(-1)^k \binom{n}{k} - \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor}(-1)^k \binom{2k}{n}$

Now, we have to handle each term separately.

The first term is $\sum_{k=0}^{n}(-1)^k \binom{n}{k}$, which we already know is zero.

The second term is $\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor}(-1)^k \binom{2k}{n}$. We can compute this using the formula for the binomial coefficient:
$\binom{2k}{n} = \frac{(2k)!}{n!(2k-n)!}$

1. Plug the value of $\binom{2k}{n}$ into the sum, and simplify it: $\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor}(-1)^k \frac{(2k)!}{n!(2k-n)!}$

Bonus: a ChatGPT "proof" (2024)

This expression is a bit complex and might not simplify nicely. It might be helpful to consider specific cases or further manipulation based on the value of n.

Let me know if you need further assistance!

Bonus: a ChatGPT "proof" (2024)

You

Can you simplify further? Ideally, an expression with no sum

ChatGPT

To simplify the expression $\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \frac{(2k)!}{n!(2k-n)!}$ further, we can rewrite the factorial terms in terms of binomial coefficients and then manipulate the expression to remove the summation.

First, recall that $(2k)! = (2k)(2k-1)(2k-2)\ldots(2)(1)$.

We can express $(2k)!$ as $(2k)! = (2k)(2k-1)(2k-2)\ldots(2)(1) = (2k) \times (2k-1) \times \ldots \times (k+1) \times k!$.

Now, we can rewrite the expression:

$$(-1)^k \frac{(2k)!}{n!(2k-n)!} = (-1)^k \frac{(2k)(2k-1)\ldots(k+1)k!}{n!(2k-n)!}$$

Using the definition of binomial coefficients, we can express $(2k)!$ and $(2k-n)!$ in terms of binomial coefficients:

$$(2k)! = (2k)(2k-1)\ldots(k+1)k! = \binom{2k}{k} k!$$
$$(2k-n)! = (2k-n)(2k-n-1)\ldots(2)(1) = \binom{2k-n}{0} (2k-n)!$$

Bonus: a ChatGPT "proof" (2024)

Substituting these into the expression:

$$(-1)^k \frac{\binom{2k}{k} k!}{n! \binom{2k-n}{0} (2k-n)!}$$

$$= (-1)^k \frac{\binom{2k}{k}}{\binom{2k-n}{0} n!}$$

$$= (-1)^k \frac{\binom{2k}{k}}{\binom{2k-n}{2k-n} n!}$$

$$= (-1)^k \frac{\binom{2k}{k}}{\binom{2k-n}{2k-n} \binom{n}{n-k}}$$

$$= (-1)^k \binom{2k}{k} \binom{n}{n-k} \binom{2k-n}{2k-n}^{-1}$$

Now, $\binom{2k-n}{2k-n}^{-1}$ is equivalent to $\binom{2k-n}{n-2k}$, so we can rewrite the expression as:

$$(-1)^k \binom{2k}{k} \binom{n}{n-k} \binom{2k-n}{n-2k}$$

This expression contains no summation. It gives the simplified form of the original summation expression.

Bonus: a ChatGPT "proof" (2024)

You

Can you find the sum over k from 0 to n of (-1)^k times n choose k times 2k choose n? The final result should be a simple expression of n

ChatGPT

Certainly! We'll approach this problem using a combinatorial identity and some algebraic manipulation.

The given sum is:

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{2k}{n}$$

First, let's rewrite the binomial coefficients in terms of factorials:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$
$$\binom{2k}{n} = \frac{(2k)!}{n!(2k-n)!}$$

Substitute these into the sum:

$$\sum_{k=0}^{n}(-1)^k\frac{n!}{k!(n-k)!}\frac{(2k)!}{n!(2k-n)!}$$

Now, let's simplify:

$$\sum_{k=0}^{n}(-1)^k \frac{n!}{k!(n-k)!}\frac{(2k)!}{n!(2k-n)!}$$
$$= \sum_{k=0}^{n}(-1)^k \frac{n!}{k!(n-k)!}\frac{(2k)(2k-1)(2k-2)\ldots(2)(1)}{n!(2k-n)(2k-n-1)\ldots(2)(1)}$$

$$= \sum_{k=0}^{n}(-1)^k \frac{n!}{k!(n-k)!}\frac{(2k)!}{n!(2k-n)!}$$
$$= \sum_{k=0}^{n}(-1)^k \frac{n!}{k!(n-k)!}\frac{(2k)!}{n!(2k-n)!}$$

$$= \sum_{k=0}^{n}(-1)^k \frac{n!}{k!(n-k)!}\frac{(2k)(2k-1)(2k-2)\ldots(2)(1)}{n!(2k-n)(2k-n-1)\ldots(2)(1)}$$

$$= \sum_{k=0}^{n}(-1)^k \frac{n!}{k!(n-k)!}\frac{(2k)!}{n!(2k-n)!}$$
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$$= \sum_{k=0}^{n}(-1)^k \frac{n!}{k!(n-k)!}\frac{(2k)!}{n!(2k-n)!}$$
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$$= \sum_{k=0}^{n}(-1)^k \frac{n!}{k!(n-k)!}\frac{(2k)!}{n!(2k-n)!}$$

Bonus: a ChatGPT “proof” (2024)

$$= \sum_{k=0}^{n} (-1)^k \frac{n!}{k!(n-k)!} \frac{(2k)(2k-1)(2k-2)\ldots(2)(1)}{n!(2k-n)(2k-n-1)\ldots(2)(1)}$$

$$= \sum_{k=0}^{n} (-1)^k \frac{n!}{k!(n-k)!} \frac{(2k)!}{n!(2k-n)!}$$

$$= \sum_{k=0}^{n} (-1)^k \frac{n!}{k!(n-k)!} \frac{(2k)(2k-1)(2k-2)\ldots(2)(1)}{n!(2k-n)(2k-n-1)\ldots(2)(1)}$$

$$= \sum_{k=0}^{n} (-1)^k \frac{n!}{k!(n-k)!} \frac{(2k)!}{n!(2k-n)!}$$

$$= \sum_{k=0}^{n} (-1)^k \frac{n!}{k!(n-k)!} \frac{(2k)(2k-1)(2k-2)\ldots(2)(1)}{n!(2k-n)(2k-n-1)\ldots(2)(1)}$$

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$$= \sum_{k=0}^{n} (-1)^k \frac{n!}{k!(n-k)!} \frac{(2k)(2k-1)(2k-2)\ldots(2)(1)}{n!(2k-n)(2k-n-1)\ldots(2)(1)}$$

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$$= \sum_{k=0}^{n} (-1)^k \frac{n!}{k!(n-k)!} \frac{(2k)!}{n!(2k-n)!}$$

$$= \sum_{k=0}^{n} (-1)^k \frac{n!}{k!(n-k)!} \frac{(2k)(2k-1)(2k-2)\ldots(2)(1)}{n!(2k-n)(2k-n-1)\ldots(2)(1)}$$

Bonus: a ChatGPT "proof" (2024)

You

There is still a sum involved in the answer, and I would like the answer to contain no sum, just a simple formula in terms of n

ChatGPT

Apologies for the oversight. Let's try to find a simplified expression without a sum.

We're looking at the expression:

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{2k}{n}$$

We know that:

$$\binom{n}{k}=\frac{n!}{k!(n-k)!}$$

$$\binom{2k}{n}=\frac{(2k)!}{n!(2k-n)!}$$

Let's simplify these expressions:

$$\binom{n}{k}=\frac{n(n-1)(n-2)\ldots(n-k+1)}{k!}$$

$$\binom{2k}{n}=\frac{(2k)(2k-1)(2k-2)\ldots(2k-n+1)}{n!}$$

Now, let's substitute these back into the original expression:

Bonus: a ChatGPT "proof" (2024)

$$\sum_{k=0}^{n}(-1)^k \frac{n(n-1)(n-2)\ldots(n-k+1)}{k!} \frac{(2k)(2k-1)(2k-2)\ldots(2k-n+1)}{n!}$$

We can simplify this further:

$$\frac{n(n-1)(n-2)\ldots(n-k+1)}{k!} = \frac{n!}{k!(n-k)!}$$

$$\frac{(2k)(2k-1)(2k-2)\ldots(2k-n+1)}{n!} = \frac{(2k)!}{n!(2k-n)!}$$

So, the expression becomes:

$$\sum_{k=0}^{n}(-1)^k \frac{n!}{k!(n-k)!} \frac{(2k)!}{n!(2k-n)!}$$

$$= \sum_{k=0}^{n}(-1)^k \binom{n}{k}\binom{2k}{n}$$

$$= \sum_{k=0}^{n}(-1)^k \frac{n!}{k!(n-k)!} \frac{(2k)!}{n!(2k-n)!}$$

$$= \sum_{k=0}^{n}(-1)^k \frac{n(n-1)(n-2)\ldots(n-k+1)}{k!} \frac{(2k)(2k-1)(2k-2)\ldots(2k-n+1)}{n!}$$

$$= \sum_{k=0}^{n}(-1)^k \frac{n!}{k!(n-k)!} \frac{(2k)!}{n!(2k-n)!}$$

$$= \sum_{k=0}^{n}(-1)^k \binom{n}{k}\binom{2k}{n}$$

$$= \sum_{k=0}^{n}(-1)^k \frac{n!}{k!(n-k)!} \frac{(2k)!}{n!(2k-n)!}$$

Bonus: a ChatGPT "proof" (2024)

$$= \sum_{k=0}^{n} (-1)^k \frac{n(n-1)(n-2)\dots(n-k+1)}{k!} \frac{(2k)(2k-1)(2k-2)\dots(2k-n+1)}{n!}$$

$$= \sum_{k=0}^{n} (-1)^k \frac{n!}{k!(n-k)!} \frac{(2k)!}{n!(2k-n)!}$$

$$= \sum_{k=0}^{n} (-1)^k \frac{n(n-1)(n-2)\dots(n-k+1)}{k!} \frac{(2k)(2k-1)(2k-2)\dots(2k-n+1)}{n!}$$

$$= \sum_{k=0}^{n} (-1)^k \frac{n!}{k!(n-k)!} \frac{(2k)!}{n!(2k-n)!}$$

$$= \sum_{k=0}^{n} (-1)^k \frac{n(n-1)(n-2)\dots(n-k+1)}{k!} \frac{(2k)(2k-1)(2k-2)\dots(2k-n+1)}{n!}$$

$$= \sum_{k=0}^{n} (-1)^k \frac{n!}{k!(n-k)!} \frac{(2k)!}{n!(2k-n)!}$$

$$= \sum_{k=0}^{n} (-1)^k \frac{n(n-1)(n-2)\dots(n-k+1)}{k!} \frac{(2k)(2k-1)(2k-2)\dots(2k-n+1)}{n!}$$

$$= \sum_{k=0}^{n} (-1)^k \frac{n!}{k!(n-k)!} \frac{(2k)!}{n!(2k-n)!}$$

$$= \sum_{k=0}^{n} (-1)^k \frac{n(n-1)(n-2)\dots(n-k+1)}{k!} \frac{(2k)(2k-1)(2k-2)\dots(2k-n+1)}{n!}$$

$$= \sum_{k=0}^{n} (-1)^k \frac{n!}{k!(n-k)!} \frac{(2k)!}{n!(2k-n)!}$$

$$= \sum_{k=0}^{n} (-1)^k \frac{n(n-1)(n-2)\dots(n-k+1)}{k!} \frac{(2k)(2k-1)(2k-2)\dots(2k-n+1)}{n!}$$

$$= \sum_{k=0}^{n} (-1)^k \frac{n!}{k!(n-k)!} \frac{(2k)!}{n!(2k-n)!}$$

Give (and prove!) a simple formula for
sum_{k=0}^n (-1)^k*binomial(n,k)*binomial(2k,n)

Thought for 61 seconds

To find a simple formula for the sum

$$S(n) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{2k}{n}$$

and prove it, let's first compute the values of $S(n)$ for small n to look for a pattern.

Computing Small Values:

For $n = 0$:

$$S(0) = (-1)^0 \binom{0}{0} \binom{0}{0} = 1 \times 1 \times 1 = 1$$

For $n = 1$:

$$S(1) = (-1)^0 \binom{1}{0} \binom{0}{1} + (-1)^1 \binom{1}{1} \binom{2}{1} = 0 + (-1) \times 1 \times 2 = -2$$

Bonus: a ChatGPT proof (2025)

For $n = 2$:

$$S(2) = (-1)^0 \binom{2}{0}\binom{0}{2} + (-1)^1 \binom{2}{1}\binom{2}{2} + (-1)^2 \binom{2}{2}\binom{4}{2} = 0 - 2 + 6 = 4$$

For $n = 3$:

$$S(3) = (-1)^0 \binom{3}{0}\binom{0}{3} + (-1)^1 \binom{3}{1}\binom{2}{3} + (-1)^2 \binom{3}{2}\binom{4}{3} + (-1)^3 \binom{3}{3}\binom{6}{3} = 0 + 0 + 12 - 20 = -8$$

For $n = 4$:

$$S(4) = (-1)^0 \binom{4}{0}\binom{0}{4} + (-1)^1 \binom{4}{1}\binom{2}{4} + (-1)^2 \binom{4}{2}\binom{4}{4} + (-1)^3 \binom{4}{3}\binom{6}{4} + (-1)^4 \binom{4}{4}\binom{8}{4} = 0 + 0 + 6 - 60 + 70 = 16$$

The values of $S(n)$ for $n = 0, 1, 2, 3, 4$ are $1, -2, 4, -8, 16$, respectively. Observing these values, we notice the pattern $S(n) = (-1)^n \times 2^n$.

Bonus: a ChatGPT proof (2025)

Proving the Formula:

Our goal is to prove that:

$$S(n) = (-1)^n \times 2^n$$

Proof Using Generating Functions:

1. **Consider the Sum as a Coefficient in a Generating Function:**

   We can write $\binom{2k}{n}$ as the coefficient of z^n in $(1+z)^{2k}$:

   $$\binom{2k}{n} = [z^n](1+z)^{2k}$$

   where $[z^n]$ denotes the coefficient of z^n.

2. **Express the Sum Using Coefficients:**

   The sum becomes:

   $$S(n) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} [z^n](1+z)^{2k}$$

   Swap the sum and the coefficient (since the sums are finite and coefficients are linear operators):

   $$S(n) = [z^n] \sum_{k=0}^{n} (-1)^k \binom{n}{k} (1+z)^{2k}$$

3. **Simplify the Summation:**

   Note that:

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}(1+z)^{2k} = \left[\sum_{k=0}^{n}\binom{n}{k}(-(1+z)^2)^k\right] = \left(1-(1+z)^2\right)^n$$

   Simplify the expression inside:

$$1-(1+z)^2 = 1-(1+2z+z^2) = -(2z+z^2)$$

   Therefore:

$$\left(1-(1+z)^2\right)^n = (-(2z+z^2))^n = (-1)^n(2z+z^2)^n$$

4. **Extract the Coefficient of z^n:**

   Factor out z^n:

$$(-1)^n(2z+z^2)^n = (-1)^n z^n (2+z)^n$$

5. **Expand $(2+z)^n$:**

   Using the binomial theorem:

$$(2+z)^n = \sum_{k=0}^{n}\binom{n}{k}2^{n-k}z^k$$

Bonus: a ChatGPT proof (2025)

6. **Multiply and Extract the Coefficient:**

   Multiply z^n by the expansion:

$$(-1)^n z^n \sum_{k=0}^{n} \binom{n}{k} 2^{n-k} z^k = (-1)^n \sum_{k=0}^{n} \binom{n}{k} 2^{n-k} z^{n+k}$$

   The coefficient of z^n corresponds to $k = 0$:

$$\text{Coefficient of } z^n = (-1)^n \binom{n}{0} 2^{n-0} = (-1)^n \times 1 \times 2^n = (-1)^n 2^n$$

7. **Conclusion:**

   Therefore, we have shown that:

$$S(n) = (-1)^n \times 2^n$$

Final Answer:

An explicit formula: S(n) = $(-1)^n \times 2^n$; that is, $\sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{2k}{n} = (-1)^n 2^n.$

[math.CO] 5 Feb 2024

Enumeration of multi-rooted plane trees

Anwar Al Ghabra[1], K. Gopala Krishna[2], Patrick Labelle[3], and Vasilisa Shramchenko[1]

Abstract

We give closed form expressions for the numbers of multi-rooted plane trees with specified degrees of root vertices. This results in an infinite number of integer sequences some of which are known to have an alternative interpretation. We also propose recursion relations for numbers of such trees as well as for the corresponding generating functions. Explicit expressions for the generating functions corresponding to plane trees having two and three roots are derived. As a by-product, we obtain a new binomial identity and a conjecture relating hypergeometric functions.

