Computer algebra for combinatorics

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Inria

MPRI, COMPALG

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Computer Algebra for Enumerative Combinatorics

Enumerative Combinatorics: science of counting

Area of mathematics primarily concerned with counting discrete objects.

▷ Main outcome: theorems

Computer Algebra: effective mathematics

Area of computer science primarily concerned with the algorithmic manipulation of algebraic objects.

▷ Main outcome: algorithms

Computer Algebra for Enumerative Combinatorics

Today: Algorithms for proving Theorems on Lattice Paths Combinatorics.

An (innocent looking) combinatorial question

Let $\mathscr{S} = \{\uparrow, \leftarrow, \searrow\}$. An $\mathscr{S}$-walk is a path in $\mathbb{Z}^2$ using only steps from $\mathscr{S}$. Show that, for any integer n, the following quantities are equal:

(i) number a_n of n-steps $\mathscr{S}$-walks confined to the upper half plane $\mathbb{Z} \times \mathbb{N}$ that start and finish at the origin $(0,0)$ (*excursions*);

(ii) number b_n of n-steps $\mathscr{S}$-walks confined to the quarter plane $\mathbb{N}^2$ that start at the origin $(0,0)$ and finish on the diagonal of $\mathbb{N}^2$ (*diagonal walks*).

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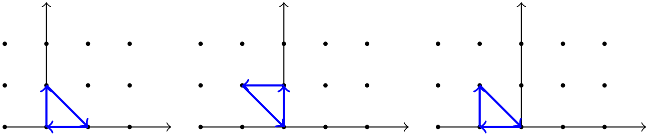
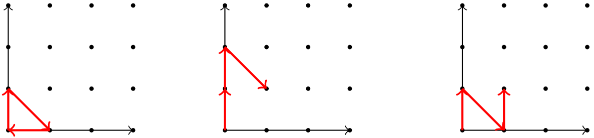
For instance, for $n = 3$, this common value is $a_3 = b_3 = 3$:

(i)



(ii)



Teasers

Teaser 1: This "exercise" is non-trivial

Teaser 2: It can be solved using Experimental Math and Computer Algebra

Teaser 3: ...by two robust and efficient algorithmic techniques,
Guess-and-Prove and Creative Telescoping

Why count walks?

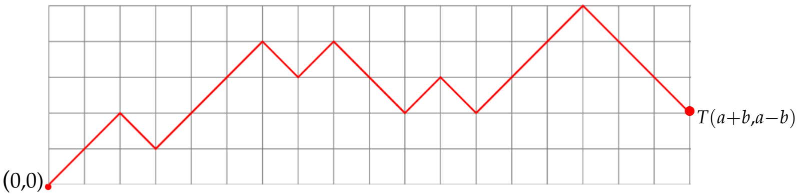
Many objects can be encoded by (confined) walks:

- probability theory (voting, games of chance, branching processes, ...)
- discrete mathematics (permutations, trees, words, urns, ...)
- statistical physics (Ising model, ...)
- operations research (queueing theory, ...)

Counting walks is an old topic: the ballot problem [Bertrand, 1887]

Suppose that candidates A and B are running in an election. If a votes are cast for A and b votes are cast for B, where $a > b$, then the probability that A stays ahead of B throughout the counting of the ballots is $(a-b)/(a+b)$.

Lattice path reformulation: find the number of paths in $\mathbb{Z}^2$ with a upsteps $\nearrow$ and b downsteps $\searrow$ that start at the origin and never touch the x-axis



▷ Without the constraint, the number of such paths is $\binom{a+b}{a}$

$\longrightarrow$ a Guess-and-Prove proof in a few slides

Counting walks is an old topic: the ballot problem [Bertrand, 1887]

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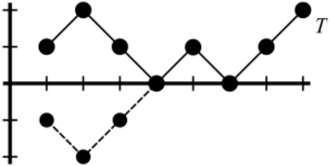
Lattice path reformulation: find the number of paths in $\mathbb{Z}^2$ with $a-1$ upsteps $\nearrow$ and b downsteps $\searrow$ that start at $(1,1)$ and never touch the x-axis

Reflection principle [Aebly, 1923]: *paths in* $\mathbb{Z}^2$ *from* $(1,1)$ *to* $T(a+b, a-b)$ *that do touch the x-axis* are in bijection with *paths in* $\mathbb{Z}^2$ *from* $(1,-1)$ *to* T



Answer: $\underbrace{(\textit{paths in } \mathbb{Z}^2 \textit{ from } (1,1) \textit{ to } T)}_{\binom{a+b-1}{a-1}} - \underbrace{(\textit{paths in } \mathbb{Z}^2 \textit{ from } (1,-1) \textit{ to } T)}_{\binom{a+b-1}{b-1}}$

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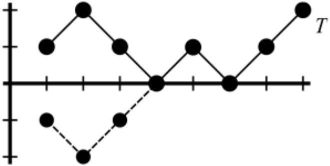
Reflection principle [Aebly, 1923]: *paths in $\mathbb{Z}^2$ from $(1, 1)$ to $T(a + b, a - b)$ that do touch the x-axis* are in bijection with *paths in $\mathbb{Z}^2$ from $(1, -1)$ to T*



Answer: $\underbrace{(\textit{paths in } \mathbb{Z}^2 \textit{ from } (1,1) \textit{ to } T) - (\textit{paths in } \mathbb{Z}^2 \textit{ from } (1,-1) \textit{ to } T)}$

$$\binom{a+b-1}{a-1} - \binom{a+b-1}{b-1} = \frac{a-b}{a+b}\binom{a+b}{a}$$

Lot of recent activity; many recent contributors:

Arquès, Bacher, Banderier, Beaton, Bernardi, Biane, Bonnet, Bostan, Bousquet-Mélou, Buchacher, Budd, Chyzak, Cori, Courtiel, Denisov, Dreyfus, Du, Duchon, Dulucq, Duraj, Elvey-Price, Fayolle, Fisher, Flajolet, Franceschi, Fusy, Garbit, Gessel, Gouyou-Beauchamps, Guttmann, Guy, Hardouin, van Hoeij, Hou, Iasnogorodski, Johnson, Kauers, Kenyon, Koutschan, Krattenthaler, Kreweras, Kurkova, Lecouvey, Malyshev, Melczer, Miller, Mishna, Niederhausen, Owczarek, Pech, Petkovšek, Prellberg, Raschel, Rechnitzer, Roques, Sagan, Salvy, Sheffield, Singer, Tarrago, Trotignon, Verron, Viennot, Wachtel, Wallner, Wang, Wilf, D. Wilson, M. Wilson, Xu, Yatchak, Yeats, Zeilberger, . . .

etc.

. . . but it is still a very hot topic

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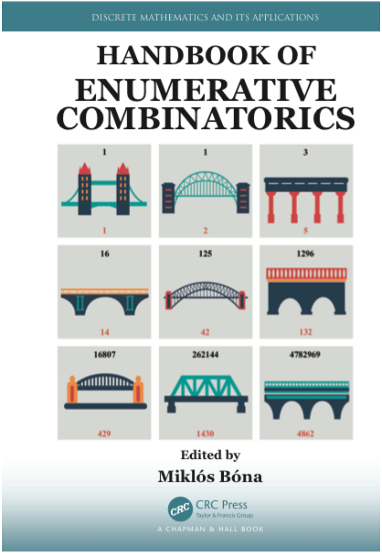
etc.

~~**Specific question**~~
~~**Ad hoc solution**~~ ⟶ **Systematic approach**



Chapter 10

Lattice Path Enumeration

Christian Krattenthaler

Universität Wien

CONTENTS

LaBRI — université de BORDEAUX

THÈSE

PRÉSENTÉE À

L'UNIVERSITÉ DE BORDEAUX

ÉCOLE DOCTORALE
DE MATHÉMATIQUES ET INFORMATIQUE

par **Pierre BONNET**

POUR OBTENIR LE GRADE DE

DOCTEUR

SPÉCIALITÉ : INFORMATIQUE

La longue marche à travers le quart de plan

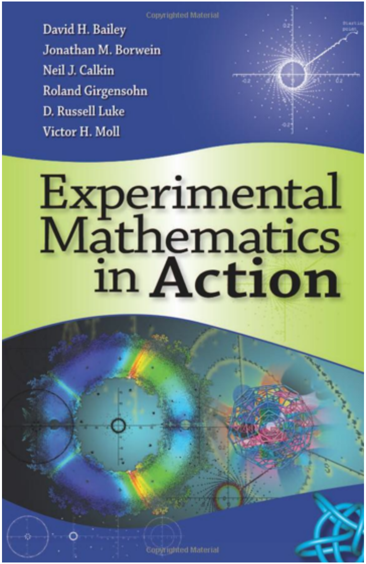
Sous la direction de **Mireille BOUSQUET-MÉLOU** et **Charlotte HARDOUIN**

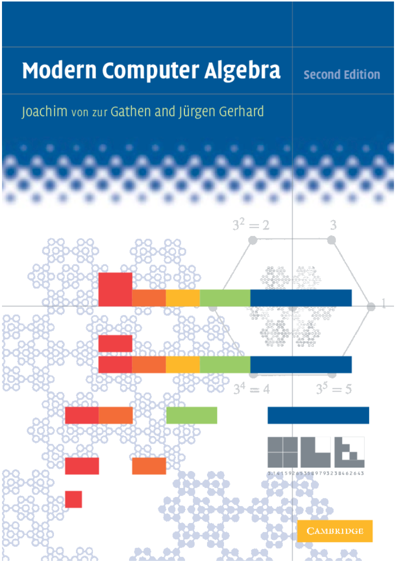
Soutenue le 10 Février 2026

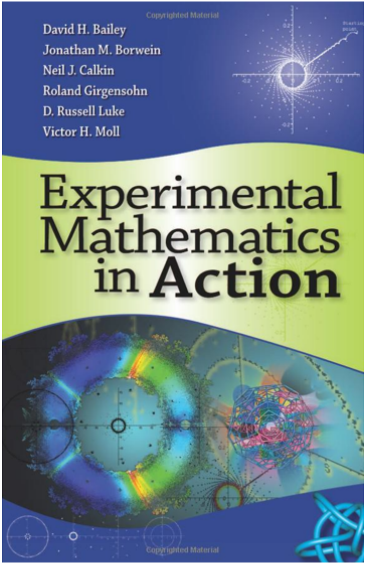
Devant la commission d'examen composée de :

Mme Frédérique BASSINO	Professeure, Université Sorbonne Paris Nord	Examinatrice
M. Alin BOSTAN	Directeur de recherche, INRIA Saclay	Rapporteur
Mme Mireille BOUSQUET-MÉLOU	Directrice de recherche, CNRS, Université de Bordeaux	Directrice
M. Vincent DELECROIX	Chargé de recherche, CNRS, Université de Bordeaux	Examinateur
Mme Charlotte HARDOUIN	Professeure, Université de Toulouse	Directrice
M. Kilian RASCHEL	Directeur de recherche, CNRS, Université d'Angers	Rapporteur
M. Julien ROQUES	Professeur, Université Lyon 1	Examinateur

Our approach: Experimental Mathematics using Computer Algebra






David H. Bailey
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Roland Girgensohn
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Experimental Mathematics in Action
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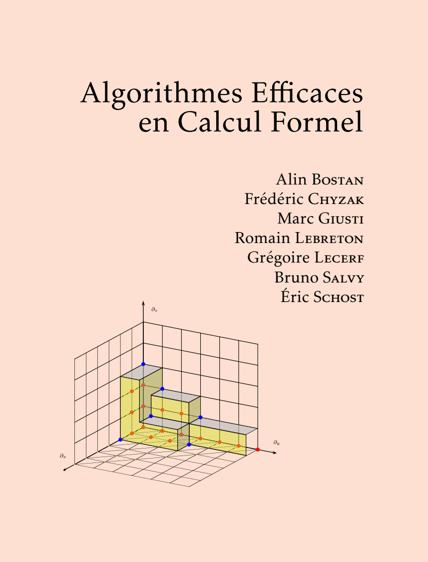

Algorithmes Efficaces
en Calcul Formel
Alin Bostan
Frédéric Chyzak
Marc Giusti
Romain Lebreton
Grégoire Lecerf
Bruno Salvy
Éric Schost

Lattice walks with small steps in the quarter plane

▷ Walks in $\mathbb{N}^2$ starting at $(0,0)$ and using steps in a fixed subset $\mathscr{S}$ of

$$\{\swarrow, \leftarrow, \nwarrow, \uparrow, \nearrow, \rightarrow, \searrow, \downarrow\}.$$

▷ Counting sequence: $q_{\mathscr{S}}(n)$ = number of $\mathscr{S}$-walks of length n

▷ Length generating function:

$$Q_{\mathscr{S}}(t) = \sum_{n=0}^{\infty} q_{\mathscr{S}}(n) t^n \in \mathbb{Z}[[t]]$$

Lattice walks with small steps in the quarter plane

▷ Walks in $\mathbb{N}^2$ starting at $(0,0)$ and using steps in a fixed subset $\mathscr{S}$ of

$$\{\swarrow, \leftarrow, \nwarrow, \uparrow, \nearrow, \rightarrow, \searrow, \downarrow\}.$$

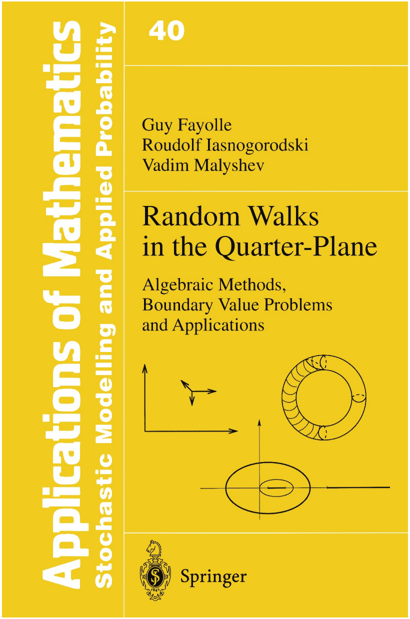
▷ Refinement: $q_{\mathscr{S}}(i,j;n)$ = number of $\mathscr{S}$-walks of length n ending at (i,j)

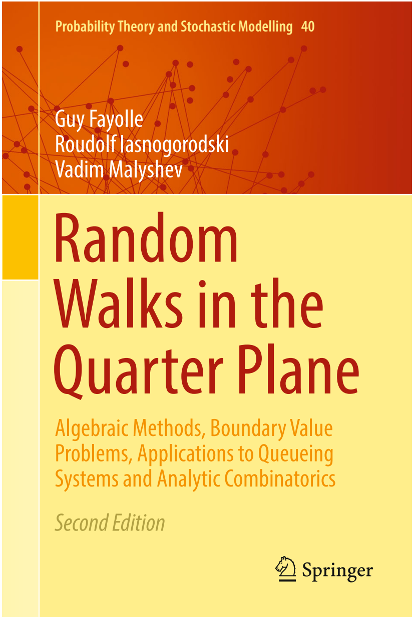
▷ Full generating function (with "catalytic" variables x, y):

$$Q_{\mathscr{S}}(x,y;t) = \sum_{i,j,n=0}^{\infty} q_{\mathscr{S}}(i,j;n) x^i y^j t^n \in \mathbb{Z}[[x,y,t]]$$

▷ Actually: $Q_{\mathscr{S}}(x,y;t) \in \mathbb{Z}[x,y][[t]]$ and $Q_{\mathscr{S}}(1,1;t) = Q_{\mathscr{S}}(t)$

Entire books dedicated to small-steps walks in the quarter plane!





