N-th term of a linearly recurrent sequence

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Exercise 1

The aim of this exercise is to prove algorithmically the following identity:

$$\sqrt[3]{\sqrt[3]{2}-1} = \sqrt[3]{\frac{1}{9}} - \sqrt[3]{\frac{2}{9}} + \sqrt[3]{\frac{4}{9}}.$$

Let $a = \sqrt[3]{2}$ and $b = \sqrt[3]{1/9}$.

(a) Determine a polynomial annihilating $c = 1 - a + a^2$, using a resultant.

(b) Deduce a polynomial annihilating the RHS, using another resultant.

(c) Show that the polynomial computed in (b) also annihilates the LHS.

(d) Conclude.

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(a) Determine a polynomial annihilating $c = 1 - a + a^2$, using a resultant.

▷ Maple solution (all resultant computations *can* be done by hand):

```
> (* (a): a = 2^(1/3), c = 1 - 2^(1/3) + 4^(1/3) *)
> Pa := A^3 - 2:
> Pc := resultant(Pa, C - (1-A+A^2), A);
```

$$C^3 - 3\,C^2 + 9\,C - 9$$

Exercise 1(b)

The aim of this exercise is to prove algorithmically the following identity:

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Let $a = \sqrt[3]{2}$ and $b = \sqrt[3]{1/9}$.

(b) Deduce a polynomial annihilating the RHS, using another resultant.

```
> (* (a): b = (1/9)^(1/3), c = 1 - 2^(1/3) + 4^(1/3), RHS = b*c *)
> Pb:=B^3-1/9;
> numer(subs(B = RHS/C, Pb));
```

$$-C^3 + 9\,\mathrm{RHS}^3$$

```
> PR:=primpart(resultant(Pc, %, C));
```

$$\mathrm{RHS}^9 + 3\,\mathrm{RHS}^6 + 3\,\mathrm{RHS}^3 - 1$$

Exercise 1(c)

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(c) Show that the polynomial computed in (b) also annihilates the LHS.

```
> (* (c): LHS = (a-1)^(1/3) *)
> PL := resultant(x - 1, subs(A = LHS^3 + x, Pa), x);
```

$$\mathrm{LHS}^9 + 3\,\mathrm{LHS}^6 + 3\,\mathrm{LHS}^3 - 1$$

```
> evalb(subs(LHS = x, PL) = subs(RHS = x, PR));
```

true

Exercise 1(d)

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(d) Conclude.

```
> (* (d): PL is non-decreasing *)
> factor(diff(PL, LHS));
```

$$9\,\mathsf{LHS}^2\left(\mathsf{LHS}+1\right)^2\left(\mathsf{LHS}^2-\mathsf{LHS}+1\right)^2$$

- The polynomial annihilating both LHS and RHS has exactly 1 real root

Exercise 2

Let $\mathbb{K}$ be a field of characteristic zero. Consider $F \in \mathbb{K}[[x]]$ with $F(0) = 1$.

(a) What is the complexity of computing $\sqrt{F}$, by using $\sqrt{F} = \exp(\frac{1}{2}\log F)$?

(b) Describe a Newton iteration that directly computes $\sqrt{F}$, without appealing to successive logarithm and exponential computations.

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1. $O(\mathsf{M}(N))$ for computing the first N terms
2. $\Phi(F, G) := G^2 - F$ to get $G = \sqrt{F}$ provides $\mathcal{N}(G) = (G + F/G)/2$.
   If $G = \sqrt{F} + O(X^n)$, then $\exists H,\ F = G^2(1 + X^n H)$, so $\mathcal{N}(G) = G + \frac{1}{2}GX^n H$.
   Next, $\sqrt{F} = G(1 + \frac{1}{2}X^n H + O(X^{2n}))$, so $\mathcal{N}(G) = \sqrt{F} + O(X^{2n})$.

   Computationally: given $F = T + O(X^N)$, recursively compute $\sqrt{F} = U + O(X^{\lceil N/2 \rceil})$, then return $U + \mathrm{rem}(T/U, X^N)/2$.

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3. $C(N) \leq C(N/2) + O(\mathsf{M}(N))$ leads to $O(\mathsf{M}(N))$.