MSC: 05A19, 05C05, 11Y55

Keywords: rooted maps; generating functions; ribbon graphs; integer sequences; plane trees; combinatorial identities.

Bonus: a new exercise

Conjecture 2. *Let n, r, s be integers and $n \geq 0$. Then*

$$\sum_{k=0}^{n} \binom{2n-2k+s}{n-k} \binom{2k+r}{k} = \sum_{k=0}^{[\frac{n}{2}]} \binom{2n+2+r+s}{n-2k}, \tag{34}$$

where $[\frac{n}{2}]$ denotes the integer part of $\frac{n}{2}$.

▷ Exercise: prove that both sides satisfy the recurrence

$$\begin{gathered} 4\,(2n+r+s+3)\,(2n+r+s+2)\,a_n \\ -\Big(8n\,(n+r+s)+(r+s)^2+13(2n+r+s)+22\Big)\,a_{n+1} \\ +\,(n+2)\,(n+s+r+2)\,a_{n+2}=0 \end{gathered}$$

with

$$a_0 = 1 \quad \text{and} \quad a_1 = r+s+4.$$

Enumerative Combinatorics: science of counting

Area of mathematics primarily concerned with counting discrete objects.

▷ Main outcome: theorems

Computer Algebra: effective mathematics

Area of computer science primarily concerned with the algorithmic manipulation of algebraic objects.

▷ Main outcome: algorithms

Computer Algebra for Enumerative Combinatorics

Today: Algorithms for proving Theorems on Lattice Paths Combinatorics.

An (innocent looking) combinatorial question

Let $\mathscr{S} = \{\uparrow, \leftarrow, \searrow\}$. An $\mathscr{S}$-walk is a path in $\mathbb{Z}^2$ using only steps from $\mathscr{S}$. Show that, for any integer n, the following quantities are equal:

(i) number a_n of n-steps $\mathscr{S}$-walks confined to the upper half plane $\mathbb{Z} \times \mathbb{N}$ that start and finish at the origin $(0,0)$ (*excursions*);

(ii) number b_n of n-steps $\mathscr{S}$-walks confined to the quarter plane $\mathbb{N}^2$ that start at the origin $(0,0)$ and finish on the diagonal of $\mathbb{N}^2$ (*diagonal walks*).

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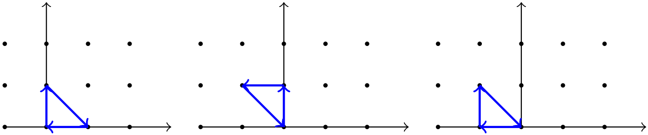
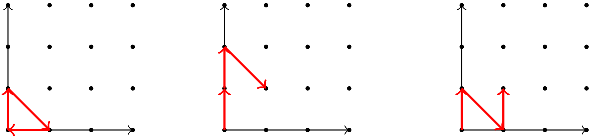
(ii) number b_n of n-steps $\mathscr{S}$-walks confined to the quarter plane $\mathbb{N}^2$ that start at the origin $(0,0)$ and finish on the diagonal of $\mathbb{N}^2$ (*diagonal walks*).

For instance, for $n = 3$, this common value is $a_3 = b_3 = 3$:

(i)



(ii)



Teasers

Teaser 1: This "exercise" is non-trivial

Teaser 2: It can be solved using Experimental Math and Computer Algebra

Teaser 3: ...by two robust and efficient algorithmic techniques,
Guess-and-Prove and Creative Telescoping

Why count walks?

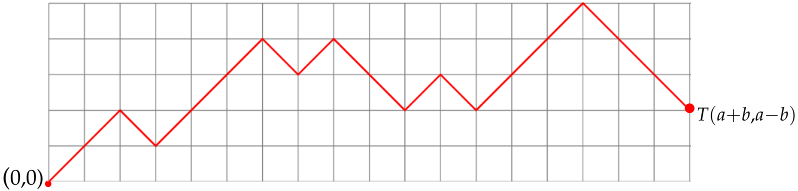
Many objects can be encoded by (confined) walks:

- probability theory (voting, games of chance, branching processes, ...)
- discrete mathematics (permutations, trees, words, urns, ...)
- statistical physics (Ising model, ...)
- operations research (queueing theory, ...)

Counting walks is an old topic: the ballot problem [Bertrand, 1887]

Suppose that candidates A and B are running in an election. If a votes are cast for A and b votes are cast for B, where $a > b$, then the probability that A stays ahead of B throughout the counting of the ballots is $(a-b)/(a+b)$.

Lattice path reformulation: find the number of paths in $\mathbb{Z}^2$ with a upsteps $\nearrow$ and b downsteps $\searrow$ that start at the origin and never touch the x-axis



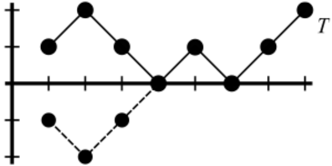
▷ Without the constraint, the number of such paths is $\binom{a+b}{a}$

$\longrightarrow$ a Guess-and-Prove proof in a few slides

Suppose that candidates A and B are running in an election. If a votes are cast for A and b votes are cast for B, where $a > b$, then the probability that A stays ahead of B throughout the counting of the ballots is $(a-b)/(a+b)$.

Lattice path reformulation: find the number of paths in $\mathbb{Z}^2$ with $a-1$ upsteps $\nearrow$ and b downsteps $\searrow$ that start at $(1,1)$ and never touch the x-axis

Reflection principle [Aebly, 1923]: *paths in* $\mathbb{Z}^2$ *from* $(1,1)$ *to* $T(a+b, a-b)$ *that do touch the x-axis* are in bijection with *paths in* $\mathbb{Z}^2$ *from* $(1,-1)$ *to* T



Answer: $\underbrace{(\textit{paths in } \mathbb{Z}^2 \textit{ from } (1,1) \textit{ to } T)}_{\binom{a+b-1}{a-1}} - \underbrace{(\textit{paths in } \mathbb{Z}^2 \textit{ from } (1,-1) \textit{ to } T)}_{\binom{a+b-1}{b-1}}$

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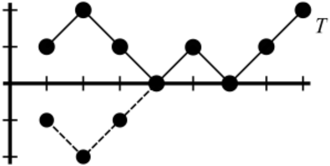

Answer: $\underbrace{(\textit{paths in } \mathbb{Z}^2 \textit{ from } (1,1) \textit{ to } T) - (\textit{paths in } \mathbb{Z}^2 \textit{ from } (1,-1) \textit{ to } T)}$

$$\binom{a+b-1}{a-1} - \binom{a+b-1}{b-1} = \frac{a-b}{a+b}\binom{a+b}{a}$$

...but it is still a very hot topic

Lot of recent activity; many recent contributors:

Arquès, Bacher, Banderier, Beaton, Bernardi, Biane, Bostan, Bousquet-Mélou, Buchacher, Budd, Chyzak, Cori, Courtiel, Denisov, Dreyfus, Du, Duchon, Dulucq, Duraj, Fayolle, Fisher, Flajolet, Fusy, Garbit, Gessel, Gouyou-Beauchamps, Guttmann, Guy, Hardouin, van Hoeij, Hou, Iasnogorodski, Johnson, Kauers, Kenyon, Koutschan, Krattenthaler, Kreweras, Kurkova, Lecouvey, Malyshev, Melczer, Miller, Mishna, Niederhausen, Owczarek, Pech, Petkovšek, Prellberg, Raschel, Rechnitzer, Roques, Sagan, Salvy, Sheffield, Singer, Tarrago, Trotignon, Verron, Viennot, Wachtel, Wallner, Wang, Wilf, D. Wilson, M. Wilson, Xu, Yatchak, Yeats, Zeilberger, ...

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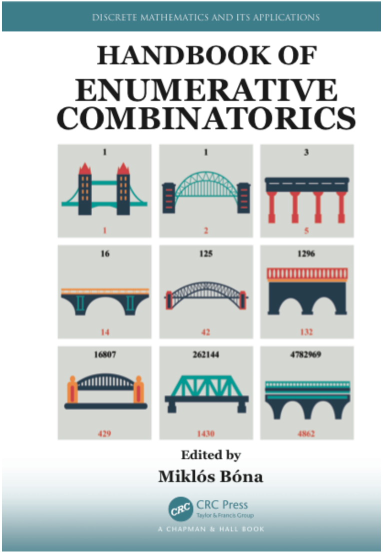
etc.

~~**Specific question**~~

~~**Ad hoc solution**~~ ⟶ **Systematic approach**

...but it is still a very hot topic



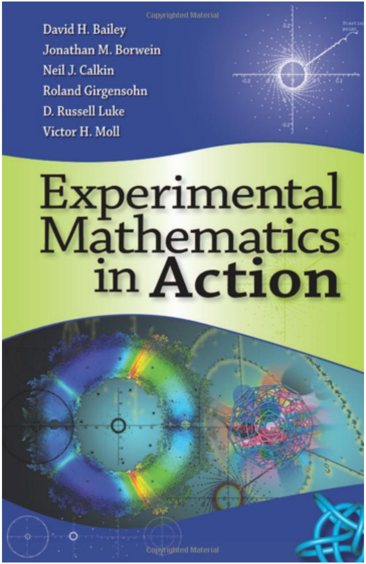
Chapter 10

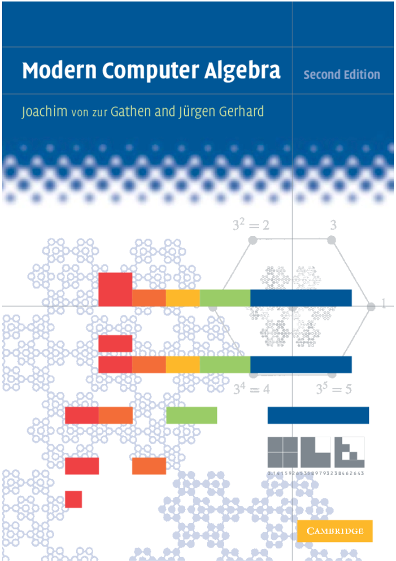
Lattice Path Enumeration

Christian Krattenthaler
Universität Wien

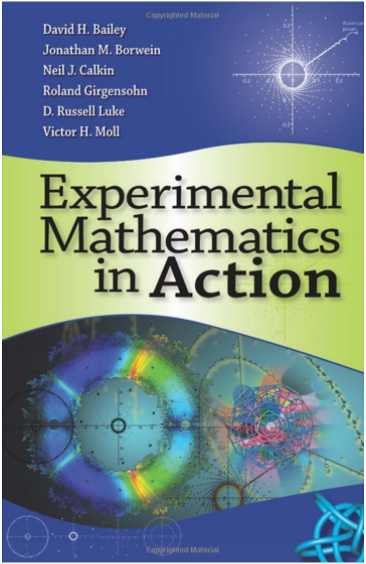
CONTENTS

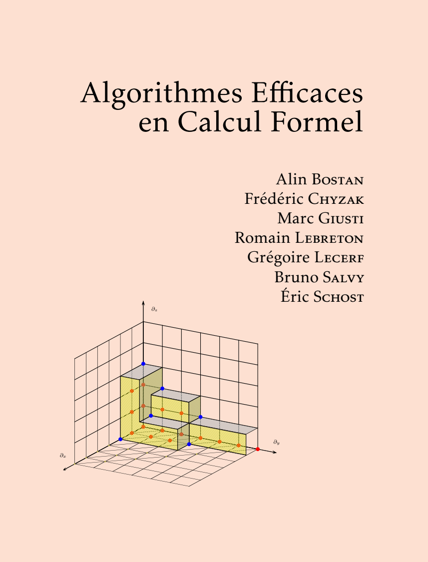
Our approach: Experimental Mathematics using Computer Algebra





Our approach: Experimental Mathematics using Computer Algebra





Lattice walks with small steps in the quarter plane

▷ Walks in $\mathbb{N}^2$ starting at $(0,0)$ and using steps in a fixed subset $\mathscr{S}$ of

$$\{\swarrow, \leftarrow, \nwarrow, \uparrow, \nearrow, \rightarrow, \searrow, \downarrow\}.$$

▷ Counting sequence: $q_{\mathscr{S}}(n)$ = number of $\mathscr{S}$-walks of length n

▷ Length generating function:

$$Q_{\mathscr{S}}(t) = \sum_{n=0}^{\infty} q_{\mathscr{S}}(n) t^n \ \in \mathbb{Z}[[t]]$$

Lattice walks with small steps in the quarter plane

▷ Walks in $\mathbb{N}^2$ starting at $(0,0)$ and using steps in a fixed subset $\mathscr{S}$ of

$$\{\swarrow, \leftarrow, \nwarrow, \uparrow, \nearrow, \rightarrow, \searrow, \downarrow\}.$$

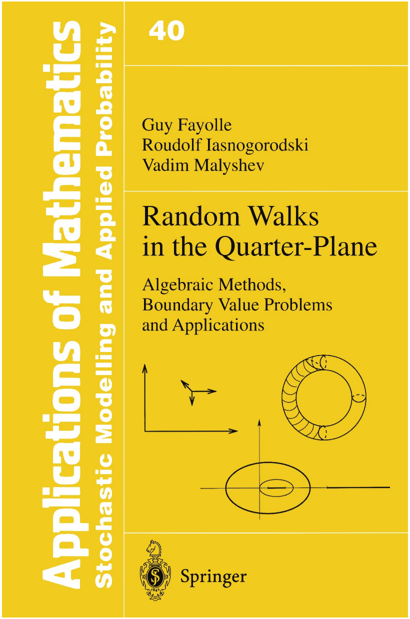
▷ Refinement: $q_{\mathscr{S}}(i,j;n)$ = number of $\mathscr{S}$-walks of length n ending at (i,j)

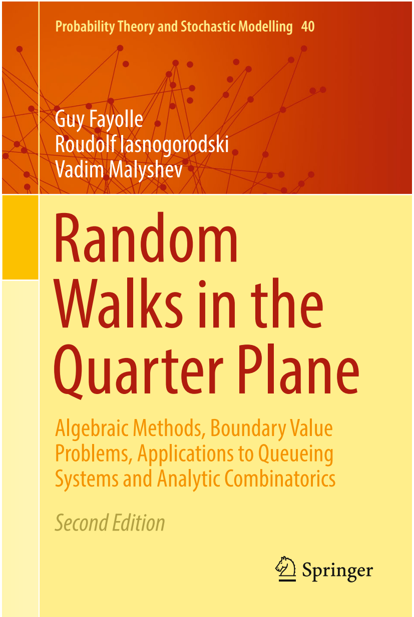
▷ Full generating function (with "catalytic " variables x,y):

$$Q_{\mathscr{S}}(x,y;t) = \sum_{i,j,n=0}^{\infty} q_{\mathscr{S}}(i,j;n) x^i y^j t^n \in \mathbb{Z}[[x,y,t]]$$

▷ Actually: $Q_{\mathscr{S}}(x,y;t) \in \mathbb{Z}[x,y][[t]]$ and $Q_{\mathscr{S}}(1,1;t) = Q_{\mathscr{S}}(t)$

Entire books dedicated to small-steps walks in the quarter plane!