Small-steps models of interest

Among the 2^8 step sets $\mathscr{S} \subseteq \{-1,0,1\}^2 \setminus \{(0,0)\}$, some are:

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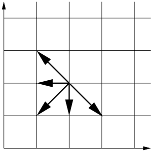

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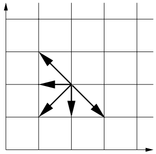

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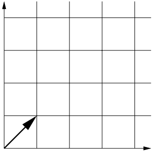

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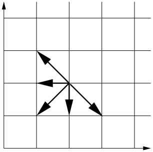

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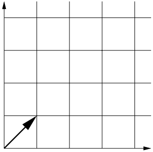

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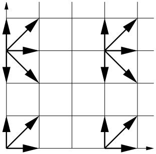

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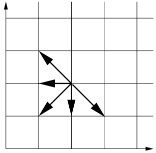

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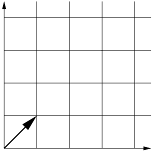

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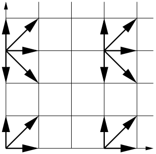

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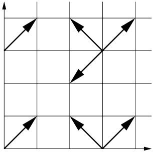

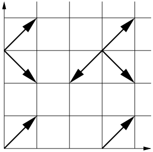

symmetrical.

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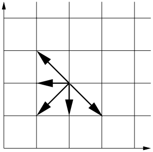

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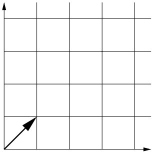

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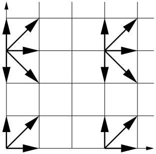

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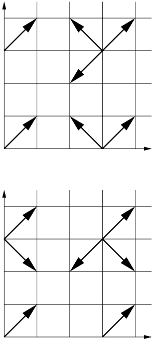
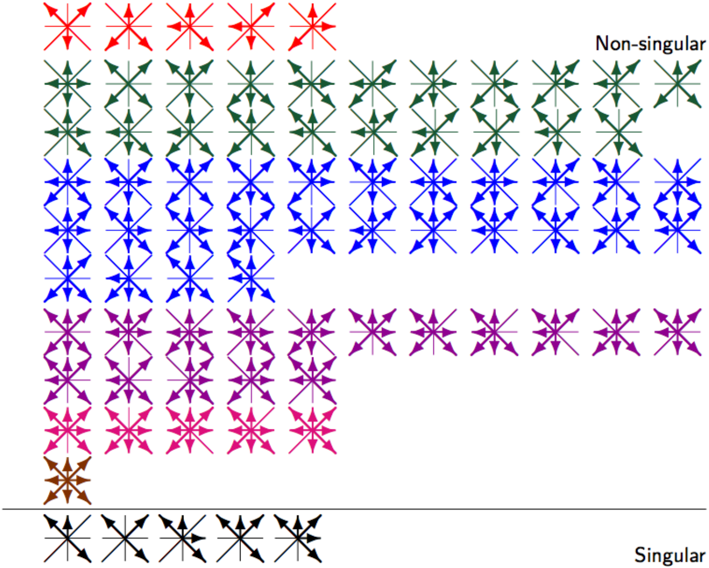


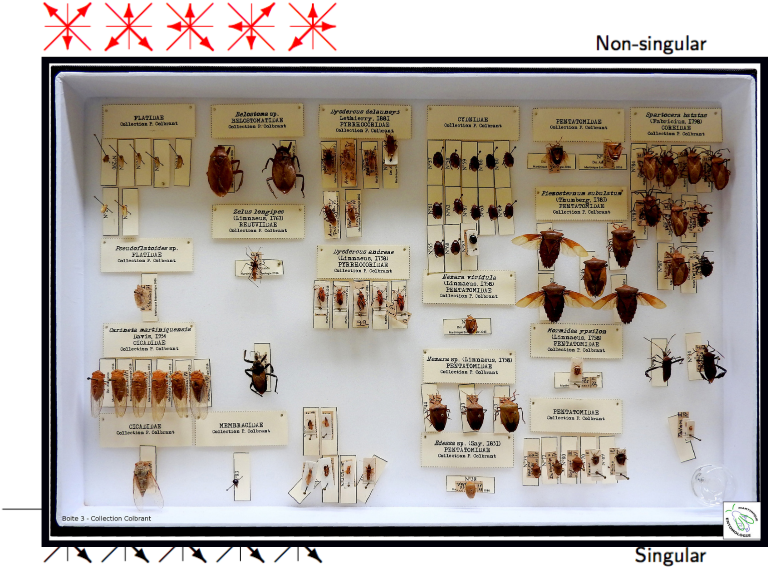
symmetrical.

One is left with 79 interesting distinct models.

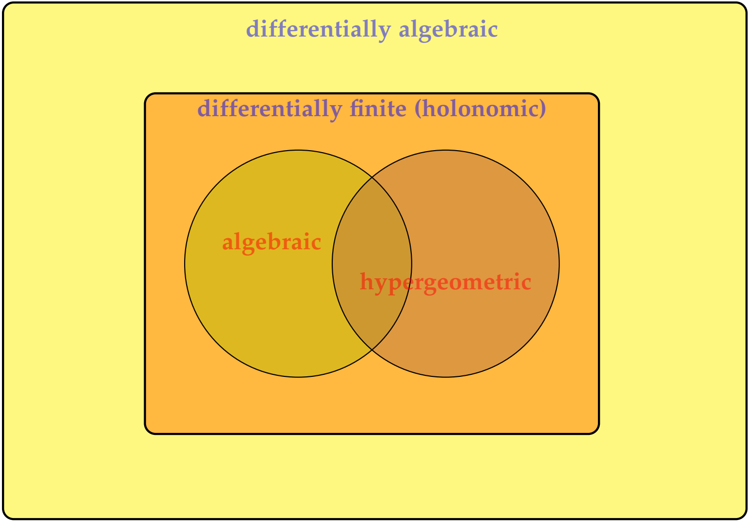
The 79 small-steps quadrant models

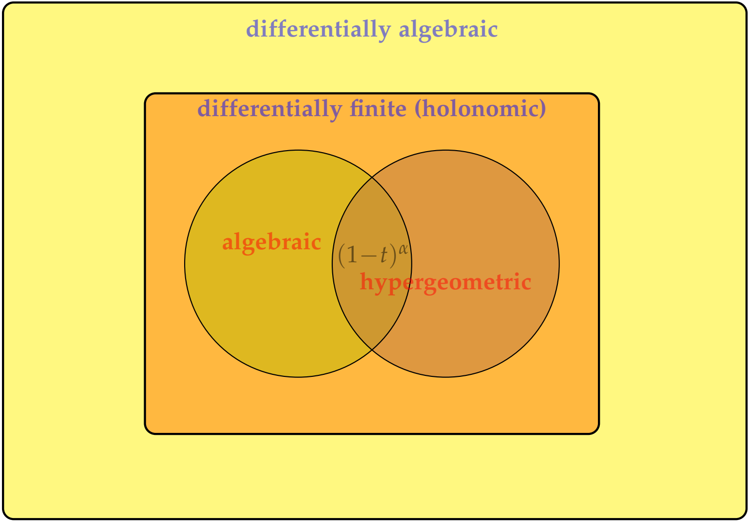


Task: classify their generating functions!

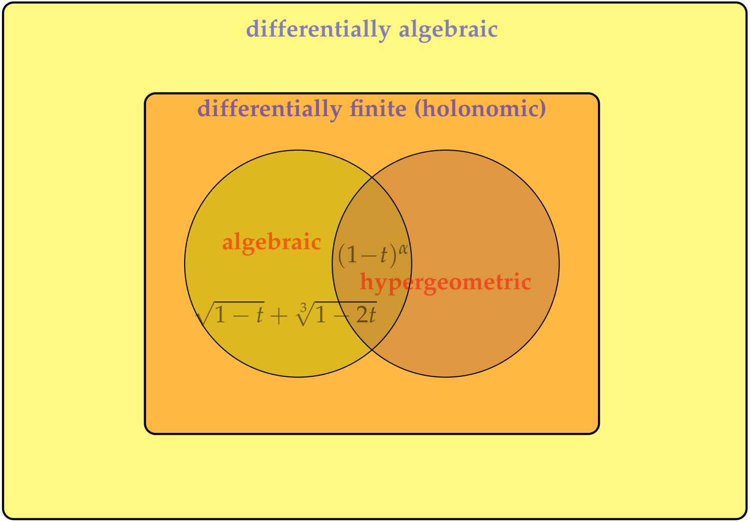


Classification criterion: properties of generating functions




differentially algebraic
differentially finite (holonomic)
algebraic
$(1-t)^\alpha$
hypergeometric

Classification criterion: properties of generating functions



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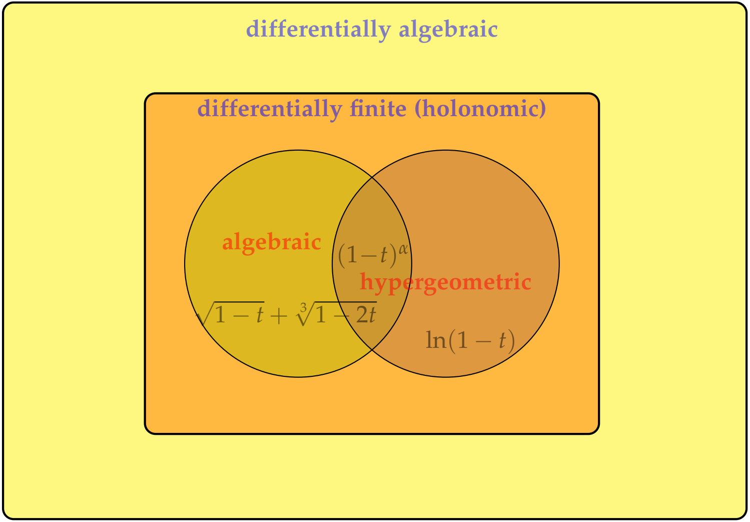

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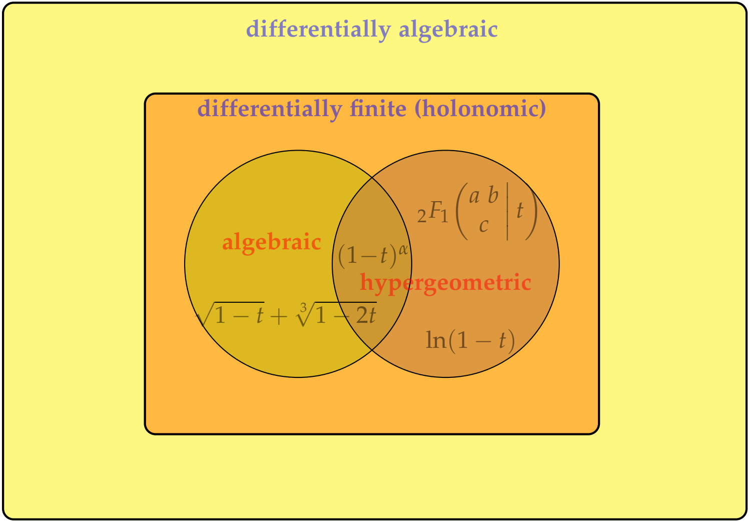
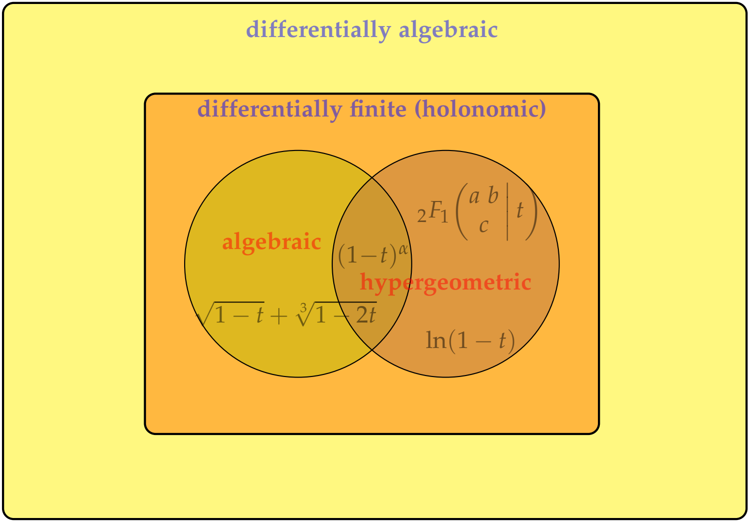


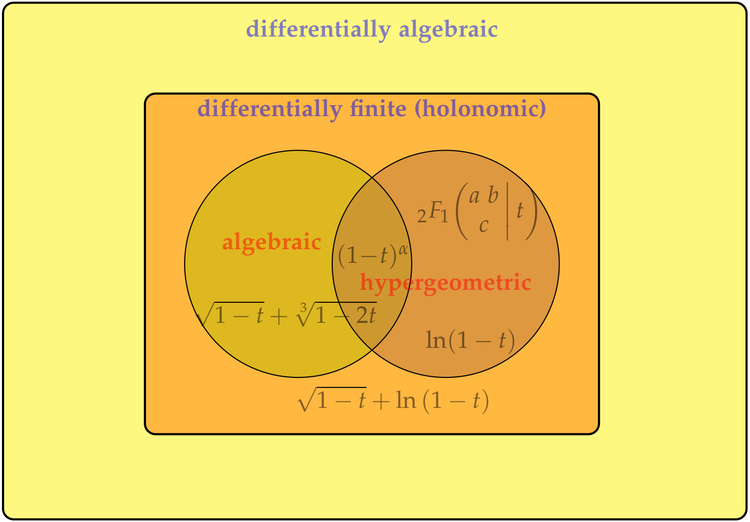
$$ {}_2F_1\left(\begin{matrix} a\ b \\ c \end{matrix} \middle| t\right) := \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{t^n}{n!}, \quad \text{where} \quad (a)_n = a(a+1)\cdots(a+n-1). $$