COMPUTING TERMS OF RECURRENT SEQUENCES

Goal, motivation, examples, main results

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Given a sequence $(u_n)_{n\geq 0}$ in a ring R, and $N \in \mathbb{N}$, compute u_N fast

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Two variants:

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and

Given $(u_n)_n$ in $\mathbb{Z}^{\mathbb{N}}$ and $(N_\ell)_{\ell=1}^s \in \mathbb{N}^s$, compute $(u_{N_\ell} \bmod N_\ell)_{\ell=1}^s$ fast

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p-curvature

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  - *matrix powering* M^N; more generally, $P(M)$ [Giesbrecht, 1995]

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  - *counting points on elliptic curves* over $\mathbb{F}_q$ [Schoof-Elkies-Atkin, 1992–1998]

Overview: naive algorithms

Seq. Term	Arith. size	Arith. cost	Method	Bit size	Bit cost	Method
q^N	1	$O(N)$	iterative algorithm	N	$\tilde{O}(N^2)$	iterative algorithm

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q^N	1	$O(\log N)$	binary powering	N	$\tilde{O}(N)$	binary powering

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▷ For $R = \mathbb{F}_p$: $M_N \in R$ in $O(\log N)$ ops. in R; same for any u_N with *algebraic* GF $\sum_n u_n x^n$ [B., Christol, Dumas, 2016], [B., Caruso, Christol, Dumas, 2019]

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- First part of this course focusses on the first two rows
- 27/01/2025: two middle rows

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COMPUTING TERMS OF C-RECURSIVE SEQUENCES

Main question

Given a sequence $(u_n)_{n\geq 0}$ *in a field* $\mathbb{K}$*, and* $N \in \mathbb{N}$*, compute* u_N *fast*

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First part: input sequence is C-recursive, given by initial terms $u_0, \ldots, u_{d-1}$ and a linear recurrence with constant coefficients $(c_0, \ldots, c_{d-1}) \in \mathbb{K}^d$

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- Def. $\Gamma(x) := x^d - \sum_{i=0}^{d-1} c_i x^i$ is called *characteristic polynomial* for $(u_n)_{n\geq 0}$

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▷ Application: modular exponentiation in $R = \mathbb{K}[x]/(Q)$:

- if $P, Q \in \mathbb{K}[x]_{<d}$, then $P^N \bmod Q$ in $O(\mathsf{M}(d) \log N)$ ops. in $\mathbb{K}$

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▷ Application: modular exponentiation in $R = \mathbb{Z}/A\mathbb{Z}$:

- N-th decimal of $\frac{1}{A}$ via $(10^{N-1} \bmod A)$ in $O(\mathsf{M}_{\mathbb{Z}}(\log A) \log N)$ bit ops.