Small-steps models of interest

Among the 2^8 step sets $\mathscr{S} \subseteq \{-1,0,1\}^2 \setminus \{(0,0)\}$, some are:

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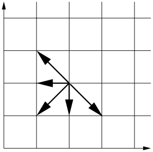

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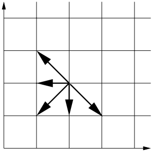

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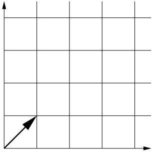

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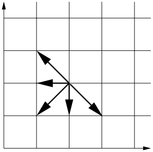

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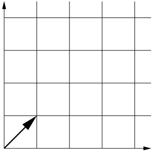

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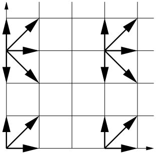

intrinsic to the half plane,

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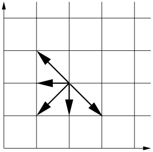

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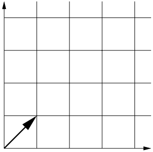

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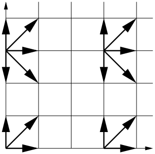

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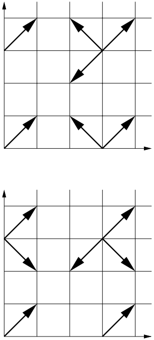
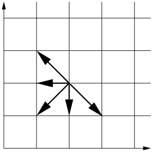


symmetrical.

Among the 2^8 step sets $\mathscr{S} \subseteq \{-1,0,1\}^2 \setminus \{(0,0)\}$, some are:



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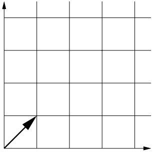

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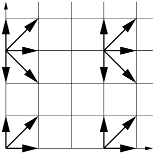

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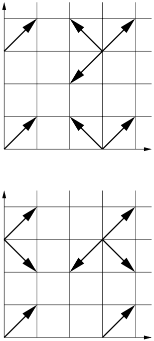
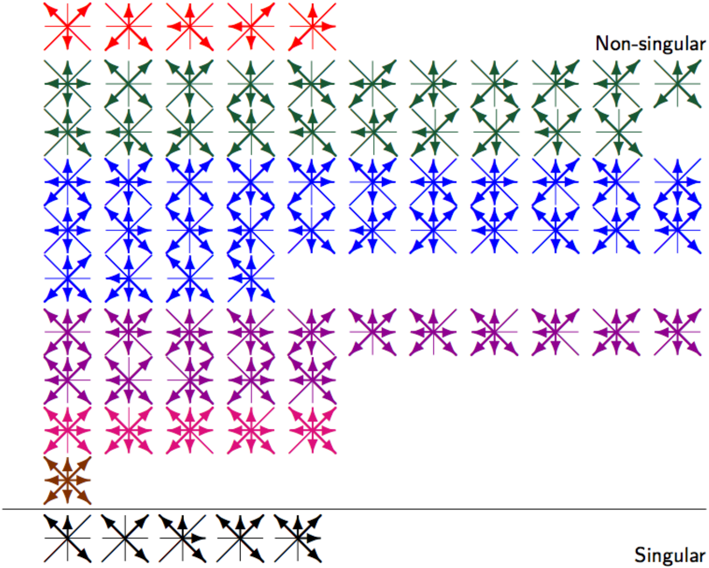


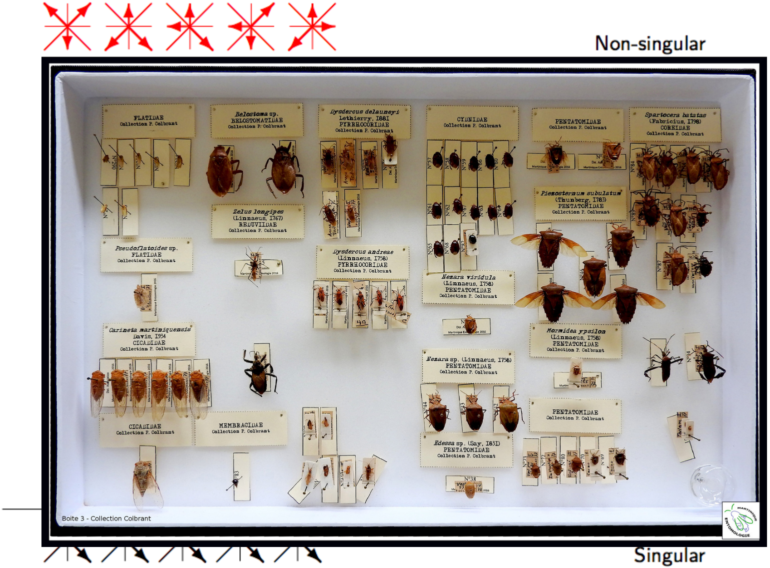
symmetrical.

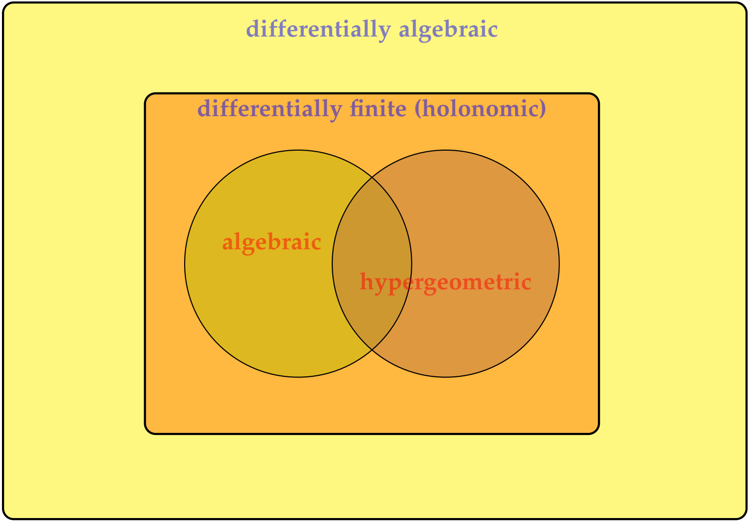
One is left with 79 interesting distinct models.

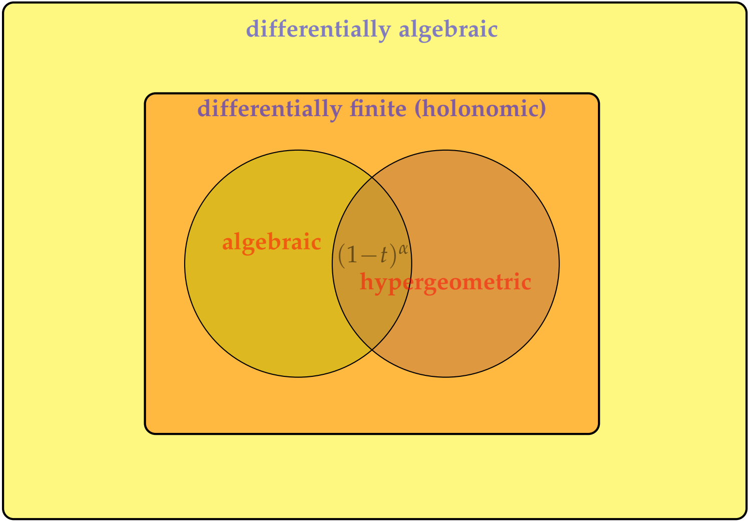
The 79 small-steps quadrant models



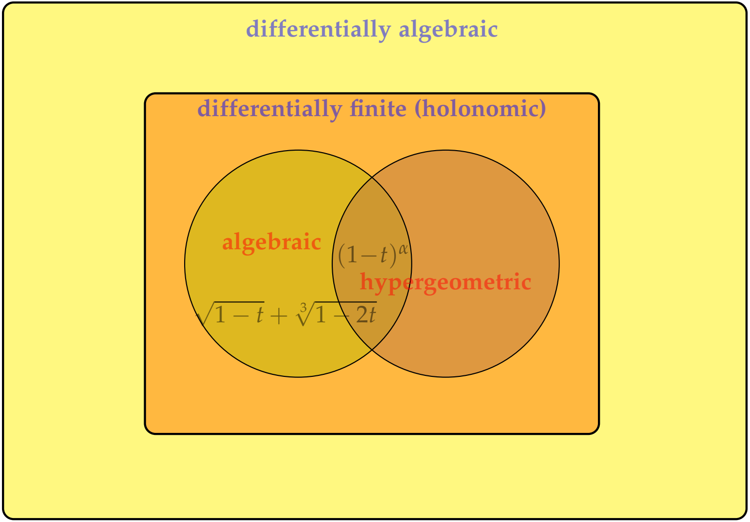
Task: classify their generating functions!



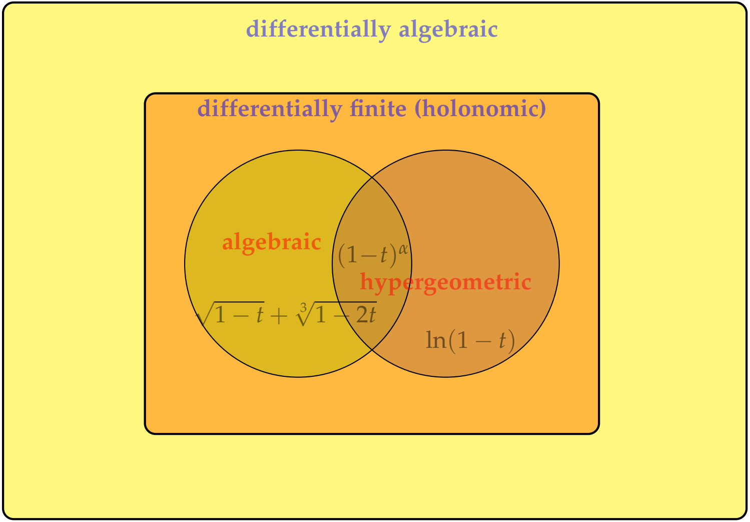

differentially algebraic
differentially finite (holonomic)
algebraic
hypergeometric


differentially algebraic
differentially finite (holonomic)
algebraic
$(1-t)^\alpha$
hypergeometric

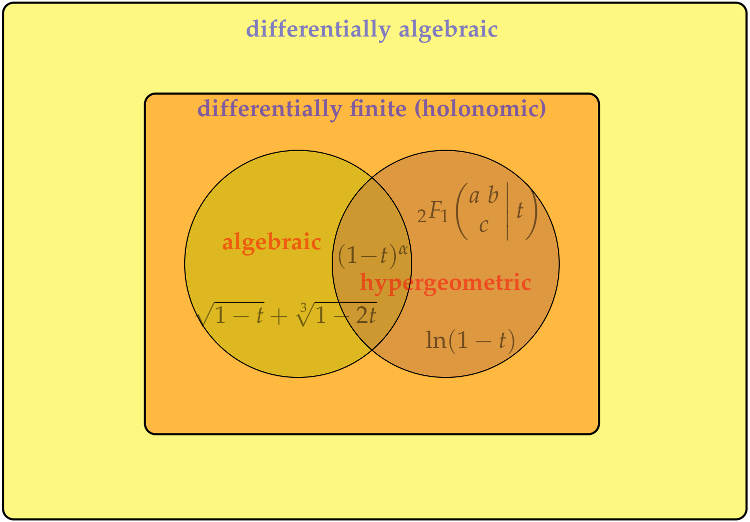
Classification criterion: properties of generating functions



Classification criterion: properties of generating functions

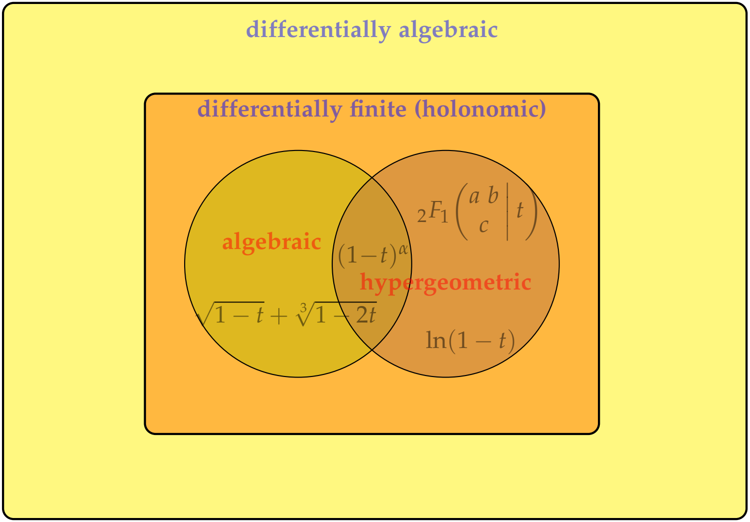


Classification criterion: properties of generating functions



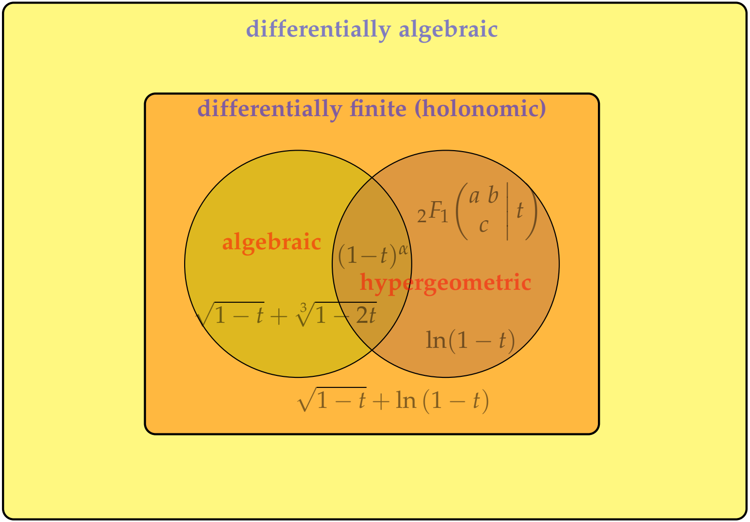
$${}_2F_1\left(\begin{matrix} a\ b \\ c \end{matrix} \middle| t\right) := \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{t^n}{n!}, \quad \text{where} \quad (a)_n = a(a+1)\cdots(a+n-1).$$

Classification criterion: properties of generating functions



E.g., $(1-t)^\alpha = {}_2F_1\left(\begin{matrix} -\alpha\ 1 \\ 1 \end{matrix} \middle| t\right), \quad \ln(1-t) = -t \cdot {}_2F_1\left(\begin{matrix} 1\ 1 \\ 2 \end{matrix} \middle| t\right) = -\sum_{n=1}^{\infty} \frac{t^n}{n}$

Classification criterion: properties of generating functions



$$_2F_1\left(\begin{matrix} a\ b \\ c \end{matrix} \middle| t\right) := \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{t^n}{n!}, \quad \text{where} \quad (a)_n = a(a+1)\cdots(a+n-1).$$