Classification criterion: properties of generating functions



E.g., $(1-t)^\alpha = {}_2F_1\left(\begin{matrix} -\alpha\ 1 \\ 1 \end{matrix}\,\middle|\, t\right)$, $\ln(1-t) = -t\cdot {}_2F_1\left(\begin{matrix} 1\ 1 \\ 2 \end{matrix}\,\middle|\, t\right) = -\sum_{n=1}^{\infty} \frac{t^n}{n}$

Classification criterion: properties of generating functions



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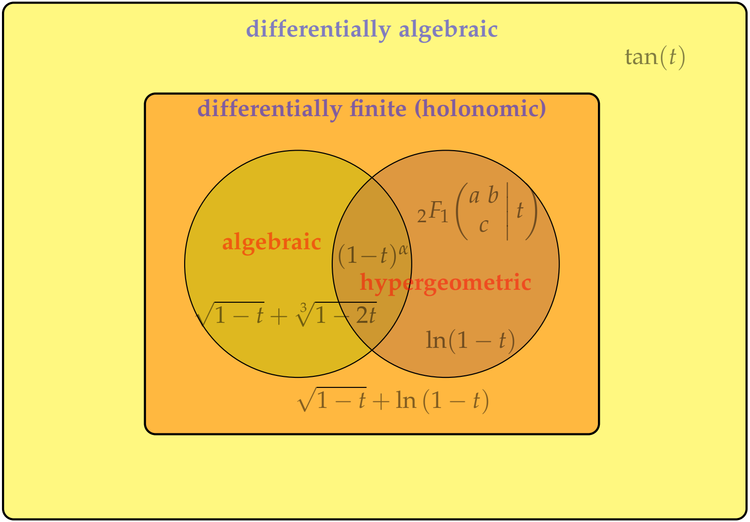

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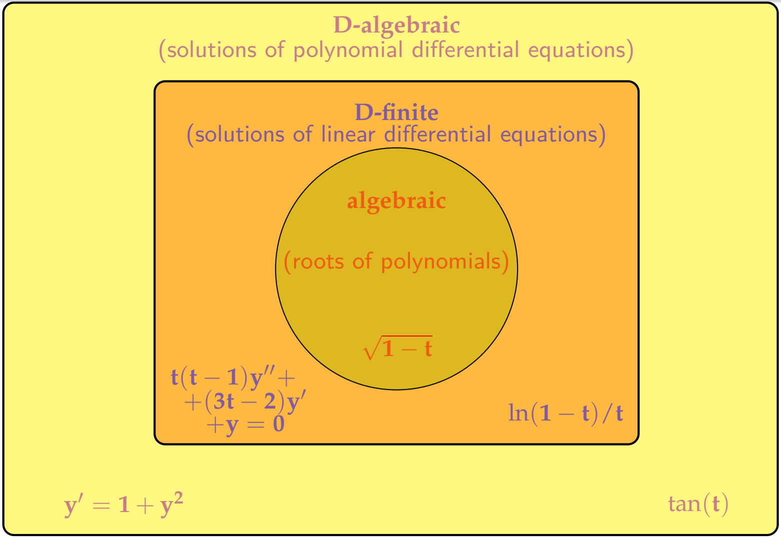
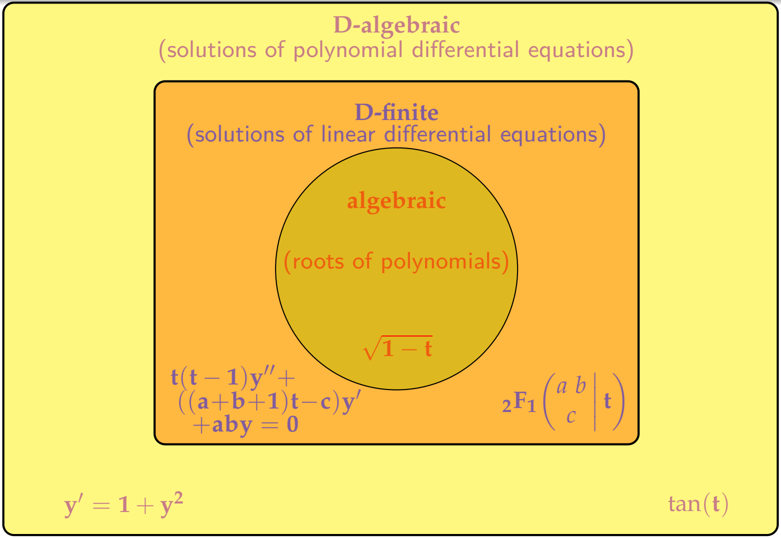


D-transcendental $\quad \Gamma(t) = \int_0^\infty x^{t-1} e^{-x}\, dx$

Classification criterion: properties of generating functions

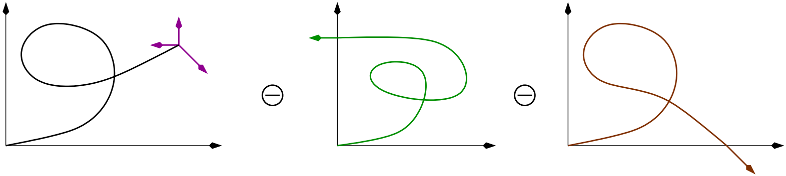


$${}_2F_1\left(\begin{matrix}a\ b\\ c\end{matrix}\,\middle|\,t\right) := \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n}\frac{t^n}{n!}, \quad \text{where} \quad (a)_n = a(a+1)\cdots(a+n-1).$$

Generating function: $Q(x,y) \equiv Q(x,y;t) = \sum_{i,j,n=0}^{\infty} q(i,j;n) x^i y^j t^n \in \mathbb{Z}[[x,y,t]]$

Recursive construction yields the *kernel equation*

$$Q(x,y) = 1 + t\left(y + \frac{1}{x} + x\frac{1}{y}\right)Q(x,y) - t\frac{1}{x}Q(0,y) - tx\frac{1}{y}Q(x,0)$$

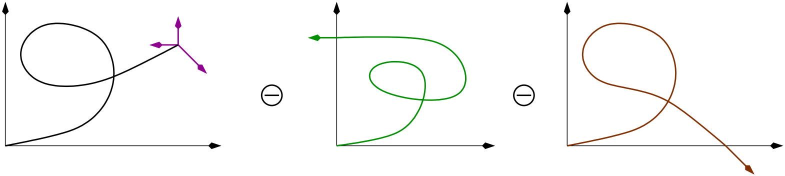


Algebraic reformulation of main task: solving a functional equation

Generating function: $Q(x,y) \equiv Q(x,y;t) = \sum_{i,j,n=0}^{\infty} q(i,j;n) x^i y^j t^n \in \mathbb{Z}[[x,y,t]]$

Recursive construction yields the *kernel equation*

$$\left(1 - t\left(y + \frac{1}{x} + x\frac{1}{y}\right)\right) xyQ(x,y) = xy - tyQ(0,y) - tx^2Q(x,0)$$



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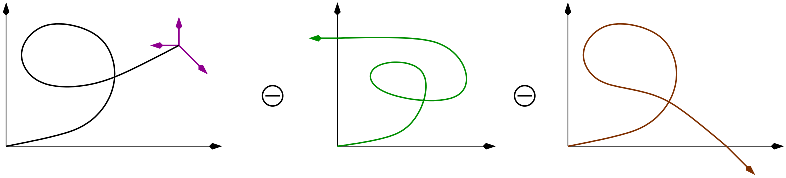

New task: Solve this functional equation!

Algebraic reformulation of main task: solving a functional equation

Generating function: $Q(x,y) \equiv Q(x,y;t) = \sum_{i,j,n=0}^{\infty} q(i,j;n)x^i y^j t^n \in \mathbb{Z}[[x,y,t]]$

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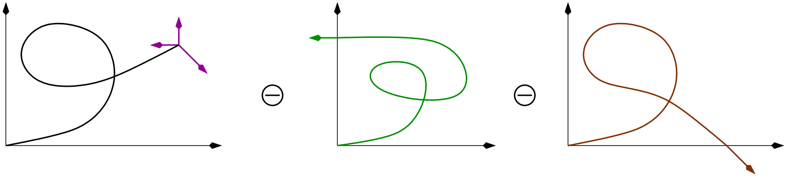
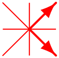
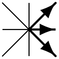


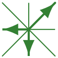
New task: For the other models – solve 78 similar equations!

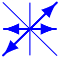
"Special" models of walks in the quarter plane

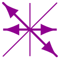
Dyck: 

Motzkin: 

Pólya: 

Kreweras: 

Gessel: 

Gouyou-Beauchamps: 

King walks: 

Tandem walks: 

▷ Kernel equation:

$$(y - tx(1 + y^2)) \cdot Q(x, y) = y - tx \cdot Q(x, 0)$$

▷ Kernel method [Knuth, 1968]:

- let $y_0 \in \mathbb{Q}[x][[t]]$ be the power series root of $K = y - tx(1 + y^2)$

$$y_0 = \frac{1 - \sqrt{1 - 4t^2x^2}}{2tx} = tx + t^3x^3 + 2t^5x^5 + \cdots \in \mathbb{Q}[x][[t]]$$

- plug $y = y_0$ in the kernel equation $\Longrightarrow$ $Q(x, 0) = \dfrac{y_0}{tx}$
- conclude algebraicity:

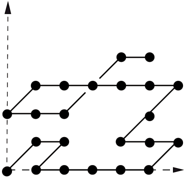
$$Q(x, y) = \frac{y - y_0}{K} = \frac{\sqrt{1 - 4t^2x^2} + 2txy - 1}{2tx(y - tx(1 + y^2))}$$

▷ Same method proves *algebraicity* for all models intrinsic to the half plane

- $g(n)$ = number of n-steps $\{\nearrow, \swarrow, \leftarrow, \rightarrow\}$-walks in $\mathbb{N}^2$
  1, 2, 7, 21, 78, 260, 988, 3458, 13300, 47880, ...

Question: What is the nature of the generating function

$$G(t) = \sum_{n=0}^{\infty} g(n)\, t^n \; ?$$



- $g(i,j;n)$ = number of n-steps $\{\nearrow, \swarrow, \leftarrow, \rightarrow\}$-walks in $\mathbb{N}^2$ from $(0,0)$ to (i,j)

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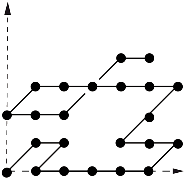

A difficult model: Gessel walks (2000) $\mathscr{S} = \{\nearrow, \swarrow, \leftarrow, \rightarrow\}$

- $g(i,j;n)$ = number of n-steps $\{\nearrow, \swarrow, \leftarrow, \rightarrow\}$-walks in $\mathbb{N}^2$ from $(0,0)$ to (i,j)

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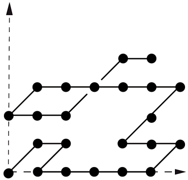

Theorem [B., Kauers, 2010]

$G(x,y;t)$ is an algebraic function†.

▷ computer-driven discovery/proof via *algorithmic Guess-and-Prove*

---

† Minimal polynomial $P(G(x,y;t);x,y,t) = 0$ has $> 10^{11}$ terms; $\approx 30\,$Gb (6 DVDs!)

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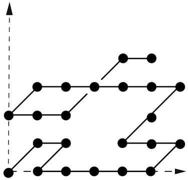
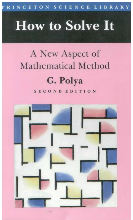


Corollary [B., Kauers, 2010] (former conjecture of Gessel's)

$$(3n+1)\, g(2n) = (12n+2)\, g(2n-1) \text{ and } (n+1)\, g(2n+1) = (4n+2)\, g(2n)$$

▷ computer-driven discovery/proof via *algorithmic Guess-and-Prove*



Guessing and Proving

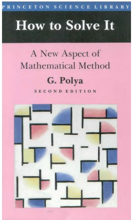
George Pólya

What is "scientific method"? Philosophers and non-philosophers have discussed this question and have not yet finished discussing it. Yet as a first introduction it can be described in three syllables:

Guess and test.

Mathematicians too follow this advice in their research although they sometimes refuse to confess it. They have, however, something which the other scientists cannot really have. For mathematicians the advice is

First guess, then prove.



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First guess, then prove.

generate data → make conjectures → prove them

Guess-and-Prove: a toy example

Question: Find $B_{i,j}$:= the number of $\{\rightarrow, \uparrow\}$-walks in $\mathbb{N}^2$ from $(0,0)$ to (i,j)

Guess-and-Prove: a toy example

Question: Find $B_{i,j} :=$ the number of $\{\rightarrow, \uparrow\}$-walks in $\mathbb{N}^2$ from $(0,0)$ to (i,j)

1. There are 2 ways to get to (i,j), either from $(i-1,j)$, or from $(i,j-1)$:

$$B_{i,j} = B_{i-1,j} + B_{i,j-1}$$

2. There is only one way to get to a point on an axis: $B_{i,0} = B_{0,j} = 1$

▷ These two rules completely determine all the numbers $B_{i,j}$

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(I) Generate data:

⋮							
1	7	28	84	210	462	924	
1	6	21	56	126	252	462	
1	5	15	35	70	126	210	
1	4	10	20	35	56	84	
1	3	6	10	15	21	28	
1	2	3	4	5	6	7	
1	1	1	1	1	1	1	…

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1	3	6	10	15	21	28	$\longrightarrow \frac{(i+1)(i+2)}{2}$
1	2	3	4	5	6	7	$\longrightarrow i+1$
1	1	1	1	1	1	1	$\longrightarrow 1$

(II) Guess:

Guess-and-Prove: a toy example

Question: Find $B_{i,j}$:= the number of $\{\rightarrow,\uparrow\}$-walks in $\mathbb{N}^2$ from $(0,0)$ to (i,j)

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$$B_{i,j} \stackrel{?}{=} \frac{(i+j)!}{i!\,j!}$$

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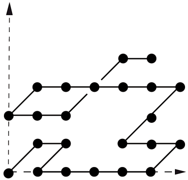
(III) Prove: If $C_{i,j} \overset{\text{def}}{=} \frac{(i+j)!}{i!j!}$, then

$$\frac{C_{i-1,j}}{C_{i,j}} + \frac{C_{i,j-1}}{C_{i,j}} = \frac{i}{i+j} + \frac{j}{i+j} = 1$$

and $C_{i,0} = C_{0,j} = 1$.