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▷ The following also computes the right answer. Can you see why?

```
> n := irem(N,A-1);
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6

Schönhage's Golden Rules

RULE 1: *Do care about the size of* $\mathbf{0}$ *!*

Fast polynomial division and modular exponentiation [Strassen, 1973]

Pb: Given $F \in \mathbb{K}[x]_{<2d}$ and $Q \in \mathbb{K}[x]_d$ compute (U, R) in Euclidean division

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Formalization: Let $D = \deg(F)$. Then $\deg(U) = D - d < d$, $\deg(R) < d$ and

$$\underbrace{F(1/x)x^D}_{\mathrm{rev}(F)} = \underbrace{Q(1/x)x^d}_{\mathrm{rev}(Q)} \cdot \underbrace{U(1/x)x^{D-d}}_{\mathrm{rev}(U)} + \underbrace{R(1/x)x^{\deg(R)}}_{\mathrm{rev}(R)} \cdot x^{D-\deg(R)}$$

Fast polynomial division and modular exponentiation [Strassen, 1973]

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Algorithm: Complexity

1. Compute $A = 1/\mathrm{rev}(Q) \mod x^{D-d+1}$ — $3\,\mathsf{M}(d) + O(d)$
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▷ Corollary: Modular exponentiation $x^N \bmod Q$ in $\sim 3\,\mathsf{M}(d) \log N$ ops. in $\mathbb{K}$

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▷ A bit optimistic (did not count "-and-multiply"...); OK if $P = x$

Schönhage's Golden Rules

RULE 2: *Do not waste a factor of two !*

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$$u_N = e \cdot v_N = e \cdot (C^T)^N \cdot v_0 = \left(C^N \cdot e^T\right)^T \cdot v_0 = \langle x^N \bmod \Gamma, v_0 \rangle,$$

where $e = [1 \quad 0 \quad \cdots \quad 0]$. $\quad \sim 3\,\mathsf{M}(d)\log N$

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- ▷ [B., Mori, 2021]: different ideas / algorithms $\quad \sim 2\,\mathsf{M}(d)\log N$

Example: N-th term of the Fibonacci sequence

L'INTERMÉDIAIRE

DES

MATHÉMATICIENS

DIRIGÉ PAR

C.-A. LAISANT, DOCTEUR ÈS SCIENCES, | ÉMILE LEMOINE, INGÉNIEUR CIVIL,

ANCIENS ÉLÈVES DE L'ÉCOLE POLYTECHNIQUE.

TOME VI. — 1899.

PARIS,

GAUTHIER-VILLARS, IMPRIMEUR-LIBRAIRE

DU BUREAU DES LONGITUDES, DE L'ÉCOLE POLYTECHNIQUE,

55, Quai des Grands-Augustins, 55.

1899

Je serais également heureux d'avoir des renseignements bibliographiques (postérieurs à Delambre) sur les études *scientifiques* consacrées aux cadrans verticaux dans l'antiquité. PAUL TANNERY.

1539. **[H1b]** Il peut arriver qu'une équation différentielle admette une intégrale particulière imaginaire. La connaissance de celle-ci peut-elle être de quelque utilité pour l'intégration de l'équation donnée? H. BROCARD.

1540. **[I2b]** b étant un nombre composé, quelles sont les valeurs de b qui rendent le produit $1.2.3\ldots(b-1)$ non divisible par b? G. DE ROCQUIGNY.

1541. **[I25a]** Quel est le procédé le plus expéditif pour calculer un terme très éloigné dans la série de Fibonacci :

0, 1, 1, 2, 3, 5, 8, ...?

G. DE ROCQUIGNY.

1542. **[E1a]** Est-il exact, et dans ce cas comment pourrait-on le démontrer, que :

1° L'expression

$$\Phi_n(x) = n^{x-1} - \frac{x}{1}(n-1)^{x-1} + \frac{x(x-1)}{1.2}(n-2)^{x-1} - \ldots,$$

où n désigne un entier et x une quantité positive quelconque, tend vers zéro en même temps que n vers l'infini;

2° La loi de décroissance des quantités $\Phi_n(x)$ est assez rapide pour que la série