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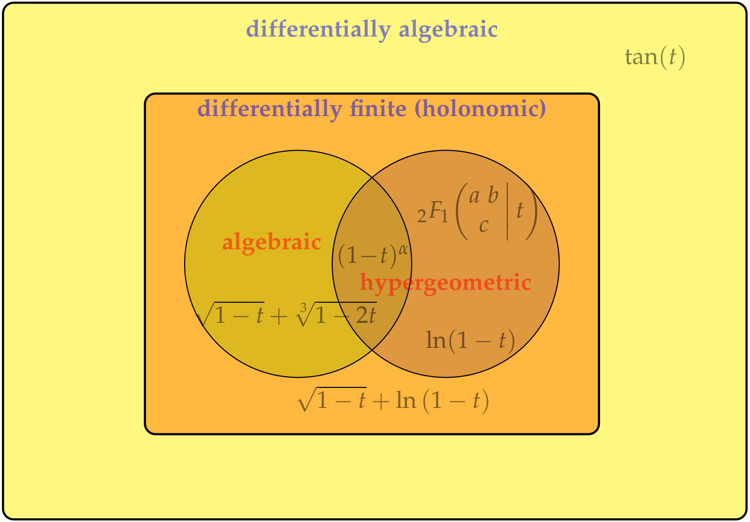

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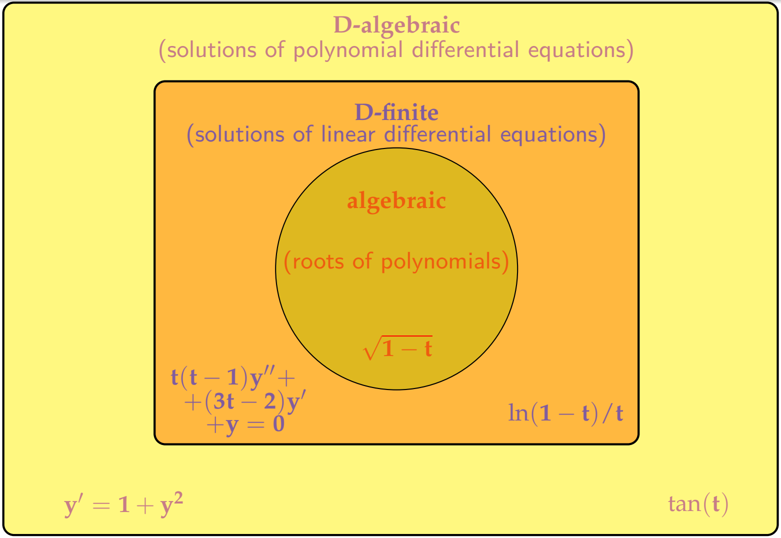
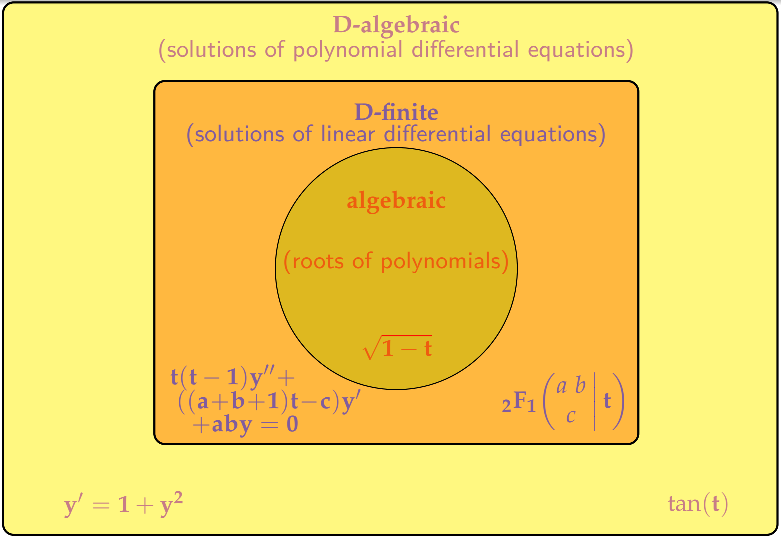


D-transcendental $\Gamma(t) = \int_0^\infty x^{t-1} e^{-x}\, dx$

Classification criterion: properties of generating functions



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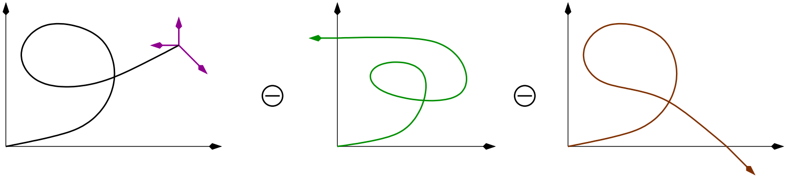
Generating function: $Q(x,y) \equiv Q(x,y;t) = \sum_{i,j,n=0}^{\infty} q(i,j;n) x^i y^j t^n \in \mathbb{Z}[[x,y,t]]$

Recursive construction yields the *kernel equation*

$$Q(x,y) = 1 + t\left(y + \frac{1}{x} + x\frac{1}{y}\right)Q(x,y) - t\frac{1}{x}Q(0,y) - tx\frac{1}{y}Q(x,0)$$



Algebraic reformulation of main task: solving a functional equation

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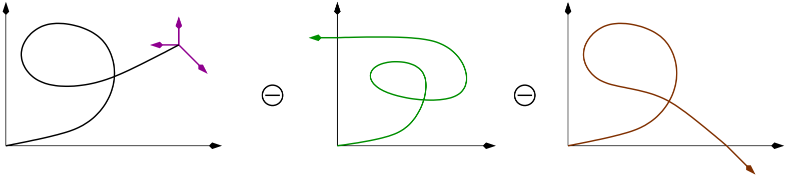
$$\left(1 - t\left(y + \frac{1}{x} + x\frac{1}{y}\right)\right) xyQ(x,y) = xy - tyQ(0,y) - tx^2Q(x,0)$$



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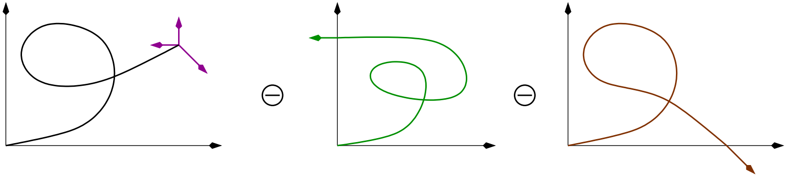

New task: Solve this functional equation!

Algebraic reformulation of main task: solving a functional equation

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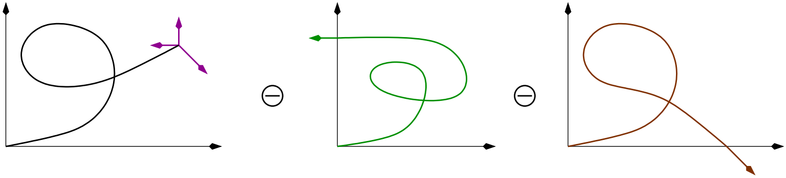
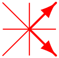
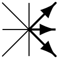
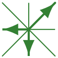


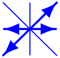
New task: For the other models – solve 78 similar equations!

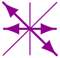
Dyck: 

Motzkin: 

Pólya: 

Kreweras: 

Gessel: 

Gouyou-Beauchamps: 

King walks: 

Tandem walks: 

▷ Kernel equation:

$$(y - tx(1 + y^2)) \cdot Q(x, y) = y - tx \cdot Q(x, 0)$$

▷ Kernel method [Knuth, 1968]:

- let $y_0 \in \mathbb{Q}[x][[t]]$ be the power series root of $K = y - tx(1 + y^2)$

$$y_0 = \frac{1 - \sqrt{1 - 4t^2x^2}}{2tx} = tx + t^3x^3 + 2t^5x^5 + \cdots \in \mathbb{Q}[x][[t]]$$

- plug $y = y_0$ in the kernel equation $\implies$ $Q(x, 0) = \frac{y_0}{tx}$
- conclude algebraicity:

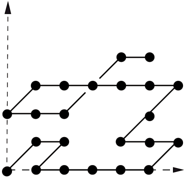
$$Q(x, y) = \frac{y - y_0}{K} = \frac{\sqrt{1 - 4t^2x^2} + 2txy - 1}{2tx(y - tx(1 + y^2))}$$

▷ Same method proves *algebraicity* for all models intrinsic to the half plane

A difficult model: Gessel walks (2000)

$\mathscr{S} =$

- $g(n)$ = number of n-steps $\{\nearrow, \swarrow, \leftarrow, \rightarrow\}$-walks in $\mathbb{N}^2$
  1, 2, 7, 21, 78, 260, 988, 3458, 13300, 47880, ...



Question: What is the nature of the generating function

$$G(t) = \sum_{n=0}^{\infty} g(n)\, t^n \text{ ?}$$

- $g(i,j;n)$ = number of n-steps $\{\nearrow, \swarrow, \leftarrow, \rightarrow\}$-walks in $\mathbb{N}^2$ from $(0,0)$ to (i,j)

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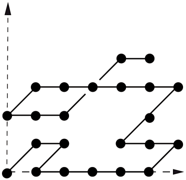

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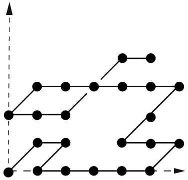

Theorem [B., Kauers, 2010]

$G(x,y;t)$ is an algebraic function†.

▷ computer-driven discovery/proof via *algorithmic Guess-and-Prove*

† Minimal polynomial $P(G(x,y;t); x,y,t) = 0$ has $> 10^{11}$ terms; $\approx 30\,\mathrm{Gb}$ (6 DVDs!)

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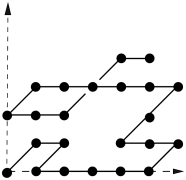
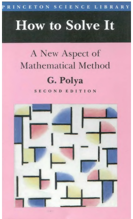


Corollary [B., Kauers, 2010] (former conjecture of Gessel's)

$$(3n+1)\,g(2n) = (12n+2)\,g(2n-1) \text{ and } (n+1)\,g(2n+1) = (4n+2)\,g(2n)$$

▷ computer-driven discovery/proof via *algorithmic Guess-and-Prove*



Guessing and Proving

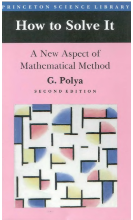
George Pólya

What is "scientific method"? Philosophers and non-philosophers have discussed this question and have not yet finished discussing it. Yet as a first introduction it can be described in three syllables:

Guess and test.

Mathematicians too follow this advice in their research although they sometimes refuse to confess it. They have, however, something which the other scientists cannot really have. For mathematicians the advice is

First guess, then prove.



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First guess, then prove.

generate data → make conjectures → prove them

Guess-and-Prove: a toy example

Question: Find $B_{i,j}$:= the number of $\{\rightarrow, \uparrow\}$-walks in $\mathbb{N}^2$ from $(0,0)$ to (i,j)

Guess-and-Prove: a toy example

Question: Find $B_{i,j} :=$ the number of $\{\rightarrow,\uparrow\}$-walks in $\mathbb{N}^2$ from $(0,0)$ to (i,j)

1. There are 2 ways to get to (i,j), either from $(i-1,j)$, or from $(i,j-1)$:

$$B_{i,j} = B_{i-1,j} + B_{i,j-1}$$

2. There is only one way to get to a point on an axis: $B_{i,0} = B_{0,j} = 1$

▷ These two rules completely determine all the numbers $B_{i,j}$

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(I) Generate data:

⋮							
1	7	28	84	210	462	924	
1	6	21	56	126	252	462	
1	5	15	35	70	126	210	
1	4	10	20	35	56	84	
1	3	6	10	15	21	28	
1	2	3	4	5	6	7	
1	1	1	1	1	1	1	…

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1	4	10	20	35	56	84	$\longrightarrow \cdots$
1	3	6	10	15	21	28	$\longrightarrow \frac{(i+1)(i+2)}{2}$
1	2	3	4	5	6	7	$\longrightarrow i+1$
1	1	1	1	1	1	1	$\longrightarrow 1$

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$$B_{i,j} \stackrel{?}{=} \frac{(i+j)!}{i!\,j!}$$

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(III) Prove: If $C_{i,j} \overset{\text{def}}{=} \frac{(i+j)!}{i!j!}$, then

$$\frac{C_{i-1,j}}{C_{i,j}} + \frac{C_{i,j-1}}{C_{i,j}} = \frac{i}{i+j} + \frac{j}{i+j} = 1$$

and $C_{i,0} = C_{0,j} = 1$.

Thus $B_{i,j} = C_{i,j}$

- $g(i,j;n)$ = number of n-steps $\{\nearrow, \swarrow, \leftarrow, \rightarrow\}$-walks in $\mathbb{N}^2$ from $(0,0)$ to (i,j)

Question: What is the nature of the generating function

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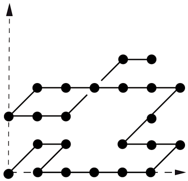

Answer: [B., Kauers, 2010] $G(x,y;t)$ is an algebraic function†.

Approach: $\longrightarrow$ very general and robust!

1. Generate data: compute G to precision t^{1200} ($\approx$ 1.5 billion coeffs!)
2. Guess: conjecture polynomial equations for $G(x,0;t)$ and $G(0,y;t)$ (degree 24 each, coeffs. of degree $(46,56)$, with 80-bits digits coeffs.)
3. Prove: multivariate resultants of (very big) polynomials (30 pages each)

† Minimal polynomial $P(G(x,y;t);x,y,t) = 0$ has $> 10^{11}$ terms; $\approx$ 30 Gb (6 DVDs!)

A typical Guess-and-Prove algorithmic proof

Theorem ["Gessel excursions are algebraic"]

$$g(t) := G(0,0;\sqrt{t}) = \sum_{n=0}^{\infty} \frac{(5/6)_n (1/2)_n}{(5/3)_n (2)_n} (16t)^n \quad \text{is algebraic.}$$

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Proof: First guess a polynomial $P(t,T)$ in $\mathbb{Q}[t,T]$, then prove that P admits the power series $g(t) = \sum_{n=0}^{\infty} g_n t^n$ as a root.