Thus $B_{i,j} = C_{i,j}$

- $g(i,j;n)$ = number of n-steps $\{\nearrow, \swarrow, \leftarrow, \rightarrow\}$-walks in $\mathbb{N}^2$ from $(0,0)$ to (i,j)



Question: What is the nature of the generating function

$$G(x,y;t) = \sum_{i,j,n=0}^{\infty} g(i,j;n)\, x^i y^j t^n \ ?$$

Answer: [B., Kauers, 2010] $G(x,y;t)$ is an algebraic function†.

Approach: $\longrightarrow$ very general and robust!

1. Generate data: compute G to precision t^{1200} ($\approx$ 1.5 billion coeffs!)
2. Guess: conjecture polynomial equations for $G(x,0;t)$ and $G(0,y;t)$ (degree 24 each, coeffs. of degree $(46,56)$, with 80-bits digits coeffs.)
3. Prove: multivariate resultants of (very big) polynomials (30 pages each)

† Minimal polynomial $P(G(x,y;t);x,y,t) = 0$ has $> 10^{11}$ terms; $\approx$ 30 Gb (6 DVDs!)

A typical Guess-and-Prove algorithmic proof

Theorem [“Gessel excursions are algebraic”]

$$g(t) := G(0,0;\sqrt{t}) = \sum_{n=0}^{\infty} \frac{(5/6)_n (1/2)_n}{(5/3)_n (2)_n} (16t)^n \quad \text{is algebraic.}$$

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3. $r(t) = \sum_{n=0}^{\infty} r_n t^n$ being algebraic, it is D-finite, and so (r_n) is P-recursive:

$$(n+2)(3n+5)r_{n+1} - 4(6n+5)(2n+1)r_n = 0, \qquad r_0 = 1$$

$\Longrightarrow$ solution $r_n = \frac{(5/6)_n (1/2)_n}{(5/3)_n (2)_n} 16^n = g_n$, thus $g(t) = r(t)$ is algebraic.

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```
> P:=gfun:-listtoalgeq([seq(pochhammer(5/6,n)*pochhammer(1/2,n)/
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```

▷ Steps 1 & 3 rely on polynomial linear algebra (*Hermite-Padé approximants*).

Sur la généralisation des fractions continues algébriques.

(*Par M.* Ch. Hermite, *membre de l'Institut, à Paris.*)

[Extrait d'une lettre à M. Pincherle (*).]

. . . . . . Le problème que j'ai en vue est lo suivant: Etant donné n séries $S_1, S_2, \ldots S_n$ procédant suivant les puissances d'une variable x, déterminer les polynômes $X_1, X_2, \ldots X_n$ des degrés $\mu_1, \mu_2, \ldots \mu_n$ de manière à avoir

$$S_1 X_1 + S_2 X_2 + \cdots + S_n X_n = S x^{\mu_1 + \mu_2 + \cdots + \mu_n + n - 1},$$

où S est une série de même nature que S_1, S_2, etc. La question ainsi posée est entièrement déterminée, et une remarque de calcul intégral en donne la complète solution dans le cas particulier où les séries sont de simples exponentielles. C'est ce que je vais montrer, je me proposerai ensuite de faire sortir, en vue du cas général, les enseignements que contient cette solution.

Sur la généralisation des fractions continues algébriques;

Par M. H. PADÉ,

Docteur ès Sciences mathématiques,
Professeur au lycée de Lille.

INTRODUCTION.

M. Hermite s'est, dans un travail récemment paru (1), occupé de la généralisation des fractions continues algébriques. La question est de déterminer les polynomes $X_1, X_2, \dots, X_n$, de degrés $\mu_1, \mu_2, \dots, \mu_n$, qui satisfont à l'équation

$$S_1 X_1 + S_2 X_2 + \dots + S_n X_n = S\, x^{\mu_1+\mu_2+\dots+\mu_n+n-1},$$

$S_1, S_2, \dots, S_n$ étant des séries entières données, et S une série également entière. Ou plutôt, il s'agit d'obtenir un algorithme qui permette le calcul de proche en proche de ces systèmes de n polynomes, et qui

Definition: A Hermite-Padé approximant of type $\mathbf{d} = (d_1, \ldots, d_n) \in \mathbb{N}^n$ for $\mathbf{F} = (f_1, \ldots, f_n) \in \mathbb{K}[[x]]^n$ is a $\mathbf{P} = (P_1, \ldots, P_n) \in \mathbb{K}[x]^n \setminus \{0\}$ such that:

(1) $P_1 f_1 + \cdots + P_n f_n = O(x^\sigma)$ with $\sigma = \sum_i (d_i + 1) - 1$,

(2) $\deg(P_i) \leq d_i$ for all i.

Hermite-Padé approximants

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▷ Very useful concept in number theory (irrationality/transcendence):

- [Hermite, 1873]: e is transcendent; [Lindemann, 1882]: π is transc.
- [Apéry, 1978; Beukers, 1981]: $\zeta(3) = \sum_{n \geq 1} \frac{1}{n^3}$ is irrational; [Rivoal, 2000]: there are infinitely many k such that $\zeta(2k+1) \notin \mathbb{Q}$.

▷ Very useful tool in computer algebra

- algebraic approximants when $f_\ell = A^{\ell - 1}$ for a given $A \in \mathbb{K}[[x]]$
- differential approximants when $f_\ell = A^{(\ell - 1)}$ for a given $A \in \mathbb{K}[[x]]$

Definition: A Hermite-Padé approximant of type $\mathbf{d} = (d_1, \ldots, d_n) \in \mathbb{N}^n$ for $\mathbf{F} = (f_1, \ldots, f_n) \in \mathbb{K}[[x]]^n$ is a $\mathbf{P} = (P_1, \ldots, P_n) \in \mathbb{K}[x]^n \setminus \{0\}$ such that:

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▷ Many fast algorithms: [Beckermann, Labahn, 1994], [Giorgi, Jeannerod, Villard, 2003], [B., Jeannerod, Schost, 2007, 2008], [Zhou, Labahn, 2012], [Jeannerod, Neiger, Villard, 2020], [Rosenkilde, Storjohann, 2016, 2021], etc.

▷ `gfun` (Maple), `Guess.m` (Mathematica), `ore_algebra` (SageMath), etc.

Algorithmic classification of models with D-finite $Q_{\mathscr{S}}(t) := Q_{\mathscr{S}}(1,1;t)$

	OEIS	$\mathscr{S}$	Pol size	LDE size	Rec size		OEIS	$\mathscr{S}$	Pol size	LDE size	Rec size
1	A005566		—	(3, 4)	(2, 2)	13	A151275		—	(5, 24)	(9, 18)
2	A018224		—	(3, 5)	(2, 3)	14	A151314		—	(5, 24)	(9, 18)
3	A151312		—	(3, 8)	(4, 5)	15	A151255		—	(4, 16)	(6, 8)
4	A151331		—	(3, 6)	(3, 4)	16	A151287		—	(5, 19)	(7, 11)
5	A151266		—	(5, 16)	(7, 10)	17	A001006		(2, 2)	(2, 3)	(2, 1)
6	A151307		—	(5, 20)	(8, 15)	18	A129400		(2, 2)	(2, 3)	(2, 1)
7	A151291		—	(5, 15)	(6, 10)	19	A005558		—	(3, 5)	(2, 3)
8	A151326		—	(5, 18)	(7, 14)						
9	A151302		—	(5, 24)	(9, 18)	20	A151265		(6, 8)	(4, 9)	(6, 4)
10	A151329		—	(5, 24)	(9, 18)	21	A151278		(6, 8)	(4, 12)	(7, 4)
11	A151261		—	(4, 15)	(5, 8)	22	A151323		(4, 4)	(2, 3)	(2, 1)
12	A151297		—	(5, 18)	(7, 11)	23	A060900		(8, 9)	(3, 5)	(2, 3)

Equation sizes = (order, degree)

▷ Computerized discovery: enumeration + guessing [B., Kauers, 2009]

Algorithmic classification of models with D-finite $Q_{\mathscr{S}}(t) := Q_{\mathscr{S}}(1,1;t)$

	OEIS	$\mathscr{S}$	Pol size	LDE size	Rec size		OEIS	$\mathscr{S}$	Pol size	LDE size	Rec size
1	A005566		—	(3, 4)	(2, 2)	13	A151275		—	(5, 24)	(9, 18)
2	A018224		—	(3, 5)	(2, 3)	14	A151314		—	(5, 24)	(9, 18)
3	A151312		—	(3, 8)	(4, 5)	15	A151255		—	(4, 16)	(6, 8)
4	A151331		—	(3, 6)	(3, 4)	16	A151287		—	(5, 19)	(7, 11)
5	A151266		—	(5, 16)	(7, 10)	17	A001006		(2, 2)	(2, 3)	(2, 1)
6	A151307		—	(5, 20)	(8, 15)	18	A129400		(2, 2)	(2, 3)	(2, 1)
7	A151291		—	(5, 15)	(6, 10)	19	A005558		—	(3, 5)	(2, 3)
8	A151326		—	(5, 18)	(7, 14)						
9	A151302		—	(5, 24)	(9, 18)	20	A151265		(6, 8)	(4, 9)	(6, 4)
10	A151329		—	(5, 24)	(9, 18)	21	A151278		(6, 8)	(4, 12)	(7, 4)
11	A151261		—	(4, 15)	(5, 8)	22	A151323		(4, 4)	(2, 3)	(2, 1)
12	A151297		—	(5, 18)	(7, 11)	23	A060900		(8, 9)	(3, 5)	(2, 3)

Equation sizes = (order, degree)

- ▷ Computerized discovery: enumeration + guessing [B., Kauers, 2009]
- ▷ 1–22: DF confirmed by human proofs in [Bousquet-Mélou, Mishna, 2010]
- ▷ 23: DF confirmed by a human proof in [B., Kurkova, Raschel, 2017]
- ▷ All: explicit eqs. proved via CA [B., Chyzak, van Hoeij, Kauers, Pech, 2017]

Algorithmic classification of models with D-finite $Q_{\mathscr{S}}(t) := Q_{\mathscr{S}}(1,1;t)$

	OEIS	$\mathscr{S}$	algebraic?	asymptotics
1	A005566		N	$\frac{4}{\pi}\frac{4^n}{n}$
2	A018224		N	$\frac{2}{\pi}\frac{4^n}{n}$
3	A151312		N	$\frac{\sqrt{6}}{\pi}\frac{6^n}{n}$
4	A151331		N	$\frac{8}{3\pi}\frac{8^n}{n}$
5	A151266		N	$\frac{1}{2}\sqrt{\frac{3}{\pi}}\frac{3^n}{n^{1/2}}$
6	A151307		N	$\frac{1}{2}\sqrt{\frac{5}{2\pi}}\frac{5^n}{n^{1/2}}$
7	A151291		N	$\frac{4}{3\sqrt{\pi}}\frac{4^n}{n^{1/2}}$
8	A151326		N	$\frac{2}{\sqrt{3\pi}}\frac{6^n}{n^{1/2}}$
9	A151302		N	$\frac{1}{3}\sqrt{\frac{5}{2\pi}}\frac{5^n}{n^{1/2}}$
10	A151329		N	$\frac{1}{3}\sqrt{\frac{7}{3\pi}}\frac{7^n}{n^{1/2}}$
11	A151261		N	$\frac{12\sqrt{3}}{\pi}\frac{(2\sqrt{3})^n}{n^2}$
12	A151297		N	$\frac{\sqrt{3}B^{7/2}}{2\pi}\frac{(2B)^n}{n^2}$
13	A151275		N	$\frac{12\sqrt{30}}{\pi}\frac{(2\sqrt{6})^n}{n^2}$
14	A151314		N	$\frac{\sqrt{6}\lambda\mu C^{5/2}}{5\pi}\frac{(2C)^n}{n^2}$
15	A151255		N	$\frac{24\sqrt{2}}{\pi}\frac{(2\sqrt{2})^n}{n^2}$
16	A151287		N	$\frac{2\sqrt{2}A^{7/2}}{\pi}\frac{(2A)^n}{n^2}$
17	A001006		Y	$\frac{3}{2}\sqrt{\frac{3}{\pi}}\frac{3^n}{n^{3/2}}$
18	A129400		Y	$\frac{3}{2}\sqrt{\frac{3}{\pi}}\frac{6^n}{n^{3/2}}$
19	A005558		N	$\frac{8}{\pi}\frac{4^n}{n^2}$
	$A = 1+\sqrt{2}$, $B = 1+\sqrt{3}$, $C = 1+\sqrt{6}$, $\lambda = 7+3\sqrt{6}$,			$\mu = \sqrt{\frac{4\sqrt{6}-1}{19}}$
20	A151265		Y	$\frac{2\sqrt{2}}{\Gamma(1/4)}\frac{3^n}{n^{3/4}}$
21	A151278		Y	$\frac{3\sqrt{3}}{\sqrt{2}\Gamma(1/4)}\frac{3^n}{n^{3/4}}$
22	A151323		Y	$\frac{\sqrt{2}3^{3/4}}{\Gamma(1/4)}\frac{6^n}{n^{3/4}}$
23	A060900		Y	$\frac{4\sqrt{3}}{3\Gamma(1/3)}\frac{4^n}{n^{2/3}}$

▷ Computerized discovery: convergence acceleration + LLL [B., Kauers, '09]

Algorithmic classification of models with D-finite $Q_{\mathscr{S}}(t) := Q_{\mathscr{S}}(1,1;t)$