$$\Phi_1(x) + \Phi_2(x) + \Phi_3(x) + \ldots + \Phi_n(x) + \ldots$$

soit convergente;

3° La somme de cette série a pour limite la fonction $\Gamma(x)$?

Tout cela étant, la fonction eulérienne de deuxième $\Gamma(1+x)$ se présenterait comme la limite de l'expression

$$n^x - \frac{x}{1}(n-1)^x + \frac{x(x-1)}{1.2}(n-2)^x - \ldots.$$

E.-A. Majol.

Example: N-th term of the Fibonacci sequence

Fiduccia's algorithm (1985): binary powering in the ring $\mathbb{K}[x]/(x^2 - x - 1)$:

$$C^n = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}^n = \text{matrix of} \quad (x^n \bmod x^2 - x - 1)$$

$$\Longrightarrow \quad F_{n-2} + xF_{n-1} = \quad x^n \bmod x^2 - x - 1$$

Cost: $O(\log N)$ products in $\mathbb{K}[x]/(x^2 - x - 1) \longrightarrow O(\log N)$ ops. for F_N

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$$F_{2n-2} + xF_{2n-1} = (F_{n-2} + xF_{n-1})^2 \bmod x^2-x-1$$

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$$\text{implies} \begin{cases} F_{2n-2} &= F_{n-2}^2 + F_{n-1}^2 \\ F_{2n-1} &= F_{n-1}^2 + 2F_{n-1}F_{n-2} \end{cases}$$

$$(F_0, F_1) \to (F_2, F_3) \to (F_6, F_7) \to (F_{14}, F_{15}) \to \dots$$

Cost: $3\times$ and $3+$ per arrow

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cise en prenant un plus grand nombre de décimales, nous paraît, en effet, l'un des procédés de calcul les meilleurs. Elle permet de trouver des termes, exactement, jusqu'à une limite assez éloignée; et pour des termes très lointains et impossibles à écrire, elle a le mérite de faire connaître le nombre des chiffres. C.-A. LAISANT.

La suite de Fibonacci peut être exprimée par une autre loi que celle qui est ordinairement employée. En effet, en considérant les termes de cette suite comme provenant du calcul des réduites d'une fraction continue périodique symétrique dont tous les quotients incomplets sont égaux à l'unité, on trouve, en appliquant les formules de Lucas (*Théorie des nombres*) :

$$u_{2n} = u_n^2 + u_{n-1}^2,$$
$$u_{2n-1} = u_{n-1}(u_n + u_{n-2}) = u_n^2 - u_{n-2}^2,$$

le point de départ étant

$$u_0 = 1, \qquad u_1 = 1, \qquad u_2 = 2.$$

Ces formules permettent de calculer assez rapidement les termes de cette suite. En moins d'un quart d'heure j'ai pu calculer

$$u_{100} = 573\,147\,844\,013\,817\,084\,101.$$

Accessoirement ces formules montrent que tous les termes de rang impair sont des nombres composés sauf le terme $u_3 = 3$ et que tous les nombres premiers, que l'on ne peut rencontrer qu'aux rangs pairs, sont tous de la forme $4u + 1$ puisqu'ils sont la somme de deux carrés premiers entre eux. G. PICOU.

Les valeurs de u_{100} données par MM. *Rosace* et Picou sont toutes deus exactes, seulement M. *Rosace* prend pour premiers termes 0, 1, 1, 2, 3, ... et M. Picou 1, 1, 2, 3, ... E. LEMOINE.

1549. (**1899**, 150) (E. DUPORCQ). — *Sur une propriété nouvelle de l'ovale de Descartes.* — Soient R le rayon de courbure, θ, θ', θ'' les angles que font les rayons vecteurs avec la normale; selon que l'ovale est rapporté en coordonnées bipolaires aux foyers φ, φ'; φ, φ''; φ', φ'' on a pour le rayon de courbure les formes

$$(1)\quad R = \frac{\cos\theta + m\cos\theta'}{\frac{\cos^2\theta}{\rho} + m\frac{\cos^2\theta'}{\rho'}} = \frac{\cos\theta' + p\cos\theta''}{\frac{\cos^2\theta'}{\rho'} + p\frac{\cos^2\theta''}{\rho''}} = \frac{\cos\theta'' + q\cos\theta'}{\frac{\cos^2\theta''}{\rho''} + q\frac{\cos^2\theta'}{\rho'}}$$