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3. $r(t) = \sum_{n=0}^{\infty} r_n t^n$ being algebraic, it is D-finite, and so (r_n) is P-recursive:

$$(n+2)(3n+5)r_{n+1} - 4(6n+5)(2n+1)r_n = 0, \qquad r_0 = 1$$

$$\Longrightarrow \text{solution } r_n = \frac{(5/6)_n (1/2)_n}{(5/3)_n (2)_n} 16^n = g_n, \text{ thus } g(t) = r(t) \text{ is algebraic.}$$

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```
> P:=gfun:-listtoalgeq([seq(pochhammer(5/6,n)*pochhammer(1/2,n)/
     pochhammer(5/3,n)/pochhammer(2,n)*16^n, n=0..100)], g(t)):
> gfun:-diffeqtorec(gfun:-algeqtodiffeq(P[1], g(t)), g(t), r(n));
```

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$$\Longrightarrow \text{solution } r_n = \frac{(5/6)_n(1/2)_n}{(5/3)_n(2)_n}16^n = g_n, \text{ thus } g(t) = r(t) \text{ is algebraic.}$$

```
> P:=gfun:-listtoalgeq([seq(pochhammer(5/6,n)*pochhammer(1/2,n)/
      pochhammer(5/3,n)/pochhammer(2,n)*16^n, n=0..100)], g(t)):
> gfun:-diffeqtorec(gfun:-algeqtodiffeq(P[1], g(t)), g(t), r(n));
```

▷ Steps 1 & 3 rely on polynomial linear algebra (*Hermite-Padé approximants*).

Sur la généralisation des fractions continues algébriques.

(*Par M.* Ch. Hermite, *membre de l'Institut, à Paris.*)

[Extrait d'une lettre à M. Pincherle (*).]

. . . . . . Le problème que j'ai en vue est lo suivant: Etant donné n séries $S_1, S_2, \ldots S_n$ procédant suivant les puissances d'une variable x, déterminer les polynômes $X_1, X_2, \ldots X_n$ des degrés $\mu_1, \mu_2, \ldots \mu_n$ de manière à avoir

$$S_1 X_1 + S_2 X_2 + \cdots + S_n X_n = S x^{\mu_1 + \mu_2 + \cdots + \mu_n + n - 1},$$

où S est une série de même nature que S_1, S_2, etc. La question ainsi posée est entièrement déterminée, et une remarque de calcul intégral en donne la complète solution dans le cas particulier où les séries sont de simples exponentielles. C'est ce que je vais montrer, je me proposerai ensuite de faire sortir, en vue du cas général, les enseignements que contient cette solution.

Sur la généralisation des fractions continues algébriques;

Par M. H. PADÉ,

Docteur ès Sciences mathématiques,
Professeur au lycée de Lille.

INTRODUCTION.

M. Hermite s'est, dans un travail récemment paru (1), occupé de la généralisation des fractions continues algébriques. La question est de déterminer les polynomes $X_1, X_2, \ldots, X_n$, de degrés $\mu_1, \mu_2, \ldots, \mu_n$, qui satisfont à l'équation

$$S_1X_1 + S_2X_2 + \ldots + S_nX_n = S\,x^{\mu_1+\mu_2+\ldots+\mu_n+n-1},$$

$S_1, S_2, \ldots, S_n$ étant des séries entières données, et S une série également entière. Ou plutôt, il s'agit d'obtenir un algorithme qui permette le calcul de proche en proche de ces systèmes de n polynomes, et qui

Hermite-Padé approximants

Definition: A Hermite-Padé approximant of type $\mathbf{d} = (d_1, \ldots, d_n) \in \mathbb{N}^n$ for $\mathbf{F} = (f_1, \ldots, f_n) \in \mathbb{K}[[x]]^n$ is a $\mathbf{P} = (P_1, \ldots, P_n) \in \mathbb{K}[x]^n \setminus \{0\}$ such that:

(1) $P_1 f_1 + \cdots + P_n f_n = O(x^\sigma)$ with $\sigma = \sum_i (d_i + 1) - 1$,

(2) $\deg(P_i) \leq d_i$ for all i.

Hermite-Padé approximants

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(2) $\deg(P_i) \le d_i$ for all i.

▷ Very useful concept in number theory (irrationality/transcendence):

- [Hermite, 1873]: e is transcendent; [Lindemann, 1882]: π is transc.
- [Apéry, 1978; Beukers, 1981]: $\zeta(3) = \sum_{n \ge 1} \frac{1}{n^3}$ is irrational; [Rivoal, 2000]: there are infinitely many k such that $\zeta(2k+1) \notin \mathbb{Q}$.

▷ Very useful tool in computer algebra

- algebraic approximants when $f_\ell = A^{\ell - 1}$ for a given $A \in \mathbb{K}[[x]]$
- differential approximants when $f_\ell = A^{(\ell - 1)}$ for a given $A \in \mathbb{K}[[x]]$

Hermite-Padé approximants

Definition: A Hermite-Padé approximant of type $\mathbf{d} = (d_1, \ldots, d_n) \in \mathbb{N}^n$ for $\mathbf{F} = (f_1, \ldots, f_n) \in \mathbb{K}[[x]]^n$ is a $\mathbf{P} = (P_1, \ldots, P_n) \in \mathbb{K}[x]^n \setminus \{0\}$ such that:

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▷ Many fast algorithms: [Beckermann, Labahn, 1994], [Giorgi, Jeannerod, Villard, 2003], [B., Jeannerod, Schost, 2007, 2008], [Zhou, Labahn, 2012], [Jeannerod, Neiger, Villard, 2020], [Rosenkilde, Storjohann, 2016, 2021], etc.

▷ `gfun` (Maple), `Guess.m` (Mathematica), `ore_algebra` (SageMath), etc.

Algorithmic classification of models with D-finite $Q_{\mathscr{S}}(t) := Q_{\mathscr{S}}(1,1;t)$

	OEIS	$\mathscr{S}$	Pol size	LDE size	Rec size
1	A005566		—	(3, 4)	(2, 2)
2	A018224		—	(3, 5)	(2, 3)
3	A151312		—	(3, 8)	(4, 5)
4	A151331		—	(3, 6)	(3, 4)
5	A151266		—	(5, 16)	(7, 10)
6	A151307		—	(5, 20)	(8, 15)
7	A151291		—	(5, 15)	(6, 10)
8	A151326		—	(5, 18)	(7, 14)
9	A151302		—	(5, 24)	(9, 18)
10	A151329		—	(5, 24)	(9, 18)
11	A151261		—	(4, 15)	(5, 8)
12	A151297		—	(5, 18)	(7, 11)

	OEIS	$\mathscr{S}$	Pol size	LDE size	Rec size
13	A151275		—	(5, 24)	(9, 18)
14	A151314		—	(5, 24)	(9, 18)
15	A151255		—	(4, 16)	(6, 8)
16	A151287		—	(5, 19)	(7, 11)
17	A001006		(2, 2)	(2, 3)	(2, 1)
18	A129400		(2, 2)	(2, 3)	(2, 1)
19	A005558		—	(3, 5)	(2, 3)
20	A151265		(6, 8)	(4, 9)	(6, 4)
21	A151278		(6, 8)	(4, 12)	(7, 4)
22	A151323		(4, 4)	(2, 3)	(2, 1)
23	A060900		(8, 9)	(3, 5)	(2, 3)

Equation sizes = (order, degree)

▷ Computerized discovery: enumeration + guessing [B., Kauers, 2009]

Algorithmic classification of models with D-finite $Q_{\mathscr{S}}(t) := Q_{\mathscr{S}}(1,1;t)$

	OEIS	$\mathscr{S}$	Pol size	LDE size	Rec size		OEIS	$\mathscr{S}$	Pol size	LDE size	Rec size
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2	A018224		—	(3, 5)	(2, 3)	14	A151314		—	(5, 24)	(9, 18)
3	A151312		—	(3, 8)	(4, 5)	15	A151255		—	(4, 16)	(6, 8)
4	A151331		—	(3, 6)	(3, 4)	16	A151287		—	(5, 19)	(7, 11)
5	A151266		—	(5, 16)	(7, 10)	17	A001006		(2, 2)	(2, 3)	(2, 1)
6	A151307		—	(5, 20)	(8, 15)	18	A129400		(2, 2)	(2, 3)	(2, 1)
7	A151291		—	(5, 15)	(6, 10)	19	A005558		—	(3, 5)	(2, 3)
8	A151326		—	(5, 18)	(7, 14)						
9	A151302		—	(5, 24)	(9, 18)	20	A151265		(6, 8)	(4, 9)	(6, 4)
10	A151329		—	(5, 24)	(9, 18)	21	A151278		(6, 8)	(4, 12)	(7, 4)
11	A151261		—	(4, 15)	(5, 8)	22	A151323		(4, 4)	(2, 3)	(2, 1)
12	A151297		—	(5, 18)	(7, 11)	23	A060900		(8, 9)	(3, 5)	(2, 3)

Equation sizes = (order, degree)

- ▷ Computerized discovery: enumeration + guessing [B., Kauers, 2009]
- ▷ 1–22: DF confirmed by human proofs in [Bousquet-Mélou, Mishna, 2010]
- ▷ 23: DF confirmed by a human proof in [B., Kurkova, Raschel, 2017]
- ▷ All: explicit eqs. proved via CA [B., Chyzak, van Hoeij, Kauers, Pech, 2017]

Algorithmic classification of models with D-finite $Q_{\mathscr{S}}(t) := Q_{\mathscr{S}}(1,1;t)$

	OEIS	$\mathscr{S}$	algebraic?	asymptotics		OEIS	$\mathscr{S}$	algebraic?	asymptotics
1	A005566		N	$\frac{4}{\pi}\frac{4^n}{n}$	13	A151275		N	$\frac{12\sqrt{30}}{\pi}\frac{(2\sqrt{6})^n}{n^2}$
2	A018224		N	$\frac{2}{\pi}\frac{4^n}{n}$	14	A151314		N	$\frac{\sqrt{6}\lambda\mu C^{5/2}}{5\pi}\frac{(2C)^n}{n^2}$
3	A151312		N	$\frac{\sqrt{6}}{\pi}\frac{6^n}{n}$	15	A151255		N	$\frac{24\sqrt{2}}{\pi}\frac{(2\sqrt{2})^n}{n^2}$
4	A151331		N	$\frac{8}{3\pi}\frac{8^n}{n}$	16	A151287		N	$\frac{2\sqrt{2}A^{7/2}}{\pi}\frac{(2A)^n}{n^2}$
5	A151266		N	$\frac{1}{2}\sqrt{\frac{3}{\pi}}\frac{3^n}{n^{1/2}}$	17	A001006		Y	$\frac{3}{2}\sqrt{\frac{3}{\pi}}\frac{3^n}{n^{3/2}}$
6	A151307		N	$\frac{1}{2}\sqrt{\frac{5}{2\pi}}\frac{5^n}{n^{1/2}}$	18	A129400		Y	$\frac{3}{2}\sqrt{\frac{3}{\pi}}\frac{6^n}{n^{3/2}}$
7	A151291		N	$\frac{4}{3\sqrt{\pi}}\frac{4^n}{n^{1/2}}$	19	A005558		N	$\frac{8}{\pi}\frac{4^n}{n^2}$
8	A151326		N	$\frac{2}{\sqrt{3\pi}}\frac{6^n}{n^{1/2}}$		$A = 1+\sqrt{2}$, $B = 1+\sqrt{3}$, $C = 1+\sqrt{6}$, $\lambda = 7+3\sqrt{6}$,			$\mu = \sqrt{\frac{4\sqrt{6}-1}{19}}$
9	A151302		N	$\frac{1}{3}\sqrt{\frac{5}{2\pi}}\frac{5^n}{n^{1/2}}$	20	A151265		Y	$\frac{2\sqrt{2}}{\Gamma(1/4)}\frac{3^n}{n^{3/4}}$
10	A151329		N	$\frac{1}{3}\sqrt{\frac{7}{3\pi}}\frac{7^n}{n^{1/2}}$	21	A151278		Y	$\frac{3\sqrt{3}}{\sqrt{2}\Gamma(1/4)}\frac{3^n}{n^{3/4}}$
11	A151261		N	$\frac{12\sqrt{3}}{\pi}\frac{(2\sqrt{3})^n}{n^2}$	22	A151323		Y	$\frac{\sqrt{2}3^{3/4}}{\Gamma(1/4)}\frac{6^n}{n^{3/4}}$
12	A151297		N	$\frac{\sqrt{3}B^{7/2}}{2\pi}\frac{(2B)^n}{n^2}$	23	A060900		Y	$\frac{4\sqrt{3}}{3\Gamma(1/3)}\frac{4^n}{n^{2/3}}$

▷ Computerized discovery: convergence acceleration + LLL [B., Kauers, '09]

Algorithmic classification of models with D-finite $Q_{\mathscr{S}}(t) := Q_{\mathscr{S}}(1,1;t)$

	OEIS	$\mathscr{S}$	algebraic?	asymptotics		OEIS	$\mathscr{S}$	algebraic?	asymptotics
1	A005566		N	$\frac{4}{\pi}\frac{4^n}{n}$	13	A151275		N	$\frac{12\sqrt{30}}{\pi}\frac{(2\sqrt{6})^n}{n^2}$
2	A018224		N	$\frac{2}{\pi}\frac{4^n}{n}$	14	A151314		N	$\frac{\sqrt{6}\lambda\mu C^{5/2}}{5\pi}\frac{(2C)^n}{n^2}$
3	A151312		N	$\frac{\sqrt{6}}{\pi}\frac{6^n}{n}$	15	A151255		N	$\frac{24\sqrt{2}}{\pi}\frac{(2\sqrt{2})^n}{n^2}$
4	A151331		N	$\frac{8}{3\pi}\frac{8^n}{n}$	16	A151287		N	$\frac{2\sqrt{2}A^{7/2}}{\pi}\frac{(2A)^n}{n^2}$
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▷ Computerized discovery: convergence acceleration + LLL [B., Kauers, '09]

▷ Asympt. confirmed by human proofs via ACSV in [Melczer, Wilson, 2016]

▷ Transcendence proofs via CA [B., Chyzak, van Hoeij, Kauers, Pech, 2017]

Theorem [B., Chyzak, van Hoeij, Kauers, Pech, 2017]

Let $\mathscr{S}$ be one of the models 1–19. Then

- $Q_{\mathscr{S}}(x, y; t)$ is expressible using (integrals of) ${}_2F_1$ expressions.
- $Q_{\mathscr{S}}(x, y; t)$ is transcendental.

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- $Q_{\mathscr{S}}(t)$ is transcendental, except for $\mathscr{S} =$  and $\mathscr{S} =$ .

Example (King walks in the quarter plane, A151331)

Q 

$$(t) = \frac{1}{t}\int_0^t \frac{1}{(1+4x)^3} \cdot {}_2F_1\left(\left.\begin{matrix}\frac{3}{2} & \frac{3}{2}\\ & 2\end{matrix}\right| \frac{16x(1+x)}{(1+4x)^2}\right) dx$$

$$= 1 + 3t + 18t^2 + 105t^3 + 684t^4 + 4550t^5 + 31340t^6 + 219555t^7 + \cdots$$

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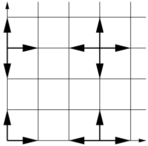
$$= 1 + 3t + 18t^2 + 105t^3 + 684t^4 + 4550t^5 + 31340t^6 + 219555t^7 + \cdots$$

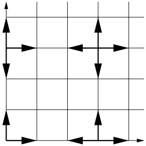
▷ Computer-driven discovery and proof; no human proof yet.

▷ Proof uses: (1) kernel method and (2) creative telescoping
+ (3) ODE factoring and (4) ODE solving.



The kernel $K(x,y;t) := 1 - t \cdot \sum_{(i,j)\in\mathscr{S}} x^i y^j = 1 - t\left(x + \frac{1}{x} + y + \frac{1}{y}\right)$ is left invariant under the change of (x,y) into the elements of

$$\mathcal{G}_{\mathscr{S}} := \left\{ (x,y), \left(\tfrac{1}{x}, y\right), \left(\tfrac{1}{x}, \tfrac{1}{y}\right), \left(x, \tfrac{1}{y}\right) \right\}$$



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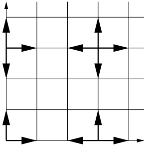
Kernel equation:

$$K(x,y;t)xyQ(x,y;t) = xy - txQ(x,0;t) - tyQ(0,y;t)$$

(1) Kernel method [Bousquet-Mélou, Mishna, 2010] $\mathscr{S} =$ 



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Kernel equation:

$$K(x,y;t)xyQ(x,y;t) = xy - txQ(x,0;t) - tyQ(0,y;t)$$
$$-K(x,y;t)\tfrac{1}{x}yQ(\tfrac{1}{x},y;t) = -\tfrac{1}{x}y + t\tfrac{1}{x}Q(\tfrac{1}{x},0;t) + tyQ(0,y;t)$$

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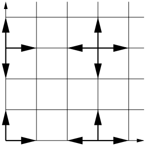

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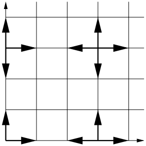
$$K(x,y;t)\tfrac{1}{x}\tfrac{1}{y}Q(\tfrac{1}{x},\tfrac{1}{y};t) = \tfrac{1}{x}\tfrac{1}{y} - t\tfrac{1}{x}Q(\tfrac{1}{x},0;t) - t\tfrac{1}{y}Q(0,\tfrac{1}{y};t)$$