	OEIS	$\mathscr{S}$	algebraic?	asymptotics		OEIS	$\mathscr{S}$	algebraic?	asymptotics
1	A005566		N	$\frac{4}{\pi}\frac{4^n}{n}$	13	A151275		N	$\frac{12\sqrt{30}}{\pi}\frac{(2\sqrt{6})^n}{n^2}$
2	A018224		N	$\frac{2}{\pi}\frac{4^n}{n}$	14	A151314		N	$\frac{\sqrt{6}\lambda\mu C^{5/2}}{5\pi}\frac{(2C)^n}{n^2}$
3	A151312		N	$\frac{\sqrt{6}}{\pi}\frac{6^n}{n}$	15	A151255		N	$\frac{24\sqrt{2}}{\pi}\frac{(2\sqrt{2})^n}{n^2}$
4	A151331		N	$\frac{8}{3\pi}\frac{8^n}{n}$	16	A151287		N	$\frac{2\sqrt{2}A^{7/2}}{\pi}\frac{(2A)^n}{n^2}$
5	A151266		N	$\frac{1}{2}\sqrt{\frac{3}{\pi}}\frac{3^n}{n^{1/2}}$	17	A001006		Y	$\frac{3}{2}\sqrt{\frac{3}{\pi}}\frac{3^n}{n^{3/2}}$
6	A151307		N	$\frac{1}{2}\sqrt{\frac{5}{2\pi}}\frac{5^n}{n^{1/2}}$	18	A129400		Y	$\frac{3}{2}\sqrt{\frac{3}{\pi}}\frac{6^n}{n^{3/2}}$
7	A151291		N	$\frac{4}{3\sqrt{\pi}}\frac{4^n}{n^{1/2}}$	19	A005558		N	$\frac{8}{\pi}\frac{4^n}{n^2}$
8	A151326		N	$\frac{2}{\sqrt{3\pi}}\frac{6^n}{n^{1/2}}$		$A=1+\sqrt{2}$, $B=1+\sqrt{3}$, $C=1+\sqrt{6}$, $\lambda=7+3\sqrt{6}$,			$\mu=\sqrt{\frac{4\sqrt{6}-1}{19}}$
9	A151302		N	$\frac{1}{3}\sqrt{\frac{5}{2\pi}}\frac{5^n}{n^{1/2}}$	20	A151265		Y	$\frac{2\sqrt{2}}{\Gamma(1/4)}\frac{3^n}{n^{3/4}}$
10	A151329		N	$\frac{1}{3}\sqrt{\frac{7}{3\pi}}\frac{7^n}{n^{1/2}}$	21	A151278		Y	$\frac{3\sqrt{3}}{\sqrt{2}\Gamma(1/4)}\frac{3^n}{n^{3/4}}$
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▷ Computerized discovery: convergence acceleration + LLL [B., Kauers, '09]

▷ Asympt. confirmed by human proofs via ACSV in [Melczer, Wilson, 2016]

▷ Transcendence proofs via CA [B., Chyzak, van Hoeij, Kauers, Pech, 2017]

Models 1–19: proofs, explicit expressions and transcendence

Theorem [B., Chyzak, van Hoeij, Kauers, Pech, 2017]

Let $\mathscr{S}$ be one of the models 1–19. Then

- $Q_{\mathscr{S}}(x, y; t)$ is expressible using (integrals of) ${}_2F_1$ expressions.
- $Q_{\mathscr{S}}(x, y; t)$ is transcendental.

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- $Q_{\mathscr{S}}(t)$ is transcendental, except for $\mathscr{S} =$  and $\mathscr{S} =$ .

Example (King walks in the quarter plane, A151331)

Q

$$(t) = \frac{1}{t} \int_0^t \frac{1}{(1+4x)^3} \cdot {}_2F_1\left(\begin{matrix} \frac{3}{2} \; \frac{3}{2} \\ 2 \end{matrix} \,\middle|\, \frac{16x(1+x)}{(1+4x)^2} \right) dx$$

$$= 1 + 3t + 18t^2 + 105t^3 + 684t^4 + 4550t^5 + 31340t^6 + 219555t^7 + \cdots$$

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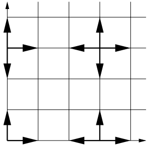
$$= 1 + 3t + 18t^2 + 105t^3 + 684t^4 + 4550t^5 + 31340t^6 + 219555t^7 + \cdots$$

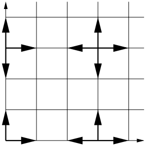
▷ Computer-driven discovery and proof; no human proof yet.

▷ Proof uses: (1) kernel method and (2) creative telescoping + (3) ODE factoring and (4) ODE solving.



The kernel $K(x,y;t) := 1 - t \cdot \sum_{(i,j)\in\mathscr{S}} x^i y^j = 1 - t\left(x + \frac{1}{x} + y + \frac{1}{y}\right)$ is left invariant under the change of (x,y) into the elements of

$$\mathcal{G}_{\mathscr{S}} := \left\{(x,y), \left(\tfrac{1}{x}, y\right), \left(\tfrac{1}{x}, \tfrac{1}{y}\right), \left(x, \tfrac{1}{y}\right)\right\}$$



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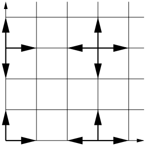
Kernel equation:

$$K(x,y;t)xyQ(x,y;t) = xy - txQ(x,0;t) - tyQ(0,y;t)$$



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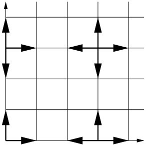

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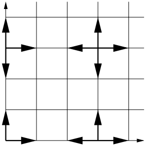
$$K(x,y;t)\tfrac{1}{x}\tfrac{1}{y}Q(\tfrac{1}{x},\tfrac{1}{y};t) = \tfrac{1}{x}\tfrac{1}{y} - t\tfrac{1}{x}Q(\tfrac{1}{x},0;t) - t\tfrac{1}{y}Q(0,\tfrac{1}{y};t)$$



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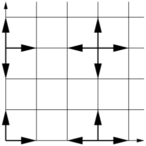
K(x,y;t)\tfrac{1}{x}\tfrac{1}{y}Q(\tfrac{1}{x},\tfrac{1}{y};t) &= \tfrac{1}{x}\tfrac{1}{y} - t\tfrac{1}{x}Q(\tfrac{1}{x},0;t) - t\tfrac{1}{y}Q(0,\tfrac{1}{y};t)\\
-K(x,y;t)x\tfrac{1}{y}Q(x,\tfrac{1}{y};t) &= -x\tfrac{1}{y} + txQ(x,0;t) + t\tfrac{1}{y}Q(0,\tfrac{1}{y};t)
\end{aligned}$$

(1) Kernel method [Bousquet-Mélou, Mishna, 2010] $\mathscr{S} =$ 



The kernel $K(x,y;t) := 1 - t \cdot \sum_{(i,j)\in\mathscr{S}} x^i y^j = 1 - t\left(x + \frac{1}{x} + y + \frac{1}{y}\right)$ is left invariant under the change of (x,y) into the elements of

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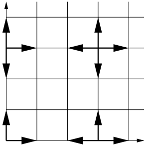
K(x,y;t)\tfrac{1}{x}\tfrac{1}{y}Q(\tfrac{1}{x},\tfrac{1}{y};t) &= \tfrac{1}{x}\tfrac{1}{y} - t\tfrac{1}{x}Q(\tfrac{1}{x},0;t) - t\tfrac{1}{y}Q(0,\tfrac{1}{y};t) \\
-K(x,y;t)x\tfrac{1}{y}Q(x,\tfrac{1}{y};t) &= -x\tfrac{1}{y} + txQ(x,0;t) + t\tfrac{1}{y}Q(0,\tfrac{1}{y};t)
\end{aligned}$$

Summing up yields the orbit equation:

$$\sum_{\theta\in\mathcal{G}} (-1)^{\theta}\theta\big(xy\,Q(x,y;t)\big) = \frac{xy - \frac{1}{x}y + \frac{1}{x}\frac{1}{y} - x\frac{1}{y}}{K(x,y;t)}$$



The kernel $K(x,y;t) := 1 - t \cdot \sum_{(i,j)\in\mathscr{S}} x^i y^j = 1 - t\left(x + \frac{1}{x} + y + \frac{1}{y}\right)$ is left invariant under the change of (x,y) into the elements of

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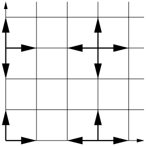
\end{aligned}$$

Taking positive parts yields:

$$[x^> y^>] \sum_{\theta\in\mathcal{G}} (-1)^\theta \theta\big(xy\,Q(x,y;t)\big) = [x^> y^>] \frac{xy - \frac{1}{x}y + \frac{1}{x}\frac{1}{y} - x\frac{1}{y}}{K(x,y;t)}$$



The kernel $K(x,y;t) := 1 - t \cdot \sum_{(i,j)\in\mathscr{S}} x^i y^j = 1 - t\left(x + \frac{1}{x} + y + \frac{1}{y}\right)$ is left invariant under the change of (x,y) into the elements of

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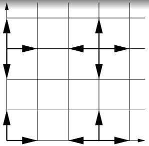
Summing up and taking positive parts yields:

$$xy\,Q(x,y;t) = [x^{>}y^{>}]\frac{xy - \frac{1}{x}y + \frac{1}{x}\frac{1}{y} - x\frac{1}{y}}{K(x,y;t)}$$



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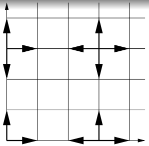
$$\text{GFun} = \text{PosPart}\left(\frac{\text{OrbitSum}}{\text{Kernel}}\right) = \oiint \text{RatFrac}$$

(1) Kernel method [Bousquet-Mélou, Mishna, 2010] $\mathscr{S} =$ 



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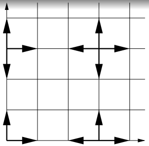

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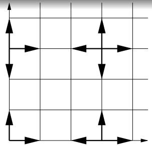
▷ Argument works if $\mathsf{OS} \neq 0$: algebraic version of the reflection principle



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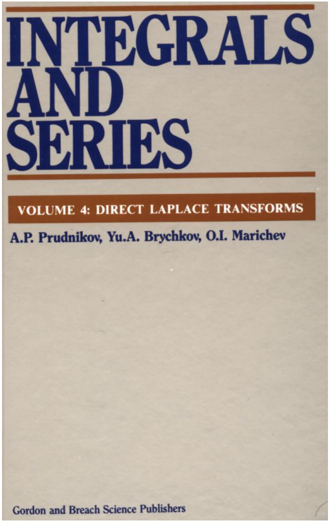
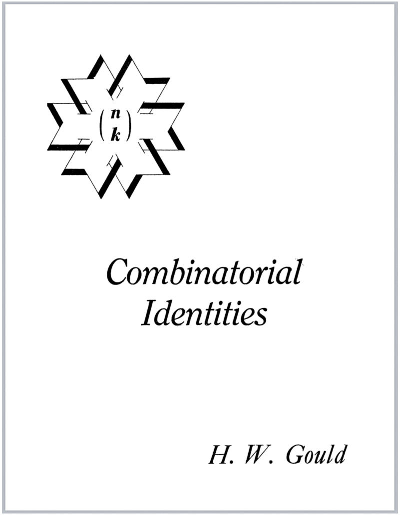
▷ Creative Telescoping finds a differential equation for GF = $\oint\!\!\oint$ RatFrac

"An algorithmic toolbox for multiple sums and integrals with parameters"





(2) Creative Telescoping

"An algorithmic toolbox for multiple sums and integrals with parameters"

DOUBLE INTEGRALS 631

6. $\int_0^\infty\int_0^\infty x^{\nu+1}y^{1-\nu}e^{-px^2-qy^2}\,\mathrm{ber}_\nu(axy)\,dx\,dy=\left(\frac{ap}{2}\right)^\nu\frac{4pq\cos(3\nu\pi/4)-a^2\sin(3\nu\pi/4)}{16p^2q^2+a^4}$ [Re $\nu>-1$].

7. $\int_0^\infty\int_0^\infty x^{4n+1}y^3e^{-px^2-qy^4}\,\mathrm{bei}_{4n}(axy)\,dx\,dy=$
$=(-1)^n\sqrt{\frac{\pi}{2}}\frac{a^{4n}}{2^{5n+3}p^{4n+1}q^{n+1}}e^{-a^4/(2^7p^2q)}D_{2n+1}\left(\frac{a^2}{4p\sqrt{2q}}\right)$.

8. $\int_0^\infty\int_0^\infty x^{4n+1}ye^{-px^2-qy^4}\,\mathrm{ber}_{4n}(axy)\,dx\,dy=$
$=(-1)^n\sqrt{\frac{\pi}{2}}\frac{a^{4n}}{2^{6n+5/2}p^{4n+1}q^{n+1/2}}e^{-a^4/(2^7p^2q)}D_{2n}\left(\frac{a^2}{4p\sqrt{2q}}\right)$.

9. $\int_0^\infty\int_0^\infty xy^3e^{-px^2-qy^4}\,\mathrm{bei}(axy)\,dx\,dy=\frac{\sqrt{\pi}\,a^2}{2^6p^2q^{3/2}}e^{-a^4/(2^4p^2q)}$.

10. $\int_0^\infty\int_0^\infty xye^{-px^4-qy^4}\,\mathrm{ber}(axy)\,dx\,dy=\frac{\sqrt{\pi}}{8p\sqrt{q}}e^{-a^4/(2^4p^2q)}$.

11. $\int_0^\infty\int_0^\infty xy^3e^{-px^2-qy^4}[\mathrm{ber}^2(axy)+\mathrm{bei}^2(axy)]\,dx\,dy=\frac{1}{8pq}e^{a^4/(16p^2q)}$.

12. $\int_0^\infty\int_0^\infty xye^{-px^4-qy^4}\,\mathrm{ber}(axy)\,dx\,dy=\frac{\pi}{16\sqrt{pq}}J_0\left(\frac{a^2}{8\sqrt{pq}}\right)$.

13. $\int_0^\infty\int_0^\infty xy\,K_0(ax)\,K_0(by)\,(\mathrm{ber}\,x\,\mathrm{ber}\,y-\mathrm{bei}\,x\,\mathrm{bei}\,y)\,dx\,dy=\frac{ab-1}{(a^2+1)(b^2+1)}$ [Re a^2, Re $b^2>1/2$].

3.2.12. Integrals containing orthogonal polynomials.

1. $\int_0^1\int_0^\pi\frac{y^{2\beta+1}}{(1-y^2)^{\beta-\alpha+1}}\sin^{2\beta}x\,P_n^{(\alpha,\beta)}(2\,|\cos a\cos b+ye^{ix}\sin a\sin b|^2-1)\,dx\,dy=$
$=\frac{\sqrt{\pi}}{2}\Gamma\left[\begin{matrix}\alpha-\beta,\ \beta+1/2\\ \alpha+1\end{matrix}\right]\frac{P_n^{(\alpha,\beta)}(\cos 2a)\,P_n^{(\alpha,\beta)}(\cos 2b)}{P_n^{(\alpha,\beta)}(1)}$ [Re $\alpha>$ Re $\beta>-1/2$].