Computing the N-th coefficient of a rational function

Duality lemma (link between C-recursive sequences and rational functions)
Let $A(x) = \sum_{n\geq 0} u_n x^n \in \mathbb{K}[[x]]$ be the generating function of $(u_n)_{n\geq 0}$.
The following assertions are equivalent:

(i) $(u_n)_{n\geq 0}$ is C-recursive, having Γ as characteristic polynomial of degree d;

(ii) $A(x)$ is rational, of the form $A = P/Q$ for some $P \in \mathbb{K}[x]_{<d}$, where $Q := \mathsf{rev}_d(\Gamma) = \Gamma(\frac{1}{x})x^d$.

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▷ Corollary: N-th Taylor coeff. of $\frac{P}{Q} \in \mathbb{K}(x)_d$ in $\sim 3\,\mathsf{M}(d)\log N$ ops. in $\mathbb{K}$

Computing the first N coefficients of a C-recursive sequence

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Example (Fibonacci): $F_{i+2} = F_{i+1} + F_i \iff \sum_i F_i x^i = \dfrac{F_0 + (F_1 - F_0)x}{1 - x - x^2}$

Computing the first N coefficients of a C-recursive sequence

Problem: Given $d, N \in \mathbb{N}$ with $N \gg d$ and the first d terms $u_0, \ldots, u_{d-1}$ of a C-recursive sequence of order d, compute the next terms $u_d, \ldots, u_N$

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A faster algorithm [Shoup, 1991]:

- Compute (P, Q) from recurrence and $u_0, \ldots, u_{d-1}$ $O(\mathsf{M}(d))$
- Compute $R(x) := 1/Q \bmod x^d$; set $c_0 := \sum_{j=0}^{d-1} u_j x^j$ $O(\mathsf{M}(d))$
- For $s = 0, \ldots, \lceil N/d \rceil - 1$ compute $c_{s+1} := -R \cdot [Q \cdot c_s]_d^{2d-1}$ $O\left(\frac{N}{d}\mathsf{M}(d)\right)$
- Return $\sum_{s=0}^{\lceil N/d \rceil} c_s(x) x^{sd} \bmod x^N$

Computing the N-th coefficient of a rational function, revisited

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$$u_N = [x^N] \frac{P(x)}{Q(x)}$$

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$$u_N = [x^N] \frac{P(x)Q(-x)}{Q(x)Q(-x)} = [x^N] \frac{U(x)}{V(x^2)}.$$

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$$u_N = \begin{cases} [x^N] \frac{U_{\mathrm{e}}(x^2)}{V(x^2)}, & \text{if } N \text{ is even} \\ [x^N] \frac{xU_{\mathrm{o}}(x^2)}{V(x^2)}, & \text{else.} \end{cases}$$

$$= \begin{cases} [x^{N/2}] \frac{U_{\mathrm{e}}(x)}{V(x)}, & \text{if } N \text{ is even} \\ [x^{(N-1)/2}] \frac{U_{\mathrm{o}}(x)}{V(x)}, & \text{else.} \end{cases}$$

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- ▷ Algorithm: repeat this reduction until $N \geq 1$ $2\,\mathsf{M}(d)\lceil \log(N+1) \rceil$

Algorithm 1 OneCoeff

Input: $P(x)$, $Q(x)$, N

Output: $[x^N] \frac{P(x)}{Q(x)}$

Assumptions: $Q(0)$ invertible and $\deg(P) < \deg(Q) =: d$

while $N \geq 1$ **do**
  $U(x) \leftarrow P(x)Q(-x)$ ▷ $U = \sum_{i=0}^{2d-1} U_i x^i$
  if N is even **then**
    $P(x) \leftarrow \sum_{i=0}^{d-1} U_{2i} x^i$
  else
    $P(x) \leftarrow \sum_{i=0}^{d-1} U_{2i+1} x^i$
  end if
  $A(x) \leftarrow Q(x)Q(-x)$ ▷ $A = \sum_{i=0}^{2d} A_i x^i$
  $Q(x) \leftarrow \sum_{i=0}^{d} A_{2i} x^i$
  $N \leftarrow \lfloor N/2 \rfloor$
end while
return $P(0)/Q(0)$

Computing the N-th term of a C-recursive sequence, revisited

Pb: Given $N \in \mathbb{N}$, $u_0, \ldots, u_{d-1} \in \mathbb{K}$, and the recurrence

$$u_{n+d} = c_{d-1}u_{n+d-1} + \cdots + c_0 u_n, \qquad n \geq 0,$$

compute u_N