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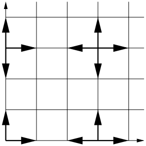
Kernel equation:

$$\begin{aligned}
K(x,y;t)xyQ(x,y;t) &= xy - txQ(x,0;t) - tyQ(0,y;t)\\
-K(x,y;t)\tfrac{1}{x}yQ(\tfrac{1}{x},y;t) &= -\tfrac{1}{x}y + t\tfrac{1}{x}Q(\tfrac{1}{x},0;t) + tyQ(0,y;t)\\
K(x,y;t)\tfrac{1}{x}\tfrac{1}{y}Q(\tfrac{1}{x},\tfrac{1}{y};t) &= \tfrac{1}{x}\tfrac{1}{y} - t\tfrac{1}{x}Q(\tfrac{1}{x},0;t) - t\tfrac{1}{y}Q(0,\tfrac{1}{y};t)\\
-K(x,y;t)x\tfrac{1}{y}Q(x,\tfrac{1}{y};t) &= -x\tfrac{1}{y} + txQ(x,0;t) + t\tfrac{1}{y}Q(0,\tfrac{1}{y};t)
\end{aligned}$$

(1) Kernel method [Bousquet-Mélou, Mishna, 2010] $\mathscr{S} = \{\leftarrow,\uparrow,\rightarrow,\downarrow\}$



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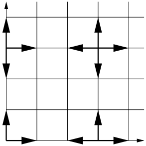
\end{aligned}$$

Summing up yields the orbit equation:

$$\sum_{\theta\in\mathcal{G}} (-1)^{\theta}\theta\big(xy\,Q(x,y;t)\big) = \frac{xy - \frac{1}{x}y + \frac{1}{x}\frac{1}{y} - x\frac{1}{y}}{K(x,y;t)}$$



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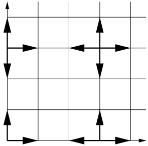
Taking positive parts yields:

$$[x^{>}y^{>}]\sum_{\theta\in\mathcal{G}}(-1)^{\theta}\theta\big(xy\,Q(x,y;t)\big) = [x^{>}y^{>}]\frac{xy - \frac{1}{x}y + \frac{1}{x}\frac{1}{y} - x\frac{1}{y}}{K(x,y;t)}$$

(1) Kernel method [Bousquet-Mélou, Mishna, 2010] $\mathscr{S} =$



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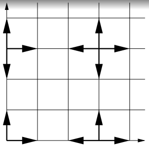

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$$\textsf{GFun} = \textsf{PosPart}\left(\frac{\textsf{OrbitSum}}{\textsf{Kernel}}\right) = \oint\!\!\oint \textsf{RatFrac}$$

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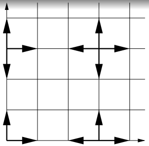

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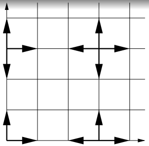
$$\mathsf{GF} = \mathsf{PosPart}\left(\frac{\mathsf{OS}}{\mathsf{Ker}}\right)$$ is D-finite [Lipshitz, 1988]



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▷ Argument works if $\text{OS} \neq 0$: algebraic version of the reflection principle

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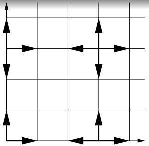

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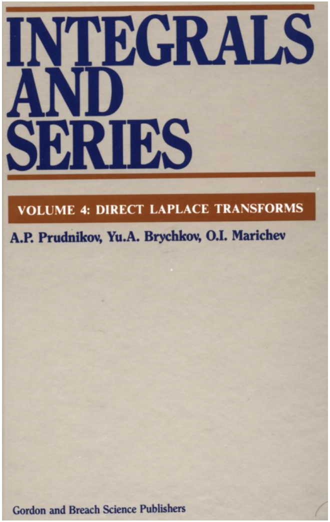
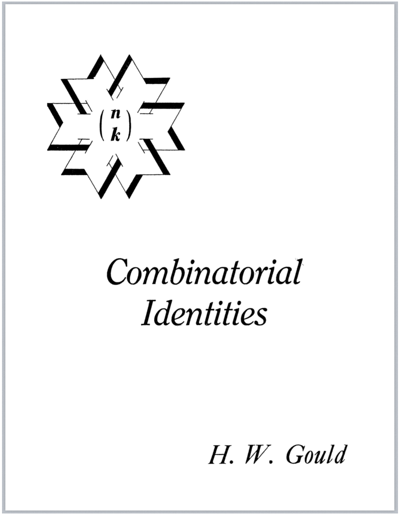
▷ Creative Telescoping finds a differential equation for GF = $\oint\!\!\oint$ RatFrac

"An algorithmic toolbox for multiple sums and integrals with parameters"





"An algorithmic toolbox for multiple sums and integrals with parameters"

DOUBLE INTEGRALS 631

6. $$\int_0^\infty\int_0^\infty x^{\nu+1}y^{1-\nu}e^{-px^2-qy^2}\operatorname{ber}_\nu(axy)\,dx\,dy=\left(\frac{ap}{2}\right)^\nu\frac{4pq\cos(3\nu\pi/4)-a^2\sin(3\nu\pi/4)}{16p^2q^2+a^4}$$ [Re $\nu > -1$].

7. $$\int_0^\infty\int_0^\infty x^{4n+1}y^3e^{-px^2-qy^4}\operatorname{bei}_{4n}(axy)\,dx\,dy=$$
$$=(-1)^n\sqrt{\frac{\pi}{2}}\frac{a^{4n}}{2^{5n+3}p^{4n+1}q^{n+1}}e^{-a^4/(2^7p^2q)}D_{2n+1}\left(\frac{a^2}{4p\sqrt{2q}}\right).$$

8. $$\int_0^\infty\int_0^\infty x^{4n+1}ye^{-px^2-qy^4}\operatorname{ber}_{4n}(axy)\,dx\,dy=$$
$$=(-1)^n\sqrt{\frac{\pi}{2}}\frac{a^{4n}}{2^{5n+5/2}p^{4n+1}q^{n+1/2}}e^{-a^4/(2^7p^2q)}D_{2n}\left(\frac{a^2}{4p\sqrt{2q}}\right).$$

9. $$\int_0^\infty\int_0^\infty xy^3e^{-px^2-qy^4}\operatorname{bei}(axy)\,dx\,dy=\frac{\sqrt{\pi}\,a^2}{2^6p^2q^{3/2}}e^{-a^4/(2^4p^2q)}.$$

10. $$\int_0^\infty\int_0^\infty xye^{-px^2-qy^4}\operatorname{ber}(axy)\,dx\,dy=\frac{\sqrt{\pi}}{8p\sqrt{q}}e^{-a^4/(2^4p^2q)}.$$

11. $$\int_0^\infty\int_0^\infty xy^3e^{-px^2-qy^4}[\operatorname{ber}^2(axy)+\operatorname{bei}^2(axy)]\,dx\,dy=\frac{1}{8pq}e^{a^4/(16p^2q)}.$$

12. $$\int_0^\infty\int_0^\infty xye^{-px^4-qy^4}\operatorname{ber}(axy)\,dx\,dy=\frac{\pi}{16\sqrt{pq}}J_0\left(\frac{a^2}{8\sqrt{pq}}\right).$$

13. $$\int_0^\infty\int_0^\infty xy\,K_0(ax)\,K_0(by)\,(\operatorname{ber}x\operatorname{ber}y-\operatorname{bei}x\operatorname{bei}y)\,dx\,dy=\frac{ab-1}{(a^2+1)(b^2+1)}$$ [Re a^2, Re $b^2 > 1/2$].

3.2.12. Integrals containing orthogonal polynomials.

1. $$\int_0^1\int_0^\pi\frac{y^{2\beta+1}}{(1-y^2)^{\beta-\alpha+1}}\sin^{2\beta}x\,P_n^{(\alpha,\beta)}(2\,|\cos a\cos b+ye^{ix}\sin a\sin b\,|^2-1)\,dx\,dy=$$
$$=\frac{\sqrt{\pi}}{2}\Gamma\begin{bmatrix}\alpha-\beta,\ \beta+1/2\\ \alpha+1\end{bmatrix}\frac{P_n^{(\alpha,\beta)}(\cos 2a)\,P_n^{(\alpha,\beta)}(\cos 2b)}{P_n^{(\alpha,\beta)}(1)}$$ [Re $\alpha >$ Re $\beta > -1/2$].

2. $$\int_{-1}^1\int_{-1}^1(1-x^2)^{\alpha-1}(1-y^2)^{\beta-1}C_{2n}^{\alpha+\beta}(ax+by)\,dx\,dy=$$
$$=\pi\Gamma\begin{bmatrix}\alpha,\ \beta\\ \alpha+1/2,\ \beta+1/2\end{bmatrix}\frac{C_{2n}^{\alpha+\beta}(0)\,P_n^{(\alpha-1/2,\,\beta-1/2)}(A)\,P_n^{(\alpha-1/2,\,\beta-1/2)}(B)}{P_n^{(\alpha-1/2,\,\beta-1/2)}(-1)\,P_n^{(\alpha-1/2,\,\beta-1/2)}(1)}$$
[$2a=\sqrt{(1-A)(1-B)}$, $2b=\sqrt{(1+A)(1+B)}$; Re α, Re $\beta > 0$].

3. $$\iint_{x^2+y^2\leqslant 1}\frac{P_m(x)\,P_n(y)}{\sqrt{1-x^2-y^2}}\,dx\,dy=\begin{cases}(-1)^{n/2}\,2\pi\dfrac{(n-1)!!}{n!!\,(2n+1)} & [m=n\ \text{even}]\\ 0 & [\text{otherwise}]\end{cases}$$

.7) $$\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}\binom{2k}{k}\frac{1}{\binom{n+k}{k}2^{2k}}=\frac{\binom{6n}{3n}}{\binom{2n}{n}}2^{-4n},$$

.8) $$\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}\binom{2k}{k}\frac{2k+1}{\binom{n+k}{k}(n+k+1)}=1,$$

.9) $$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{x+k}{k}\frac{2^{2k}}{\binom{2k}{k}(2k+1)}=\frac{2^{2n}}{\binom{2n}{n}(2n+1)}\binom{n-x-\frac12}{n},$$

7.10) $$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{n+k}{k}\frac{2^{2k}}{\binom{2k}{k}(2k+1)}=\frac{(-1)^n}{2n+1},$$

7.11) $$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{x+k}{k}\frac{2^{2k}}{\binom{2k}{k}(x+k)}=(-1)^n\frac{\binom{2x}{2n}}{x\binom{x}{n}},$$

7.12) $$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{n+k}{k}\frac{2^{2k}}{\binom{2k}{k}(n+k)}=\frac{(-1)^n}{n},\ (n\geqslant 1),$$

7.13) $$\sum_{k=0}^{n}\binom{n}{k}\binom{-n-1}{n-k}\frac{1}{\binom{k+r}{k}}=(-1)^n\,r\sum_{k=0}^{n}\binom{n}{k}^2\frac{1}{k+r},$$

-59-

(2) Creative Telescoping

"An algorithmic toolbox for multiple sums and integrals with parameters"

Example [Apéry 1978]: $A_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$ satisfies the recurrence

$$(n+1)^3 A_{n+1} + n^3 A_{n-1} = (2n+1)(17n^2+17n+5) A_n.$$

▷ Key fact used to prove that $\zeta(3) := \sum_{n\geq 1} \frac{1}{n^3} \approx 1.202056903\ldots$ is irrational.

1. Journées Arithmétiques de Marseille-Luminy, June 1978

The board of programme changes informed us that R. Apéry (Caen) would speak Thursday, 14.00 "Sur l'irrationalité de $\zeta(3)$." Though there had been earlier rumours of his claiming a proof, scepticism was general. The lecture tended to strengthen this view to rank disbelief. Those who listened casually, or who were afflicted with being non-Francophone, appeared to hear only a sequence of unlikely assertions.

7. ICM '78, Helsinki, August 1978

Neither Cohen nor I had been able to prove (5) or (5') in the intervening 2 months. After a few days of fruitless effort the specific problem was mentioned to Don Zagier (Bonn), and with irritating speed he showed that indeed the sequence $\{b'_n\}$ satisfies the recurrence (4). This more or less broke the dam and (5') and (5) were quickly conquered. Henri Cohen addressed a very well-attended meeting at 17.00 on Friday, August 18 in the language of the majority, proving (5) and explaining how this implied the

[Van der Poorten, 1979: "A proof that Euler missed"]

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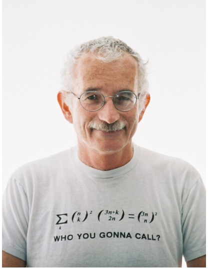
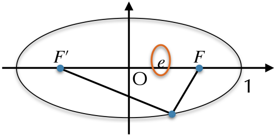


[Zeilberger, 1990: "The method of creative telescoping"]

"An algorithmic toolbox for multiple sums and integrals with parameters"

Example [Euler, 1733]: Perimeter of an ellipse of eccentricity e, semi-major axis 1

$$p(e) = 4\int_0^1 \sqrt{\frac{1-e^2u^2}{1-u^2}}\,\mathrm{d}u = 4\oint\!\!\oint \frac{\mathrm{d}x\mathrm{d}y}{1-\frac{1-e^2x^2}{(1-x^2)y^2}}$$



Principle: Find algorithmically

(2) Creative Telescoping

"An algorithmic toolbox for multiple sums and integrals with parameters"

Example [Euler, 1733]: Perimeter of an ellipse of eccentricity e, semi-major axis 1

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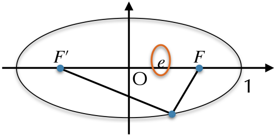

Principle: Find algorithmically

$$\left((e - e^3)\partial_e^2 + (1 - e^2)\partial_e + e\right)\left(\frac{1}{1 - \frac{1-e^2x^2}{(1-x^2)y^2}}\right) =$$

$$\partial_x \left(\frac{e(1+x-x^2-x^3)y^2(2x-3+y^2+x^2(3e^2-y^2-2))}{(y^2+x^2(e^2-y^2)-1)^2}\right)$$

$$+\, \partial_y \left(\frac{2e(e^2-1)x(1+x^3)y^3}{(y^2+x^2(e^2-y^2)-1)^2}\right)$$

▷ Conclusion: $(e - e^3) \cdot p''(e) + (1 - e^2) \cdot p'(e) + e \cdot p(e) = 0.$

(2) Creative Telescoping

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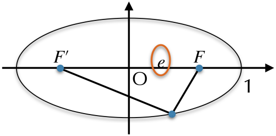

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$$\left((e-e^3)\partial_e^2 + (1-e^2)\partial_e + e\right)\left(\frac{1}{1-\frac{1-e^2x^2}{(1-x^2)y^2}}\right) =$$

$$\partial_x\left(\frac{e(1+x-x^2-x^3)y^2(2x-3+y^2+x^2(3e^2-y^2-2))}{(y^2+x^2(e^2-y^2)-1)^2}\right)$$

$$+\,\partial_y\left(\frac{2e(e^2-1)x(1+x^3)y^3}{(y^2+x^2(e^2-y^2)-1)^2}\right)$$

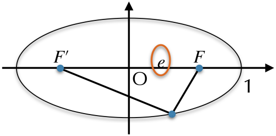
▷ Conclusion: $p(e) = \frac{\pi}{2}\cdot {}_2F_1\left(\begin{matrix}-\frac{1}{2} & \frac{1}{2}\\ & 1\end{matrix}\,\middle|\, e^2\right) = 2\pi - \frac{\pi}{2}e^2 - \frac{3\pi}{32}e^4 - \cdots.$

(2) Creative Telescoping

"An algorithmic toolbox for multiple sums and integrals with parameters"

Example [Euler, 1733]: Perimeter of an ellipse of eccentricity e, semi-major axis 1

$$p(e) = 4\int_0^1 \sqrt{\frac{1-e^2u^2}{1-u^2}}\,\mathrm{d}u = 4\oiint \frac{\mathrm{d}x\mathrm{d}y}{1-\frac{1-e^2x^2}{(1-x^2)y^2}}$$



Principle: Find algorithmically

$$\left((e-e^3)\partial_e^2 + (1-e^2)\partial_e + e\right)\left(\frac{1}{1-\frac{1-e^2x^2}{(1-x^2)y^2}}\right) =$$

$$\partial_x\left(\frac{e\left(1+x-x^2-x^3\right)y^2\left(2x-3+y^2+x^2\left(3e^2-y^2-2\right)\right)}{\left(y^2+x^2\left(e^2-y^2\right)-1\right)^2}\right)$$

$$+\,\partial_y\left(\frac{2e\left(e^2-1\right)x\left(1+x^3\right)y^3}{\left(y^2+x^2\left(e^2-y^2\right)-1\right)^2}\right)$$

▷ Drawback: Size(certificate) $\gg$ Size(telescoper).

(2) Creative Telescoping: several generations of algorithms

- 1G, *elimination-based*: [Fasenmyer, 1947], [Lipshitz, 1988], [Zeilberger, 1990], [Takayama, 1990], [Wilf, Zeilberger, 1990], [Chyzak, Salvy, 2000]
- 2G, *linear diff/rec rational solving*: [Zeilberger, 1990], [Zeilberger, 1991], [Almkvist, Zeilberger, 1990], [Chyzak, 2000], [Koutschan, 2010]
- 3G, combines *1G + 2G + linear algebra*: [Apagodu, Zeilberger, 2005], [Koutschan 2010], [Chen, Kauers 2012], [Chen, Kauers, Koutschan 2014]

▷ Advantages:

- 1G–3G: very general algorithms;
- 2G/3G algorithms are able to solve non-trivial applications.

▷ Drawbacks:

- 1G: slow;
- 2G: bad or unknown complexity;
- 1G and 3G: non-minimality of telescopers;
- 1G–3G: all compute (big) certificates.

(2) Creative Telescoping: several generations of algorithms

4G: roots in [Ostrogradsky, 1845], [Hermite, 1872] and [Picard, 1902]

- univariate:
  - rational $\int$: [B., Chen, Chyzak, Li, 2010]
  - hyperexponential $\int$: [B., Chen, Chyzak, Li, Xin, 2013]
  - hypergeometric $\sum$: [Chen, Huang, Kauers, Li, 2015], [Huang, 2016]
  - mixed $\int + \sum$: [B., Dumont, Salvy, 2016]
  - algebraic $\int$: [Chen, Kauers, Koutschan, 2016]
  - D-finite Fuchsian $\int$: [Chen, van Hoeij, Kauers, Koutschan, 2018]