2. $\int_{-1}^{1}\int_{-1}^{1}(1-x^2)^{\alpha-1}(1-y^2)^{\beta-1}C_{2n}^{\alpha+\beta}(ax+by)\,dx\,dy=$
$=\pi\Gamma\left[\begin{matrix}\alpha,\ \beta\\ \alpha+1/2,\ \beta+1/2\end{matrix}\right]\frac{C_{2n}^{\alpha+\beta}(0)\,P_n^{(\alpha-1/2,\,\beta-1/2)}(A)\,P_n^{(\alpha-1/2,\,\beta-1/2)}(B)}{P_n^{(\alpha-1/2,\,\beta-1/2)}(-1)\,P_n^{(\alpha-1/2,\,\beta-1/2)}(1)}$
[$2a=\sqrt{(1-A)(1-B)}$, $2b=\sqrt{(1+A)(1+B)}$; Re α, Re $\beta>0$].

3. $\iint_{x^2+y^2\leqslant 1}\frac{P_m(x)\,P_n(y)}{\sqrt{1-x^2-y^2}}\,dx\,dy=\begin{cases}(-1)^{n/2}\,2\pi\frac{(n-1)!!}{n!!\,(2n+1)} & [m=n\text{ even}]\\ 0 & [\text{otherwise}]\end{cases}$

7.7) $\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}\binom{2k}{k}\frac{1}{\binom{n+k}{k}2^{2k}}=\frac{\binom{6n}{3n}}{\binom{2n}{n}}2^{-4n}$,

7.8) $\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}\binom{2k}{k}\frac{2k+1}{\binom{n+k}{k}(n+k+1)}=1$,

7.9) $\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{x+k}{k}\frac{2^{2k}}{\binom{2k}{k}(2k+1)}=\frac{2^{2n}}{\binom{2n}{n}(2n+1)}\binom{n-x-\frac12}{n}$,

7.10) $\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{n+k}{k}\frac{2^{2k}}{\binom{2k}{k}(2k+1)}=\frac{(-1)^n}{2n+1}$,

7.11) $\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{x+k}{k}\frac{2^{2k}}{\binom{2k}{k}(x+k)}=(-1)^n\frac{\binom{2x}{2n}}{x\binom{x}{n}}$,

7.12) $\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{n+k}{k}\frac{2^{2k}}{\binom{2k}{k}(n+k)}=\frac{(-1)^n}{n}$, $(n\geqslant 1)$,

7.13) $\sum_{k=0}^{n}\binom{n}{k}\binom{-n-1}{n-k}\frac{1}{\binom{k+r}{k}}=(-1)^n\,r\sum_{k=0}^{n}\binom{n}{k}^2\frac{1}{k+r}$,

-59-

(2) Creative Telescoping

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▷ [J. Bernoulli 1682]: Use $\frac{1}{\binom{k}{2}} = \frac{2}{k-1} - \frac{2}{k}$ to *create* a *telescoping sum*

$$\left(\frac{2}{1} - \frac{2}{2}\right) + \left(\frac{2}{2} - \frac{2}{3}\right) + \cdots + \left(\frac{2}{2024} - \frac{2}{2026}\right) = 2 - \frac{2}{2026}.$$

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▷ [Knuth 1969] Ex. 1.2.6.63:

> *[50] Develop computer programs for simplifying sums that involve binomial coefficients.*

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▷ Creative Telescoping: how computer algebra uses (systematically and algorithmically) Bernoulli's idea to solve Knuth's 1969 exercise, and more

A more difficult example ("*Basel Problem*"): compute the value of the series

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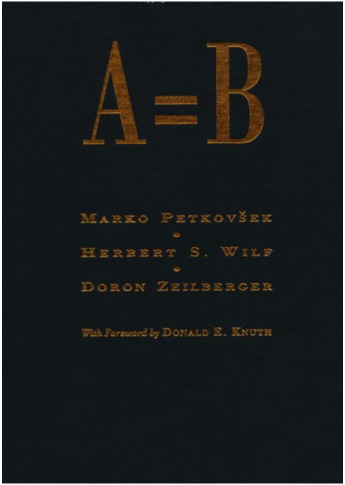
▷ One can use

$$\frac{1-(-1)^m}{2m^2} = \int_0^\pi \left(\frac{x \sin\left(\frac{(2m-1)x}{2}\right)}{4\sin\left(\frac{x}{2}\right)} - \frac{x \sin\left(\frac{(2m+1)x}{2}\right)}{4\sin\left(\frac{x}{2}\right)} \right) dx$$

to *create* a *telescoping sum* and deduce that

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \int_0^\pi \frac{x}{4} dx = \frac{\pi^2}{8}.$$

General framework in computer algebra –initiated by Zeilberger in the ’90s– for proving identities on multiple integrals and sums *with parameters*.



(2) Creative Telescoping

Case of (parametric) sums:

$$I_n := \sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

Principle: **IF** one knows Pascal's triangle:

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1} = 2\binom{n}{k} + \binom{n}{k-1} - \binom{n}{k},$$

then summing over k telescopes and yields

$$I_{n+1} = 2I_n.$$

The initial condition $I_0 = 1$ concludes the proof.

Case of (parametric) sums:

$$F_n = \sum_k u_{n,k} =?$$

IF one knows $P(n, S_n)$ (telescoper) and $Q(n, k, S_n, S_k)$ (certificate) such that

$$(P(n, S_n) + \Delta_k Q(n, k, S_n, S_k)) \cdot u_{n,k} = 0$$

(where Δ_k is the difference operator, $\Delta_k \cdot v_{n,k} = v_{n,k+1} - v_{n,k}$),
then the sum "telescopes", leading to

$$P(n, S_n) \cdot F_n = 0.$$

Case of (parametric) integrals:

$$I(t) = \oint_\gamma H(t,x)\,dx = ?$$

IF one knows $P(t,\partial_t)$ (telescoper) and $Q(t,x,\partial_t,\partial_x)$ (certificate) such that

$$\left(P(t,\partial_t) + \partial_x Q(t,x,\partial_t,\partial_x)\right) \cdot H(t,x) = 0,$$

then the integral "telescopes", leading to

$$P(t,\partial_t) \cdot I(t) = 0.$$

Zeilberger's Algorithm [1990]

Input: a hypergeometric term $u_{n,k}$, i.e., $\frac{u_{n+1,k}}{u_{n,k}}$ and $\frac{u_{n,k+1}}{u_{n,k}}$ are in $\mathbb{Q}(n,k)$

Output:

- a linear recurrence, called telescoper, (P) satisfied by $F_n = \sum_k u_{n,k}$
- a certificate (Q), for checking the result: $P(n, S_n) \cdot u_{n,k} = \Delta_k Q \cdot u_{n,k}$.

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> T := binomial(n,k);
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> tel:=Zpair[1];
```

$$S_n - 2$$

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▷ Can be checked using the certificate:

```
> cert:=Zpair[2];
> iszero:=(subs(n=n+1,T) - 2*T) - (subs(k=k+1,cert) - cert);
> simplify(convert(%,GAMMA));
```

$$0$$

Input: a hypergeometric term $u_{n,k}$, i.e., $\frac{u_{n+1,k}}{u_{n,k}}$ and $\frac{u_{n,k+1}}{u_{n,k}}$ are in $\mathbb{Q}(n,k)$

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- a certificate (Q), for checking the result: $P(n, S_n) \cdot u_{n,k} = \Delta_k Q \cdot u_{n,k}$.

Zeilberger's algorithm [Zeilberger, 1990]

Algorithm. *Input*: a biv. hypgeom. term $u_{n,k}$ *Output*: operators $P(n, S_n)$ and $Q(n,k)$

For $i = 0, 1, 2, \ldots$:

1. Call the parametrized Gosper algorithm over $\mathbb{K}(n)$ to search for $a_0, \ldots, a_s \in \mathbb{K}(n)$ and $v_{n,k} = \lambda(n,k)\, u_{n,k}$ hypergeometric w.r.t. k over $\mathbb{K}(n)$ such that

$$v_{n,k+1} - v_{n,k} = a_s(n)\, u_{n+s,k} + \cdots + a_1(n)\, u_{n+1,k} + a_0(n)\, u_{n,k}$$

2. If a solution was found, return

$$P(n, S_n) = a_s(n)\, S_n^s + \cdots + a_1(n)\, S_n + a_0(n), \quad Q(n,k) = \lambda(n,k)$$

Theorem. The output (if any) satisfies $P(n, S_n) = [(S_k - 1)\, Q(n,k)] \cdot u_{n,k}$

Almkvist-Zeilberger Algorithm [1990]

Input: a hyperexponential function $H(t, x)$, i.e., $\frac{\frac{\partial H}{\partial t}}{H}$ and $\frac{\frac{\partial H}{\partial x}}{H}$ are in $\mathbb{Q}(t, x)$
Output:

- a linear differential operator $P(t, \partial_t)$ satisfied by $I(t) = \oint_\gamma H(t, x)\, dx$
- a $G(t, x) \in \mathbb{Q}(t, x)$ such that $P(t, \partial_t) \cdot H(t, x) = \frac{\partial}{\partial x}\Big(G(t, x) \cdot H(t, x)\Big)$.

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▷ Example: compute a linear differential equation for

$$I(t) = \int_0^1 \sqrt{\frac{1-t^2x^2}{1-x^2}}\,\mathrm{d}x$$

▷ Idea: construct a "telescoping identity"

$$\Big((t-t^3)\partial_t^2 + (1-t^2)\partial_t + t\Big)\left(\sqrt{\frac{1-t^2x^2}{1-x^2}}\right) = \partial_x\left(\frac{tx(1-x^2)}{1-t^2x^2}\cdot\sqrt{\frac{1-t^2x^2}{1-x^2}}\right)$$

...and conclude, by integrating both sides w.r.t. x, that

$$(t-t^3)I''(t) + (1-t^2)I'(t) + tI(t) = 0.$$

(2) Creative Telescoping

"An algorithmic toolbox for multiple sums and integrals with parameters"

Example [Apéry 1978]: $A_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$ satisfies the recurrence

$$(n+1)^3 A_{n+1} + n^3 A_{n-1} = (2n+1)(17n^2+17n+5) A_n.$$

▷ Key fact used to prove that $\zeta(3) := \sum_{n \geq 1} \frac{1}{n^3} \approx 1.202056903\ldots$ is irrational.

1. Journées Arithmétiques de Marseille-Luminy, June 1978

The board of programme changes informed us that R. Apéry (Caen) would speak Thursday, 14.00 "Sur l'irrationalité de $\zeta(3)$." Though there had been earlier rumours of his claiming a proof, scepticism was general. The lecture tended to strengthen this view to rank disbelief. Those who listened casually, or who were afflicted with being non-Francophone, appeared to hear only a sequence of unlikely assertions.

7. ICM '78, Helsinki, August 1978

Neither Cohen nor I had been able to prove (5) or (5') in the intervening 2 months. After a few days of fruitless effort the specific problem was mentioned to Don Zagier (Bonn), and with irritating speed he showed that indeed the sequence $\{b'_n\}$ satisfies the recurrence (4). This more or less broke the dam and (5') and (5) were quickly conquered. Henri Cohen addressed a very well-attended meeting at 17.00 on Friday, August 18 in the language of the majority, proving (5) and explaining how this implied the

[Van der Poorten, 1979: "A proof that Euler missed"]

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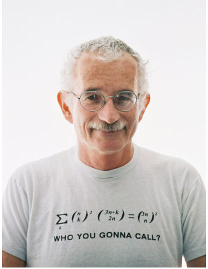
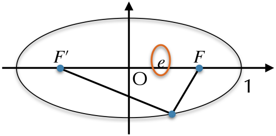


[Zeilberger, 1990: "The method of creative telescoping"]

"An algorithmic toolbox for multiple sums and integrals with parameters"

Example [Euler, 1733]: Perimeter of an ellipse of eccentricity e, semi-major axis 1

$$p(e) = 4\int_0^1 \sqrt{\frac{1-e^2u^2}{1-u^2}}\,du = 4\oint\!\!\oint \frac{dx\,dy}{1-\frac{1-e^2x^2}{(1-x^2)y^2}}$$



Principle: Find algorithmically

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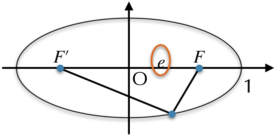

Principle: Find algorithmically

$$\left((e-e^3)\partial_e^2 + (1-e^2)\partial_e + e\right)\left(\frac{1}{1-\frac{1-e^2x^2}{(1-x^2)y^2}}\right) =$$

$$\partial_x\left(\frac{e(1+x-x^2-x^3)y^2(2x-3+y^2+x^2(3e^2-y^2-2))}{(y^2+x^2(e^2-y^2)-1)^2}\right)$$

$$+\,\partial_y\left(\frac{2e(e^2-1)x(1+x^3)y^3}{(y^2+x^2(e^2-y^2)-1)^2}\right)$$

▷ Conclusion: $(e-e^3)\cdot p''(e) + (1-e^2)\cdot p'(e) + e\cdot p(e) = 0.$

"An algorithmic toolbox for multiple sums and integrals with parameters"

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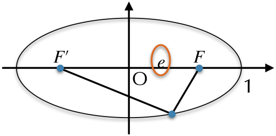

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$$+\,\partial_y\left(\frac{2e(e^2-1)x(1+x^3)y^3}{(y^2+x^2(e^2-y^2)-1)^2}\right)$$

▷ Conclusion: $p(e) = \frac{\pi}{2}\cdot {}_2F_1\left(\begin{matrix}-\frac{1}{2} & \frac{1}{2}\\ & 1\end{matrix}\,\middle|\, e^2\right) = 2\pi - \frac{\pi}{2}e^2 - \frac{3\pi}{32}e^4 - \cdots.$

(2) Creative Telescoping

"An algorithmic toolbox for multiple sums and integrals with parameters"

Example [Euler, 1733]: Perimeter of an ellipse of eccentricity e, semi-major axis 1

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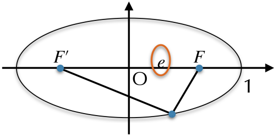

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$$+\,\partial_y\left(\frac{2e\left(e^2-1\right)x\left(1+x^3\right)y^3}{\left(y^2+x^2\left(e^2-y^2\right)-1\right)^2}\right)$$

▷ Drawback: Size(certificate) $\gg$ Size(telescoper).

(2) Creative Telescoping: several generations of algorithms

- 1G, *elimination-based*: [Fasenmyer, 1947], [Lipshitz, 1988], [Zeilberger, 1990], [Takayama, 1990], [Wilf, Zeilberger, 1990], [Chyzak, Salvy, 2000]
- 2G, *linear diff/rec rational solving*: [Zeilberger, 1990], [Zeilberger, 1991], [Almkvist, Zeilberger, 1990], [Chyzak, 2000], [Koutschan, 2010]
- 3G, combines *1G + 2G + linear algebra*: [Apagodu, Zeilberger, 2005], [Koutschan 2010], [Chen, Kauers 2012], [Chen, Kauers, Koutschan 2014]

▷ Advantages:

- 1G–3G: very general algorithms;
- 2G/3G algorithms are able to solve non-trivial applications.

▷ Drawbacks:

- 1G: slow;
- 2G: bad or unknown complexity;
- 1G and 3G: non-minimality of telescopers;
- 1G–3G: all compute (big) certificates.

(2) Creative Telescoping: several generations of algorithms

4G: roots in [Ostrogradsky, 1845], [Hermite, 1872] and [Picard, 1902]

- univariate:
  - rational $\int$: [B., Chen, Chyzak, Li, 2010]
  - hyperexponential $\int$: [B., Chen, Chyzak, Li, Xin, 2013]
  - hypergeometric $\sum$: [Chen, Huang, Kauers, Li, 2015], [Huang, 2016]
  - mixed $\int + \sum$: [B., Dumont, Salvy, 2016]
  - algebraic $\int$: [Chen, Kauers, Koutschan, 2016]
  - D-finite Fuchsian $\int$: [Chen, van Hoeij, Kauers, Koutschan, 2018]
  - D-finite $\int$: [B., Chyzak, Lairez, Salvy, 2018], [van der Hoeven, 2018]
- multiple:
  - rational bivariate $\oint\oint$: [Chen, Kauers, Singer, 2012], [Chen, Du, Kauers, 2021]
  - rational: [B., Lairez, Salvy, 2013], [Lairez 2016]
  - binomial sums: [B., Lairez, Salvy, 2017]

▷ Advantages:

- good complexity;
- minimality of telescopers;
- do not need to compute certificates;
- fast in practice.

▷ Drawback: not (yet) as general as 1G–3G algorithms.

Algorithm for the integration of rational functions [B., Lairez, Salvy, 2013]

- **Input:** $R(e, \mathbf{x})$ a rational function in e and $\mathbf{x} = x_1, \ldots, x_n$.
- **Output:** A linear ODE $T(e, \partial_e) y = 0$ satisfied by $y(e) = \oint\!\!\oint R(e, \mathbf{x}) d\mathbf{x}$.
- **Complexity:** $\mathcal{O}(D^{8n+2})$, where $D = \deg R$.
- **Output size:** T has order $\leq D^n$ in ∂_e and degree $\leq D^{3n+2}$ in e.

- Roots in [B., Chen, Chyzak, Li, 2010] ($n = 1$).
- Relies on generalized Hermite reduction and polynomial linear algebra.
- Avoids the (costly) computation of certificates, of size $\Omega(D^{n^2/2})$.
- Previous algorithms: complexity (at least) doubly exponential in n.
- Highly non-trivial extension by [Lairez, 2016]: very efficient in practice.

Theorem [B., Chyzak, van Hoeij, Kauers, Pech, 2017]

Let $\mathscr{S}$ be one of the models 1–19. Then

- $Q_{\mathscr{S}}(t)$ is expressible using (integrals of) ${}_2F_1$ expressions.
- $Q_{\mathscr{S}}(t)$ is transcendental, except for $\mathscr{S} =$  and $\mathscr{S} =$ .

Example (King walks in the quarter plane, A151331)

Q 

$$Q(t) = \frac{1}{t}\int_0^t \frac{1}{(1+4x)^3} \cdot {}_2F_1\left(\begin{matrix}\frac{3}{2} & \frac{3}{2}\\ & 2\end{matrix}\,\middle|\, \frac{16x(1+x)}{(1+4x)^2}\right) dx$$

$$= 1 + 3t + 18t^2 + 105t^3 + 684t^4 + 4550t^5 + 31340t^6 + 219555t^7 + \cdots$$

▷ Computer-driven discovery and proof; no human proof yet.

▷ Proof uses: (1) kernel method and (2) creative telescoping + (3) ODE factoring and (4) ODE solving.

Theorem (Apéry's power series is transcendental) [B., Salvy, Singer, 2025]

$$f(t) = \sum_n A_n t^n, \quad \text{where } A_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2, \quad \text{is transcendental.}$$

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Proof:

1. Creative telescoping: [Zagier, 1979], [Zeilberger, 1990]

$$(n+1)^3 A_{n+1} + n^3 A_{n-1} = (2\,n+1)\,(17\,n^2 + 17\,n + 5) A_n, \quad A_0 = 1,\ A_1 = 5$$

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   $$\mathscr{L} = (t^4 - 34t^3 + t^2)\partial_t^3 + (6t^3 - 153t^2 + 3t)\partial_t^2 + (7t^2 - 112t + 1)\partial_t + t - 5$$

Algorithmic transcendence proofs: blending Guess-and-Prove and CT

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3. Minimization: [Adamczewski, Rivoal, 2018], [B., Rivoal, Salvy, 2024]

   compute least-order $\mathscr{L}_f^{\min}$ in $\mathbb{Q}(t)\langle\partial_t\rangle$ such that $\mathscr{L}_f^{\min}(f) = 0$

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4. Local solutions of $\mathscr{L}_f^{\min}$: [Frobenius, 1873], [Chudnovsky[2], 1987]

$$\left\{1 + 5t + O(t^2),\ \ln(t) + (5\ln(t) + 12)t + O(t^2),\ \ln(t)^2 + (5\ln(t)^2 + 24\ln(t))t + O(t^2)\right\}$$

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5. Conclusion: f is transcendental†

† f algebraic would imply a full basis of algebraic solutions for $\mathscr{L}_f^{\min}$.

Summary: classification of walks with small steps in $\mathbb{N}^2$

$Q_{\mathscr{S}}$ is D-finite $\iff$ a certain group $\mathcal{G}_{\mathscr{S}}$ is finite (!)

- quadrant models $\mathscr{S}$: 79
  - $|\mathcal{G}_{\mathscr{S}}| < \infty$: 23
    - orbit sum $= 0$: 4
      - Guess-and-Prove
        - **algebraic**
    - orbit sum $\neq 0$: 19
      - Creative Telescoping
        - **D-finite**
  - $|\mathcal{G}_{\mathscr{S}}| = \infty$: 56
    - asymptotics + Gröbner Bases
      - **non-D-finite**

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  - OS $\neq 0$: 19 — kernel + CT — **D-finite transc.**
- $|\mathcal{G}_{\mathscr{S}}| = \infty$: 56
  - $\exists$ decoupling⋆: 9 — Tutte's invariants — **D-algebraic**
  - $\nexists$ decoupling: 47 — diff. Galois — **D-transcendental**

⋆ $\exists U \in \mathbb{Q}(x,t), V \in \mathbb{Q}(y,t)$ s.t. $U(x) + V(y) = xy$ on the kernel $K(x,y;t) = 0$.

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▷ Many contributors (2010–2021): Bernardi, B., Bousquet-Mélou, Chyzak, Dreyfus, Hardouin, van Hoeij, Kauers, Kurkova, Mishna, Pech, Raschel, Roques, Salvy, Singer

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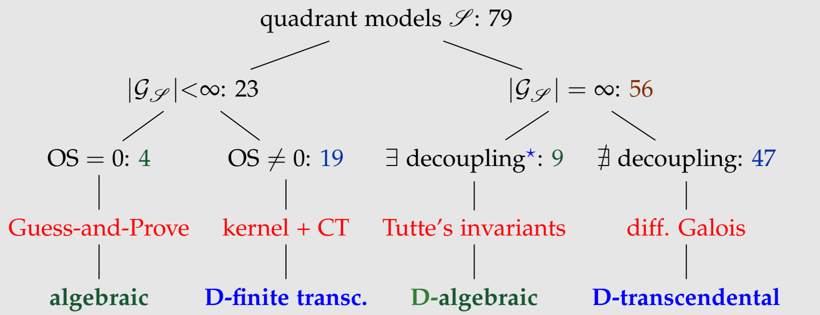

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▷ Proofs use various tools: algebra, complex analysis, probability theory, differential Galois theory, computer algebra, etc.

Conclusion

- Enumerative Combinatorics and Computer Algebra enrich one another
- Classification of $Q(x, y; t)$ fully completed for 2D small-steps walks
- Robust algorithmic methods, based on efficient algorithms:
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- Lack of "purely human" proofs for some results.
- Many beautiful open questions for 2D walk models with repeated or large steps, and in different cones, and in dimension > 2.

Thanks for your attention!

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Bonus

Beyond dimension 2: walks with small-steps in $\mathbb{N}^3$

▷ $2^{3^3-1} \approx 67$ million models, of which ≈ 11 million inherently 3D

- 3D octant models $\mathscr{S}$ with ≤ 6 steps: 20804
  - $|\mathcal{G}_{\mathscr{S}}| < \infty$: 170
    - orbit sum $\neq 0$: 108
      - Creative Telescoping
        - **D-finite**
    - orbit sum $= 0$: 62
      - 2D-reducible: 43
        - **D-finite**
      - not 2D-reducible: 19
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  - $|\mathcal{G}_{\mathscr{S}}| = \infty$: 20634
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[B., Bousquet-Mélou, Kauers, Melczer, 2016] + [Du, Hou, Wang, 2017];
completed by [Bacher, Kauers, Yatchak, 2016]

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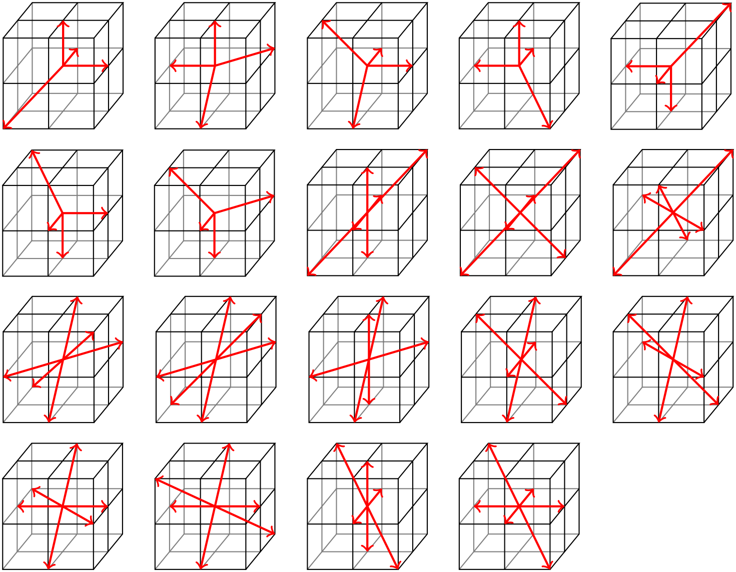
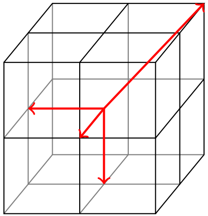
Question: differential finiteness $\Longleftrightarrow$ finiteness of the group?

Answer: probably no

19 mysterious 3D-models: finite $\mathcal{G}_{\mathscr{S}}$ and possibly non-D-finite $Q_{\mathscr{S}}$



Open question: 3D Kreweras excursions



Numerical computations [Dahne, Salvy, 2020] suggest:

$k_{4n} = C \cdot 256^n / n^{\alpha}$, for $\alpha = 3.3257570041744\ldots \notin \mathbb{Q}$,

so excursions are very probably non-D-finite quadrant models with steps in $\{-2,-1,0,1\}^2$: 13 110

- $|\mathcal{G}_{\mathscr{S}}| < \infty$: 240
  - OS $\neq 0$: 431 → **D-finite**
  - OS $= 0$: 9 → **D-finite?**
- $|\mathcal{G}_{\mathscr{S}}| = \infty$: 12 870
  - α rational: 16 → **non-D-finite?**
  - α irrational: 12 854 → **non-D-finite**

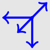
[B., Bousquet-Mélou, Melczer, 2021]

Question: differential finiteness $\iff$ finiteness of the group?

Answer: ?

Two challenging models with large steps

Conjecture 1 [B., Bousquet-Mélou, Melczer, 2021]

For the model  the excursions generating function $Q(0,0;t^{1/2})$ equals

$$\frac{1}{3t} - \frac{1}{6t} \cdot \left(\frac{1-12t}{(1+36t)^{1/3}} \cdot {}_2F_1\left(\begin{matrix} \frac{1}{6} & \frac{2}{3} \\ & 1 \end{matrix} \middle| \frac{108t(1+4t)^2}{(1+36t)^2} \right) + \sqrt{1-12t} \cdot {}_2F_1\left(\begin{matrix} -\frac{1}{6} & \frac{2}{3} \\ & 1 \end{matrix} \middle| \frac{108t(1+4t)^2}{(1-12t)^2} \right) \right).$$

Conjecture 2 [B., Bousquet-Mélou, Melczer, 2021]

For the model  the excursions generating function $Q(0,0;t)$ equals

$$\frac{(1-24\,U+120\,U^2-144\,U^3)\,(1-4\,U)}{(1-3\,U)\,(1-2\,U)^{3/2}\,(1-6\,U)^{9/2}},$$

where $U = t^4 + 53\,t^8 + 4363\,t^{12} + \cdots$ is the unique series in $\mathbb{Q}[[t]]$ satisfying

$$U\,(1-2\,U)^3\,(1-3\,U)^3\,(1-6\,U)^9 = t^4\,(1-4\,U)^4.$$

three-quadrant models $\mathscr{S}$: 74

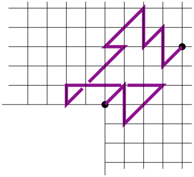
$|\mathcal{G}_{\mathscr{S}}| < \infty$: 23 — $|\mathcal{G}_{\mathscr{S}}| = \infty$: 51

D-finite? — **non-D-finite**

3 (4 ?) — **3 (19 ?)** — **1 (9 ?)** — **3 (42 ?)**

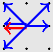
algebraic — **transcendental** — **D-algebraic** — **D-transcendental**

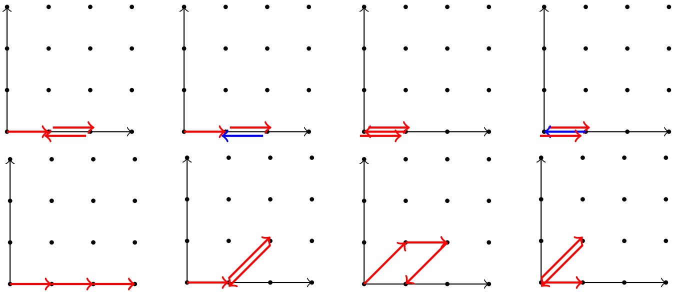
▷ Partial classification due to
[Bousquet-Mélou, 2016], [Raschel, Trotignon, 2019],
[Mustapha, 2019], [Dreyfus, Trotignon, 2020],
[Bousquet-Mélou, Wallner, 2021], [Bousquet-Mélou, 2021]



A difficult quadrant model with repeated steps

Theorem [B., Bousquet-Mélou, Kauers, Melczer, 2016]

Let $a_n = \#\Big\{$  – walks of length n in $\mathbb{N}^2$ from $(0,0)$ to $(\star,0)\Big\}$. Then $f(t) = \sum_n a_n t^n = 1 + t + 4\,t^2 + 8\,t^3 + 39\,t^4 + 98\,t^5 + \cdots$ is transcendental.



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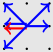
Let $a_n = \# \Bigg\{$  $-$ walks of length n in $\mathbb{N}^2$ from $(0,0)$ to $(\star,0)\Bigg\}$. Then $f(t) = \sum_n a_n t^n = 1 + t + 4\,t^2 + 8\,t^3 + 39\,t^4 + 98\,t^5 + \cdots$ is transcendental.

Proof:

1. Discover and certify a differential equation L for $f(t)$ of order 11 and degree 73 — high-tech Guess-and-Prove
2. If $\text{ord}(L_f^{\min}) \leq 10$, then $\deg_t(L_f^{\min}) \leq 580$ — apparent singularities
3. Rule out this possibility — differential Hermite-Padé approximants
4. Thus, $L_f^{\min} = L$
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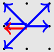
Let $a_n = \#\Big\{$  $-$ walks of length n in $\mathbb{N}^2$ from $(0,0)$ to $(\star,0)\Big\}$. Then $f(t) = \sum_n a_n t^n = 1 + t + 4\,t^2 + 8\,t^3 + 39\,t^4 + 98\,t^5 + \cdots$ is transcendental.

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▷ General minimization algorithm and application to transcendence [B., Rivoal, Salvy, 2021]

Solution of the “exercise”

Excursions in $\mathbb{Z} \times \mathbb{N}$

- The kernel equation reads (with $K(x,y) = 1 - t(y + \bar{x} + x\bar{y})$):

$$K(x,y)yH(x,y) = y - txH(x,0)$$

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$$y_0 = \frac{x - t - \sqrt{(t-x)^2 - 4t^2x^3}}{2tx} \qquad = xt + t^2 + (x^2 + \bar{x})t^3 + (3x + \bar{x}^2)t^4 + \cdots$$

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$$H(x,0) = \frac{y_0}{tx} \quad \text{and} \quad A(t) = \left[x^0\right] \frac{y_0}{tx}.$$

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- Creative telescoping then proves:

$$(27t^4 - t)A''(t) + (108t^3 - 4)A'(t) + 54t^2A(t) = 0.$$