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▷ [Fiduccia, 1985] binary powering in $\mathbb{K}[x]/(\Gamma)$, with $\Gamma = x^d - \sum_{i=0}^{d-1} c_i x^i$
$\sim 3\,\mathsf{M}(d)\log N$

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- [B., Mori, 2021]: Use $[x^N]\,\frac{P(x)}{Q(x)}$ and duality lemma $\sim 2\,\mathsf{M}(d)\log N$
- Appropriate choice is: $Q = \mathrm{rev}(\Gamma)$, and P such that $\frac{P}{Q} = \sum_{i=0}^{d-1} u_i x^i \bmod x^d$

Algorithm 2 OneTerm

Input: rec. $u_{n+d} = c_{d-1}u_{n+d-1} + \cdots + c_0 u_n$, $(n \geq 0)$, and $u_0, \ldots, u_{d-1}$, N
Output: u_N
Assumptions: $\Gamma(x) = x^d - \sum_{i=0}^{d-1} c_i x^i$ with $c_0 \neq 0$

1: $Q(x) \leftarrow x^d \Gamma(1/x)$
2: $P(x) \leftarrow (u_0 + \cdots + u_{d-1}x^{d-1}) \cdot Q(x) \bmod x^d$
3: **return** $[x^N]P(x)/Q(x)$ ▷ using Algorithm 1

▷ Advantage: faster than Fiduccia's algorithm

▷ in FFT-mode, $\sim \frac{2}{3}\,\mathrm{M}(d)\log N$ versus $\sim \frac{5}{3}\,\mathrm{M}(d)\log N$ [Shoup, NTL, 1995] and $\sim \frac{13}{12}\,\mathrm{M}(d)\log N$ [Mihăilescu, 2008]

▷ Drawback: computes a single u_N, while Fiduccia computes a whole slice

Application: new algorithm for the Fibonacci numbers

$$F_0 = 0, F_1 = 1, \quad F_{n+2} = F_{n+1} + F_n, \ n \geq 0.$$

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- Thus, the coefficient $F_N = [x^N] \frac{x}{1-x-x^2}$ is equal to

$$[x^N] \frac{x(1+x-x^2)}{1-3x^2+x^4} = \begin{cases} [x^{\frac{N}{2}}] \frac{x}{1-3x+x^2}, & \text{if } N \text{ is even,} \\ [x^{\frac{N-1}{2}}] \frac{1-x}{1-3x+x^2}, & \text{else.} \end{cases}$$

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- The computation of F_N is reduced to that of a coefficient of the form

$$[x^N] \frac{a + bx}{1 - cx + x^2} = [x^N] \frac{(a + bx)(1 + cx + x^2)}{1 - (c^2 - 2)x^2 + x^4}$$

which is equal to

$$\begin{cases} [x^{\frac{N}{2}}] \frac{a+(bc+a)x}{1-(c^2-2)x+x^2}, & \text{if } N \text{ is even,} \\ [x^{\frac{N-1}{2}}] \frac{(ac+b)+bx}{1-(c^2-2)x+x^2}, & \text{else.} \end{cases}$$

Algorithm 3 NewFibo

Input: N
Output: F_N
Assumptions: $N \geq 2$

1: $c \leftarrow 3$
2: **if** N is even **then**
3: $\quad [a, b] \leftarrow [0, 1]$
4: **else**
5: $\quad [a, b] \leftarrow [1, -1]$
6: **end if**
7: $N \leftarrow \lfloor N/2 \rfloor$
8: **while** $N > 1$ **do**
9: $\quad$ **if** N is even **then**
10: $\quad\quad b \leftarrow a + b \cdot c$
11: $\quad$ **else**
12: $\quad\quad a \leftarrow b + a \cdot c$
13: $\quad$ **end if**
14: $\quad c \leftarrow c^2 - 2$
15: $\quad N \leftarrow \lfloor N/2 \rfloor$
16: **end while**
17: **return** $b + a \cdot c$

Computation of $F_{43} = 433\,494\,437$ using the new algorithm

N	a	b	c
21	1	-1	3
10	$1 \times 3 - 1 = 2$		$3^2 - 2 = 7$
5		$(-1) \times 7 + 2 = -5$	$7^2 - 2 = 47$
2	$2 \times 47 - 5 = 89$		$47^2 - 2 = 2207$
1		$(-5) \times 2207 + 89$	$2207^2 - 2 = 4870847$
		$= -10946$	
0	$89 \times 4870847 - 10946$		
	$= 433494437$		

Algorithm 4 NewFiboPowerOfTwo

Input: N
Output: F_N
Assumptions: $N \geq 2$ and N is a power of 2

1: $[b, c] \leftarrow [1, 3]$
2: $N \leftarrow \lfloor N/2 \rfloor$
3: **while** $N > 2$ **do**
4: $\quad b \leftarrow b \cdot c$
5: $\quad c \leftarrow c^2 - 2$
6: $\quad N \leftarrow \lfloor N/2 \rfloor$
7: **end while**
8: **return** $b \cdot c$

▷ This is exactly [Cull, Holloway, 1989, Fig. 6], also [Knuth, 1969]