  - D-finite $\int$: [B., Chyzak, Lairez, Salvy, 2018], [van der Hoeven, 2018]
- multiple:
  - rational bivariate $\oiint$: [Chen, Kauers, Singer, 2012], [Chen, Du, Kauers, 2021]
  - rational: [B., Lairez, Salvy, 2013], [Lairez 2016]
  - binomial sums: [B., Lairez, Salvy, 2017]

▷ Advantages:

- good complexity;
- minimality of telescopers;
- do not need to compute certificates;
- fast in practice.

▷ Drawback: not (yet) as general as 1G–3G algorithms.

Algorithm for the integration of rational functions [B., Lairez, Salvy, 2013]

- **Input:** $R(e, \mathbf{x})$ a rational function in e and $\mathbf{x} = x_1, \ldots, x_n$.
- **Output:** A linear ODE $T(e, \partial_e)y = 0$ satisfied by $y(e) = \oint\!\!\oint R(e, \mathbf{x})d\mathbf{x}$.
- **Complexity:** $\mathcal{O}(D^{8n+2})$, where $D = \deg R$.
- **Output size:** T has order $\leq D^n$ in ∂_e and degree $\leq D^{3n+2}$ in e.

▷ Roots in [B., Chen, Chyzak, Li, 2010] ($n = 1$).

▷ Relies on generalized Hermite reduction and polynomial linear algebra.

▷ Avoids the (costly) computation of certificates, of size $\Omega(D^{n^2/2})$.

▷ Previous algorithms: complexity (at least) doubly exponential in n.

▷ Highly non-trivial extension by [Lairez, 2016]: very efficient in practice.

Models 1–19: explicit expressions and transcendence

Theorem [B., Chyzak, van Hoeij, Kauers, Pech, 2017]

Let $\mathscr{S}$ be one of the models 1–19. Then

- $Q_{\mathscr{S}}(t)$ is expressible using (integrals of) ${}_2F_1$ expressions.
- $Q_{\mathscr{S}}(t)$ is transcendental, except for $\mathscr{S} =$  and $\mathscr{S} =$ .

Example (King walks in the quarter plane, A151331)

Q$(t) = \frac{1}{t}\int_0^t \frac{1}{(1+4x)^3} \cdot {}_2F_1\left(\begin{matrix}\frac{3}{2} \quad \frac{3}{2} \\ 2\end{matrix} \,\middle|\, \frac{16x(1+x)}{(1+4x)^2}\right) dx$

$$= 1 + 3t + 18t^2 + 105t^3 + 684t^4 + 4550t^5 + 31340t^6 + 219555t^7 + \cdots$$

▷ Computer-driven discovery and proof; no human proof yet.
▷ Proof uses: (1) kernel method and (2) creative telescoping
+ (3) ODE factoring and (4) ODE solving.

Theorem (Apéry's power series is transcendental)

$$f(t) = \sum_n A_n t^n, \quad \text{where } A_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2, \text{ is transcendental.}$$

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Proof:

1. Creative telescoping:

$$(n+1)^3 A_{n+1} + n^3 A_{n-1} = (2n+1)(17n^2+17n+5) A_n, \quad A_0 = 1,\ A_1 = 5$$

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2. Conversion from recurrence to differential equation $L(f) = 0$, where

   $$L = (t^4 - 34t^3 + t^2)\partial_t^3 + (6t^3 - 153t^2 + 3t)\partial_t^2 + (7t^2 - 112t + 1)\partial_t + t - 5$$

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3. Guess-and-Prove:
   compute least-order $L_f^{\min}$ in $\mathbb{Q}(t)\langle\partial_t\rangle$ such that $L_f^{\min}(f) = 0$

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4. Basis of formal solutions of $L_f^{\min}$ at $t = 0$:

   $$\left\{1 + 5t + O(t^2),\ \ln(t) + (5\ln(t) + 12)t + O(t^2),\ \ln(t)^2 + (5\ln(t)^2 + 24\ln(t))t + O(t^2)\right\}$$

Algorithmic transcendence proofs: blending Guess-and-Prove and CT

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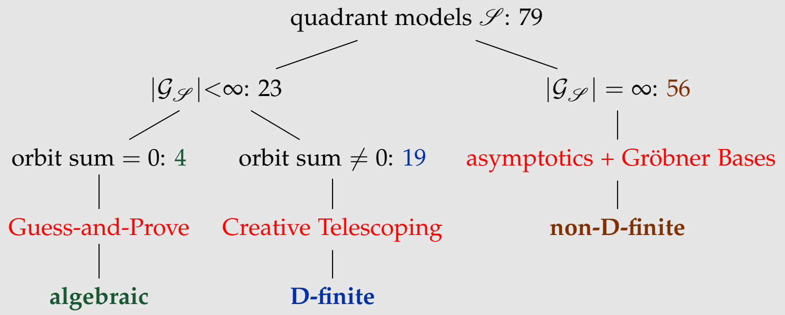
5. Conclusion: f is transcendental[†]

[†] f algebraic would imply a full basis of algebraic solutions for $L_f^{\min}$ [Tannery, 1875].

$Q_{\mathscr{S}}$ is D-finite $\iff$ a certain group $\mathcal{G}_{\mathscr{S}}$ is finite (!)



Summary: classification of walks with small steps in $\mathbb{N}^2$

$Q_{\mathscr{S}}$ is D-finite $\iff$ a certain group $\mathcal{G}_{\mathscr{S}}$ is finite (!)

quadrant models $\mathscr{S}$: 79

- $|\mathcal{G}_{\mathscr{S}}| < \infty$: 23
  - $\mathrm{OS} = 0$: 4 — Guess-and-Prove — **algebraic**
  - $\mathrm{OS} \neq 0$: 19 — kernel + CT — **D-finite transc.**
- $|\mathcal{G}_{\mathscr{S}}| = \infty$: 56
  - $\exists$ decoupling*: 9 — Tutte's invariants — **D-algebraic**
  - $\nexists$ decoupling: 47 — diff. Galois — **D-transcendental**

* $\exists U \in \mathbb{Q}(x,t), V \in \mathbb{Q}(y,t)$ s.t. $U(x) + V(y) = xy$ on the kernel $K(x,y;t) = 0$.

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▷ Many contributors (2010–2021): Bernardi, B., Bousquet-Mélou, Chyzak, Dreyfus, Hardouin, van Hoeij, Kauers, Kurkova, Mishna, Pech, Raschel, Roques, Salvy, Singer

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▷ Proofs use various tools: algebra, complex analysis, probability theory, differential Galois theory, computer algebra, etc.

Conclusion

- Enumerative Combinatorics and Computer Algebra enrich one another
- Classification of $Q(x, y; t)$ fully completed for 2D small-steps walks
- Robust algorithmic methods, based on efficient algorithms:
  - Guess-and-Prove
  - Creative Telescoping
- Brute-force and/or use of naive algorithms = hopeless.
  E.g. size of algebraic equations for $G(x, y; t) \approx$ 30Gb.

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- Brute-force and/or use of naive algorithms = hopeless.
  E.g. size of algebraic equations for $G(x, y; t) \approx$ 30Gb.

- Lack of "purely human" proofs for some results.
- Many beautiful open questions for 2D walk models with repeated or large steps, and in different cones, and in dimension > 2.

Thanks for your attention!

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- Explicit formula for the generating series of diagonal 3D Rook paths, with F. Chyzak, M. van Hoeij and L. Pech. Séminaire Lotharingien de Combinatoire, 2011.
- Non-D-finite excursions in the quarter plane, with K. Raschel and B. Salvy. Journal of Combinatorial Theory A, 2014.
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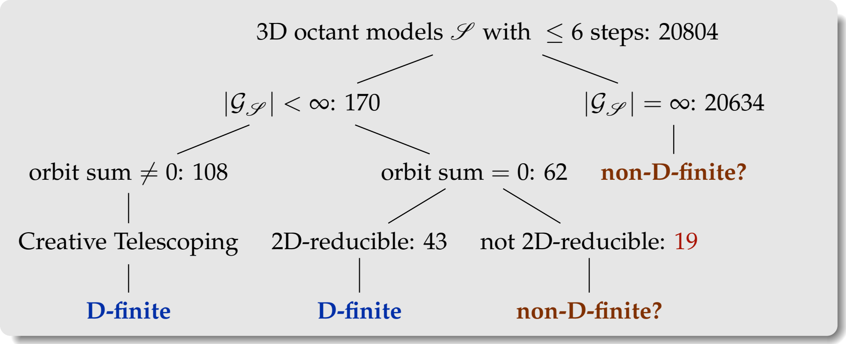
- A human proof of Gessel's lattice path conjecture, with I. Kurkova, K. Raschel, Transactions of the American Mathematical Society, 2017.
- Hypergeometric expressions for generating functions of walks with small steps in the quarter plane, with F. Chyzak, M. van Hoeij, M. Kauers and L. Pech, European Journal of Combinatorics, 2017.
- Counting walks with large steps in an orthant, with M. Bousquet-Mélou and S. Melczer, Journal of the European Mathematical Society, 2021.
- Computer Algebra in the Service of Enumerative Combinatorics, Proceedings of ISSAC'21, 2021.

Bonus

Beyond dimension 2: walks with small-steps in $\mathbb{N}^3$

▷ $2^{3^3-1} \approx 67$ million models, of which ≈ 11 million inherently 3D



[B., Bousquet-Mélou, Kauers, Melczer, 2016] + [Du, Hou, Wang, 2017];
completed by [Bacher, Kauers, Yatchak, 2016]

Beyond dimension 2: walks with small-steps in $\mathbb{N}^3$

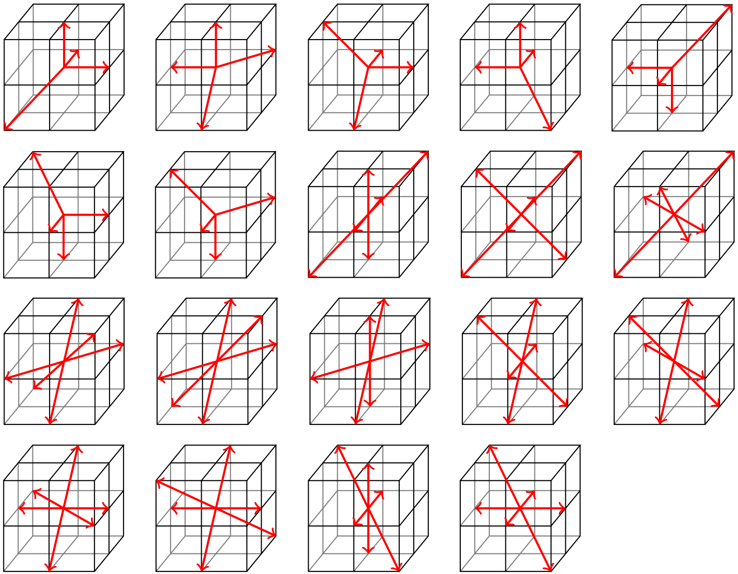
▷ $2^{3^3-1} \approx 67$ million models, of which ≈ 11 million inherently 3D

- 3D octant models $\mathscr{S}$ with ≤ 6 steps: 20804
  - $|\mathcal{G}_{\mathscr{S}}| < \infty$: 170
    - orbit sum $\neq 0$: 108
      - Creative Telescoping
        - **D-finite**
    - orbit sum $= 0$: 62
      - 2D-reducible: 43
        - **D-finite**
      - not 2D-reducible: 19
        - **non-D-finite?**
  - $|\mathcal{G}_{\mathscr{S}}| = \infty$: 20634
    - **non-D-finite?**

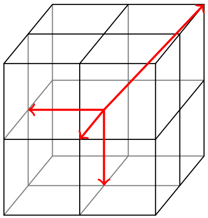
[B., Bousquet-Mélou, Kauers, Melczer, 2016] + [Du, Hou, Wang, 2017]; completed by [Bacher, Kauers, Yatchak, 2016]

Question: differential finiteness $\iff$ finiteness of the group?

Answer: probably no



Open question: 3D Kreweras excursions



Numerical computations [Dahne, Salvy, 2020] suggest:

$$k_{4n} = C \cdot 256^n / n^{\alpha}, \text{ for } \alpha = 3.3257570041744\ldots \notin \mathbb{Q},$$

so excursions are very probably non-D-finite

Beyond small steps: Walks in $\mathbb{N}^2$ with large steps

quadrant models with steps in $\{-2,-1,0,1\}^2$: 13 110

- $|\mathcal{G}_{\mathscr{S}}| < \infty$: 240
  - OS $\neq 0$: 431 → **D-finite**
  - OS $= 0$: 9 → **D-finite?**
- $|\mathcal{G}_{\mathscr{S}}| = \infty$: 12 870
  - α rational: 16 → **non-D-finite?**
  - α irrational: 12 854 → **non-D-finite**

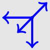
[B., Bousquet-Mélou, Melczer, 2021]

Question: differential finiteness $\iff$ finiteness of the group?

Answer: ?

Two challenging models with large steps

Conjecture 1 [B., Bousquet-Mélou, Melczer, 2021]

For the model  the excursions generating function $Q(0,0;t^{1/2})$ equals

$$\frac{1}{3t} - \frac{1}{6t} \cdot \left(\frac{1-12t}{(1+36t)^{1/3}} \cdot {}_2F_1\left(\begin{matrix} \frac{1}{6} & \frac{2}{3} \\ & 1 \end{matrix} \,\middle|\, \frac{108t(1+4t)^2}{(1+36t)^2} \right) + \sqrt{1-12t} \cdot {}_2F_1\left(\begin{matrix} -\frac{1}{6} & \frac{2}{3} \\ & 1 \end{matrix} \,\middle|\, \frac{108t(1+4t)^2}{(1-12t)^2} \right) \right).$$

Conjecture 2 [B., Bousquet-Mélou, Melczer, 2021]

For the model  the excursions generating function $Q(0,0;t)$ equals

$$\frac{(1-24\,U+120\,U^2-144\,U^3)\,(1-4\,U)}{(1-3\,U)\,(1-2\,U)^{3/2}\,(1-6\,U)^{9/2}},$$

where $U = t^4 + 53\,t^8 + 4363\,t^{12} + \cdots$ is the unique series in $\mathbb{Q}[[t]]$ satisfying

$$U\,(1-2\,U)^3\,(1-3\,U)^3\,(1-6\,U)^9 = t^4\,(1-4\,U)^4.$$

Beyond the first quadrant: three-quadrant walks with small steps

three-quadrant models $\mathscr{S}$: 74

$|\mathcal{G}_{\mathscr{S}}| < \infty$: 23

D-finite?

3 (4 ?) — **algebraic**

3 (19 ?) — **transcendental**

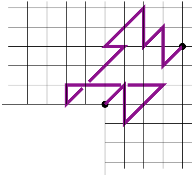
$|\mathcal{G}_{\mathscr{S}}| = \infty$: 51

non-D-finite

1 (9 ?) — **D-algebraic**

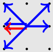
3 (42 ?) — **D-transcendental**

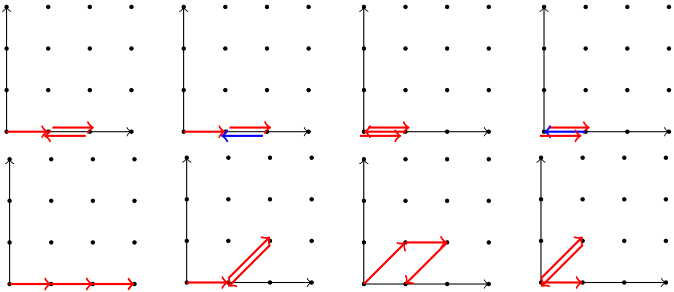
▷ Partial classification due to [Bousquet-Mélou, 2016], [Raschel, Trotignon, 2019], [Mustapha, 2019], [Dreyfus, Trotignon, 2020], [Bousquet-Mélou, Wallner, 2021], [Bousquet-Mélou, 2021]



A difficult quadrant model with repeated steps

Theorem [B., Bousquet-Mélou, Kauers, Melczer, 2016]

Let $a_n = \#$ $\Big\{$  $-$ walks of length n in $\mathbb{N}^2$ from $(0,0)$ to $(\star,0)\Big\}$. Then $f(t) = \sum_n a_n t^n = 1 + t + 4\,t^2 + 8\,t^3 + 39\,t^4 + 98\,t^5 + \cdots$ is transcendental.



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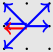
Let $a_n = \#\Big\{$  $-$ walks of length n in $\mathbb{N}^2$ from $(0,0)$ to $(\star,0)\Big\}$. Then $f(t) = \sum_n a_n t^n = 1 + t + 4\,t^2 + 8\,t^3 + 39\,t^4 + 98\,t^5 + \cdots$ is transcendental.

Proof:

1. Discover and certify a differential equation L for $f(t)$ of order 11 and degree 73 — high-tech Guess-and-Prove
2. If $\mathrm{ord}(L_f^{\min}) \leq 10$, then $\deg_t(L_f^{\min}) \leq 580$ — apparent singularities
3. Rule out this possibility — differential Hermite-Padé approximants
4. Thus, $L_f^{\min} = L$
5. L has a log singularity at $t = 0$, and so f is transcendental $\square$

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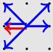
Let $a_n = \#\Big\{$  $-$ walks of length n in $\mathbb{N}^2$ from $(0,0)$ to $(\star,0)\Big\}$. Then $f(t) = \sum_n a_n t^n = 1 + t + 4\,t^2 + 8\,t^3 + 39\,t^4 + 98\,t^5 + \cdots$ is transcendental.

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▷ General minimization algorithm and application to transcendence [B., Rivoal, Salvy, 2021]

Solution of the “exercise”

Excursions in $\mathbb{Z} \times \mathbb{N}$

- The kernel equation reads (with $K(x,y) = 1 - t(y + \bar{x} + x\bar{y})$):

$$K(x,y)yH(x,y) = y - txH(x,0)$$

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Excursions in $\mathbb{Z} \times \mathbb{N}$

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$$(27t^4 - t)A''(t) + (108t^3 - 4)A'(t) + 54t^2A(t) = 0.$$