```
> Zeilberger(1/x * sqrt((t-x)^2 - 4*t^2*x^3)/(2*t^2*x^2), t, x, Dt);
```

The group of the model $\{\uparrow, \leftarrow, \searrow\}$

Step set $\mathscr{S} = \{(-1,0),(0,1),(1,-1)\}$, with characteristic polynomial

$$\chi(x,y) = \frac{1}{x} + y + x \cdot \frac{1}{y} = \bar{x} + y + x\bar{y}$$

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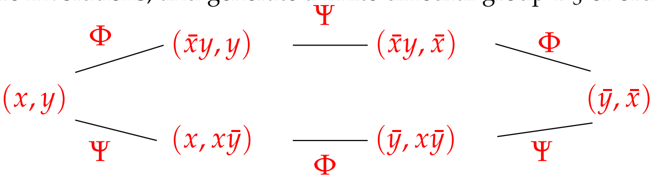
Φ and Ψ are involutions, and generate a finite dihedral group D_3 of order 6:



- Orbit equation:

$$
\begin{aligned}
& xyQ(x,y) - \bar{x}y^2Q(\bar{x}y,y) + \bar{x}^2yQ(\bar{x}y,\bar{x}) \\
& \qquad - \bar{x}\bar{y}Q(\bar{y},\bar{x}) + x\bar{y}^2Q(\bar{y},x\bar{y}) - x^2\bar{y}Q(x,x\bar{y}) = \\
& \qquad\qquad \frac{xy - \bar{x}y^2 + \bar{x}^2y - \bar{x}\bar{y} + x\bar{y}^2 - x^2\bar{y}}{1 - t(y + \bar{x} + x\bar{y})}
\end{aligned}
$$

Diagonal walks in $\mathbb{N}^2$

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- Corollary [B.-Chyzak-van Hoeij-Kauers-Pech, 2017]:

$$B(t) = [z^0]Q(z,\bar{z}) = [u^{-1}v^{-1}z^{-1}]\ \frac{\bar{u}\bar{v} - u\bar{v}^2 + u^2\bar{v} - uv + \bar{u}v^2 - \bar{u}^2v}{z(1-zu)(1-v\bar{z})(1 - t(\bar{v} + u + \bar{u}v))}$$

Diagonal walks in $\mathbb{N}^2$

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- Creative Telescoping gives a differential equation for $B(t)$:

$$(27t^4 - t)B''(t) + (108t^3 - 4)B'(t) + 54t^2B(t) = 0.$$

Conclusion

We have proved that $A(t)$ and $B(t)$ are both solutions of

$$(27t^4 - t)y''(t) + (108t^3 - 4)y'(t) + 54t^2 y(t) = 0.$$

Solving this equation proves:

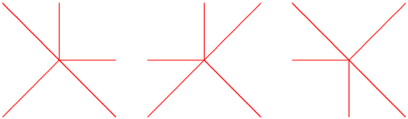
$$A(t) = B(t) = {}_2F_1\left(\begin{matrix}1/3 & 2/3\\ & 2\end{matrix} \,\middle|\, 27\,t^3\right) = \sum_{n=0}^{\infty} \frac{(3n)!}{n!^3} \frac{t^{3n}}{n+1}.$$

Thus the two sequences are equal to

$$a_{3n} = b_{3n} = \frac{(3n)!}{n!^2 \cdot (n+1)!}, \qquad \text{and} \quad a_m = b_m = 0 \quad \text{if 3 does not divide } m.$$

Example with infinite group: the scarecrows

[B., Raschel, Salvy, 2014]: $Q_{\mathscr{S}}(0,0;t)$ is not D-finite for the models



▷ For the 1st and the 3rd, the excursions sequence $[t^n]\, Q_{\mathscr{S}}(0,0;t)$

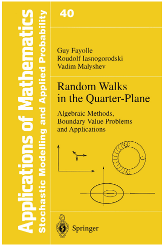
$$1, 0, 0, 2, 4, 8, 28, 108, 372, \ldots$$

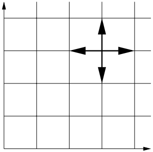
is $\sim K \cdot 5^n \cdot n^{-\alpha}$, with $\alpha = 1 + \pi / \arccos(1/4) = 3.383396\ldots$

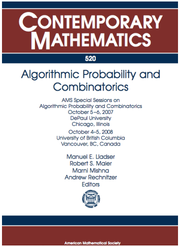
[Denisov, Wachtel, 2015]

▷ The irrationality of α prevents $Q_{\mathscr{S}}(0,0;t)$ from being D-finite.

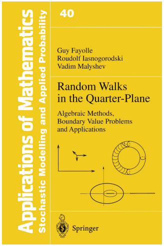
[Katz, 1970; Chudnovsky, 1985; André, 1989]

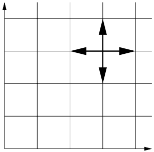


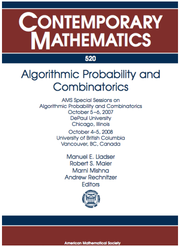




The characteristic polynomial $\chi_{\mathscr{S}} := x + \frac{1}{x} + y + \frac{1}{y}$



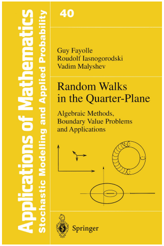


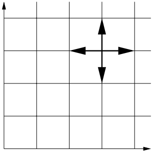


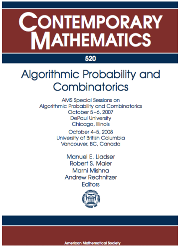
The characteristic polynomial $\chi_{\mathscr{S}} := x + \frac{1}{x} + y + \frac{1}{y}$ is left invariant under

$$\psi(x,y) = \left(x, \frac{1}{y}\right), \qquad \phi(x,y) = \left(\frac{1}{x}, y\right),$$

The group of a model: the simple walk case







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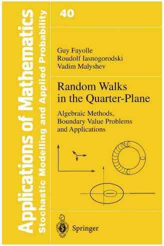
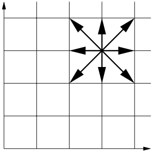
$$\psi(x,y) = \left(x, \frac{1}{y}\right), \qquad \phi(x,y) = \left(\frac{1}{x}, y\right),$$

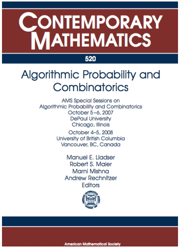
and thus under any element of the group

$$\langle \psi, \phi \rangle = \left\{ (x,y), \left(x, \frac{1}{y}\right), \left(\frac{1}{x}, \frac{1}{y}\right), \left(\frac{1}{x}, y\right) \right\}.$$

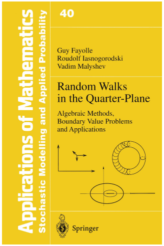
The group of a model

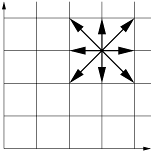


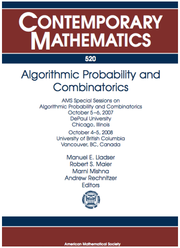




The generating polynomial $\chi_{\mathscr{S}} := \sum_{(i,j)\in\mathscr{S}} x^i y^j = \sum_{i=-1}^{1} B_i(y)x^i = \sum_{j=-1}^{1} A_j(x)y^j$







The generating polynomial $\chi_{\mathscr{S}} := \sum_{(i,j)\in\mathscr{S}} x^i y^j = \sum_{i=-1}^{1} B_i(y) x^i = \sum_{j=-1}^{1} A_j(x) y^j$

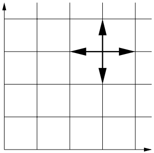
is left invariant under the birational involutions

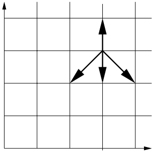
$$\psi(x,y) = \left(x, \frac{A_{-1}(x)}{A_{+1}(x)} \frac{1}{y}\right), \qquad \phi(x,y) = \left(\frac{B_{-1}(y)}{B_{+1}(y)} \frac{1}{x}, y\right),$$

and thus under any element of the (dihedral) group

$$\mathcal{G}_{\mathscr{S}} := \langle \psi, \phi \rangle.$$

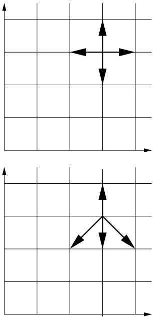
Examples of groups



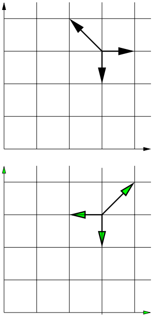


Order 4,

Examples of groups

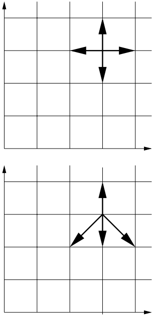


Order 4,

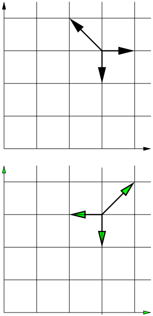


order 6,

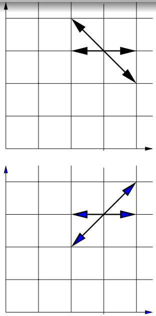
Examples of groups



Order 4,

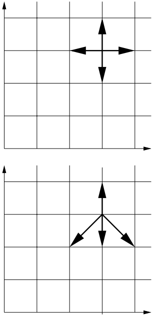


order 6,

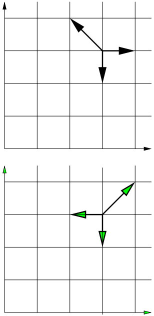


order 8,

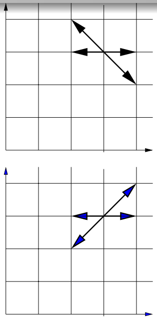
Examples of groups



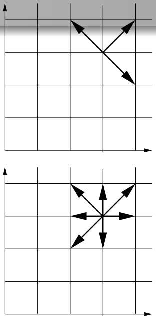
Order 4,



order 6,

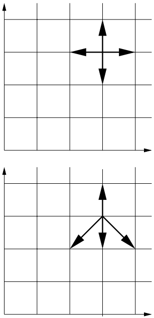


order 8,

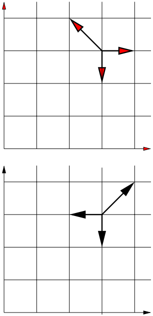


order ∞.

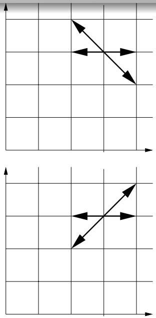
Examples of groups



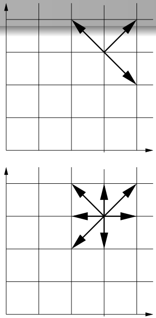
Order 4,



order 6,



order 8,



order ∞.