```
fib(n)
    f ← 1
    l ← 3
    for i = 2 to (log n − 1)
        f ← f * l
        l ← l * l − 2
    f ← f * l
    return f
```

Example: N-th term of the Fibonacci sequence

$u_{20} = 6765$; puis, pour $p = 20$, la même formule donnera

$$u_{100} = 354\,224\,848\,179\,261\,915\,075.$$

Les relations particulières qui précèdent pourraient être déduites directement de formules analogues à la formule (2), mais plus générales, telles que les suivantes :

$$u_{p+q+r-2} = u_{p-1}u_q u_r + u_p u_{q-1} u_r + u_p u_q u_{r-1} + u_{p-2} u_{q-2} u_{r-2},$$
$$u_{p+q+r-2} = u_p u_q u_r + u_p u_{q-1} u_{r-1} + u_{p-1} u_q u_{r-1} + u_{p-1} u_{q-1} u_r - u_{p-1} u_{q-1} u_{r-1};$$

Il suffit de faire $r = 1$ dans cette dernière pour retrouver la relation (2).

Rosace.

Soient les deux séries P_n (de Fibonacci) et Q_n, dont le $n^{\text{ième}}$ terme est au-dessous de l'indice n :

n	0	1	2	3	4	5	6	7	8	9	10
P_n	0	1	1	2	3	5	8	13	21	34	55
Q_n	2	1	3	4	7	11	18	29	47	76	123

La série Q_n procède des deux premiers termes 2, 1, de la même façon que la série P_n procède des termes 0 et 1. Les termes correspondants de ces deux séries présentent une remarquable analogie avec les fonctions trigonométriques *sinus* et *cosinus* :

$$P_n = \frac{1}{\sqrt{5}}[\alpha^n - (-1)^n \beta^n], \qquad Q_n = \alpha^n + (-1)^n \beta^n,$$

où $\alpha = \frac{\sqrt{5}+1}{2}$, $\beta = \frac{\sqrt{5}-1}{2}$.

De ces formules, on déduit les relations suivantes, correspondant à celles qui existent entre le *sinus* et le *cosinus* :

(1) $$Q_n^2 - 5P_n^2 = 4(-1)^n,$$
(2) $$2P_{m+n} = P_m Q_n + Q_m P_n,$$
(3) $$2P_{m-n} = (-1)^n (P_m Q_n - Q_m P_n),$$
(4) $$P_{2m} = P_m Q_m,$$
(5) $$2Q_{m+n} = Q_m Q_n + 5P_m P_n,$$
(6) $$2Q_{m-n} = (-1)^n (Q_m Q_n - 5P_m P_n),$$
(7) $$P_{-m} = (-1)^{m+1} P_m.$$
(8) $$Q_{-m} = (-1)^m Q_m,$$
(9) $$2Q_{2m} = Q_m^2 + 5P_m^2,$$
(10) $$Q_{2m} = Q_m^2 - 2(-1)^m.$$

Par combinaison de (2) et (3), nous avons

(11) $$P_{m+n} = P_m Q_n - (-1)^n P_{m-n}.$$

A l'aide de ces formules, notamment de (2), (4), (10) et (11), nous pourrons calculer très rapidement un terme P_n, d'une manière analogue à celle du sinus d'un arc multiple. Par exemple,

$$Q_{2^n} = 1, 3, 7, 47, \ldots,$$

chaque terme étant le carré du précédent, moins 2.

Pour une étude de ces séries et de quelques autres de ce genre, *voir* E. Lucas, *Théorie des fonctions numériques simplement périodiques* (*A. J. M.*, t. I, et *M. A. B.*, 1878).

E.-B. Escott (Grand Rapids, Mich.).

La série de Fibonacci, comme toute suite récurrente, est susceptible de deux modes de calcul : l'un le mode ordinaire, continu, qui n'est que l'application répétée de la formule même de récurrence; l'autre discontinu, auquel, dans le cas particulier envisagé, on est conduit par les considérations suivantes :