```
> Zeilberger(1/x * sqrt((t-x)^2 - 4*t^2*x^3)/(2*t^2*x^2), t, x, Dt);
```

The group of the model $\{\uparrow, \leftarrow, \searrow\}$

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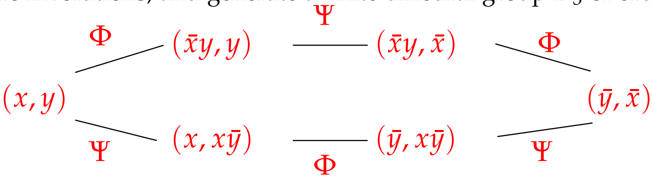
Φ and Ψ are involutions, and generate a finite dihedral group D_3 of order 6:



Diagonal walks in $\mathbb{N}^2$

- Orbit equation:

$$xyQ(x,y) - \bar{x}y^2Q(\bar{x}y,y) + \bar{x}^2yQ(\bar{x}y,\bar{x}) - \bar{x}\bar{y}Q(\bar{y},\bar{x}) + x\bar{y}^2Q(\bar{y},x\bar{y}) - x^2\bar{y}Q(x,x\bar{y}) = \frac{xy - \bar{x}y^2 + \bar{x}^2y - \bar{x}\bar{y} + x\bar{y}^2 - x^2\bar{y}}{1 - t(y + \bar{x} + x\bar{y})}$$

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- Corollary [Bousquet-Mélou & Mishna, 2010]:

$$xyQ(x,y) = [x^{>0}y^{>0}]\,\frac{xy - \bar{x}y^2 + \bar{x}^2y - \bar{x}\bar{y} + x\bar{y}^2 - x^2\bar{y}}{1 - t(y + \bar{x} + x\bar{y})}$$

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$$B(t) = [z^0]Q(z,\bar{z}) = [u^{-1}v^{-1}z^{-1}]\ \frac{\bar{u}\bar{v} - u\bar{v}^2 + u^2\bar{v} - uv + \bar{u}v^2 - \bar{u}^2v}{z(1-zu)(1-v\bar{z})(1-t(\bar{v}+u+\bar{u}v))}$$

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- Creative Telescoping gives a differential equation for $B(t)$:

$$(27t^4 - t)B''(t) + (108t^3 - 4)B'(t) + 54t^2B(t) = 0.$$

Conclusion

We have proved that $A(t)$ and $B(t)$ are both solutions of

$$(27t^4 - t)y''(t) + (108t^3 - 4)y'(t) + 54t^2y(t) = 0.$$

Solving this equation proves:

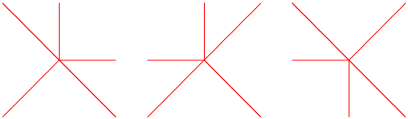
$$A(t) = B(t) = {}_2F_1\left(\begin{matrix}1/3 \quad 2/3\\ 2\end{matrix}\,\middle|\, 27\,t^3\right) = \sum_{n=0}^{\infty} \frac{(3n)!}{n!^3} \frac{t^{3n}}{n+1}.$$

Thus the two sequences are equal to

$$a_{3n} = b_{3n} = \frac{(3n)!}{n!^2 \cdot (n+1)!}, \qquad \text{and} \quad a_m = b_m = 0 \quad \text{if 3 does not divide } m.$$

Example with infinite group: the scarecrows

[B., Raschel, Salvy, 2014]: $Q_{\mathscr{S}}(0,0;t)$ is not D-finite for the models



▷ For the 1st and the 3rd, the excursions sequence $[t^n]\,Q_{\mathscr{S}}(0,0;t)$

$$1, 0, 0, 2, 4, 8, 28, 108, 372, \ldots$$

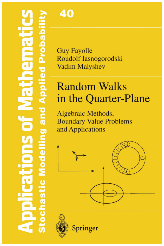
is $\sim K \cdot 5^n \cdot n^{-\alpha}$, with $\alpha = 1 + \pi / \arccos(1/4) = 3.383396\ldots$

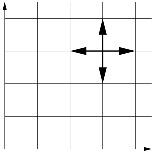
[Denisov, Wachtel, 2015]

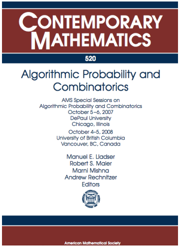
▷ The irrationality of α prevents $Q_{\mathscr{S}}(0,0;t)$ from being D-finite.

[Katz, 1970; Chudnovsky, 1985; André, 1989]

The group of a model: the simple walk case







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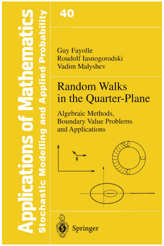

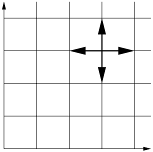

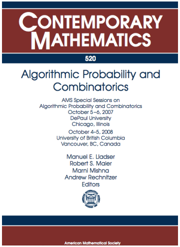

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$$\psi(x,y) = \left(x, \frac{1}{y}\right), \qquad \phi(x,y) = \left(\frac{1}{x}, y\right),$$

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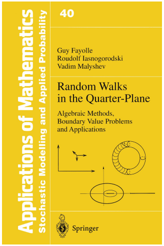

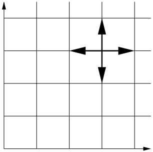

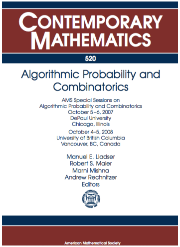

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and thus under any element of the group

$$\langle \psi, \phi \rangle = \left\{ (x,y), \left(x, \frac{1}{y}\right), \left(\frac{1}{x}, \frac{1}{y}\right), \left(\frac{1}{x}, y\right) \right\}.$$

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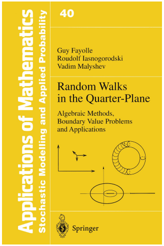

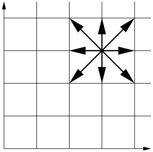

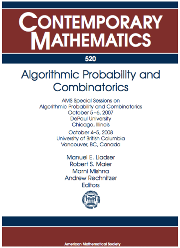

The generating polynomial $\chi_{\mathscr{S}} := \sum_{(i,j)\in\mathscr{S}} x^i y^j = \sum_{i=-1}^{1} B_i(y) x^i = \sum_{j=-1}^{1} A_j(x) y^j$

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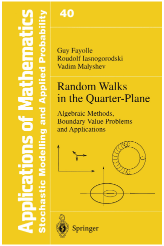

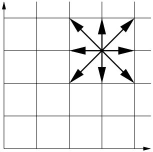

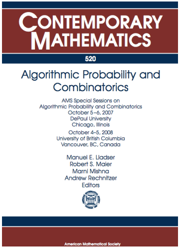

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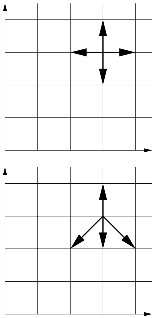
$$\psi(x,y) = \left(x, \frac{A_{-1}(x)}{A_{+1}(x)} \frac{1}{y}\right), \qquad \phi(x,y) = \left(\frac{B_{-1}(y)}{B_{+1}(y)} \frac{1}{x}, y\right),$$

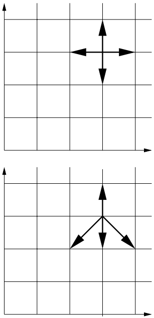
and thus under any element of the (dihedral) group

$$\mathcal{G}_{\mathscr{S}} := \langle \psi, \phi \rangle.$$



Order 4,

Examples of groups



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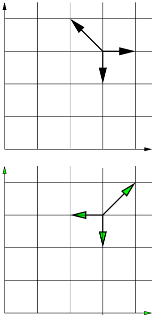

order 6,

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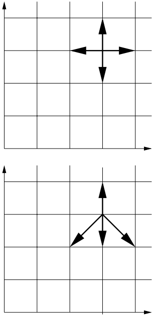

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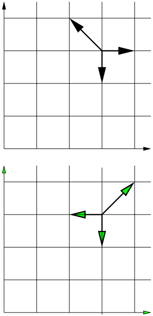

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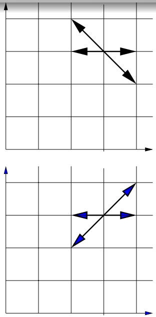
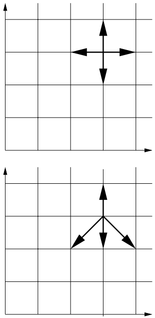


order 8,

Examples of groups



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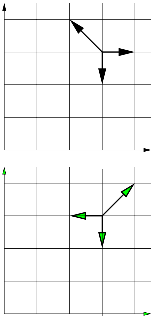

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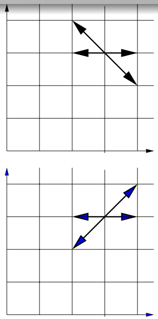

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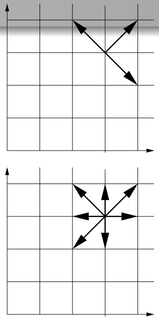

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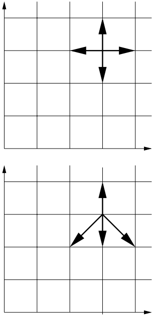

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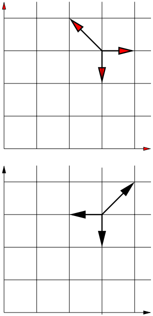

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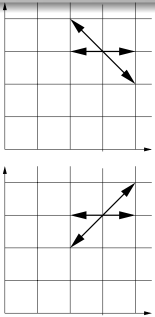

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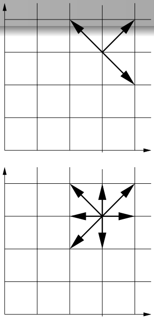

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