En partant des identités :

$$u_n + u_{n+1} - u_{n+2} = 0, \qquad \ldots, \qquad u_{n+m} + u_{n+m-1} - u_{n+m-1} = 0,$$

on a facilement $u_{n+m+1} = u_{n+1} u_{m+1} + u_n u_m$. Spécialement, pour $m = n$, j'aurai $u_{2n+1} = u_{n+1}^2 + u_n^2$ (égalité qui donne la décomposition en deux carrés d'un terme quelconque de rang impair de la série de Fibonacci), et de même, pour $m = n + 1$,

$$u_{2n+2} = u_{n+1}(u_{n+1} + 2u_n).$$

Ainsi les deux termes u_n et u_{n+1}, calculés par un moyen quelconque, suffisent à faire connaître les deux termes u_{2n+1} et u_{2n+2}, ceux-ci à faire connaître u_{4n+3} et u_{4n+4}, d'où l'on passera à u_{8n+7} et u_{8n+8}, etc. De proche en proche, on arrivera donc aux termes d'indices $(n+1)2^k - 1$ et $(n+1)2^k$, où n et k représentent des entiers à notre choix. Pour fixer les idées, soit à calculer la valeur numérique du $900^{\text{ième}}$ terme de la série. Le nombre 900 est égal à $7 \times 2^7 + 4$, j'aurai à chercher, de la façon qui vient d'être indiquée, u_{895} et u_{896}, puis, par le moyen de l'échelle de relation, je passerai successivement aux termes u_{897}, u_{898}, u_{899} et enfin u_{900}. Ainsi, les valeurs $u_{13} = 233$, $u_{14} = 377$ étant prises comme point de départ, j'en tirerai $u_{27} = 196418$, $u_{28} = 317811$, puis

$$u_{55} = 139583862445, \qquad u_{56} = 225851433717.$$

An exercise for next time (14/11/2024)

(a) Show that if $P \in \mathbb{K}[x]$ has degree d, then the sequence $(P(n))_{n \geq 0}$ is C-recursive, and admits $(x-1)^{d+1}$ as a characteristic polynomial.

(b) Deduce that P can be evaluated at the $N \gg d$ points $1, 2, \ldots, N$ in $O(N\,\mathsf{M}(d)/d)$ operations in $\mathbb{K}$.

Tellegen's transposition principle

Tellegen's transposition principle

Let $\mathbf{M}$ be a $\mathsf{m} \times \mathsf{n}$ matrix, with no zero rows and no zero columns. Any linear algorithm of complexity L that computes the matrix-vector product $\mathbf{M} \cdot \mathbf{v}$ can be transformed into a linear algorithm of complexity

$$\mathsf{L} - \mathsf{n} + \mathsf{m}$$

that computes the transposed matrix-vector product $\mathbf{M}^{\mathbf{T}} \cdot \mathbf{w}$.

▷ A precise formulation depends on the *model of computation*.

A particular case

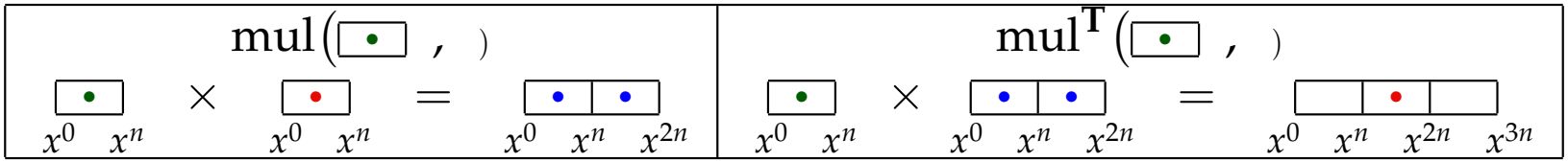
Suppose that the naive algorithm $\mathcal{N}$ is used to compute $M \cdot v$.
Define its dual $\mathcal{N}^T$ as the naive algorithm for computing $M^T \cdot w$.
Then:

$$\begin{aligned} \mathcal{N} &\text{ uses } m(n-1) \text{ ops. } \pm \text{ and } mn \text{ ops. } \times \\ \mathcal{N}^T &\text{ uses } n(m-1) \text{ ops. } \pm \text{ and } mn \text{ ops. } \times. \end{aligned}$$

$\longrightarrow$ Tellegen's theorem is trivially true for generic matrices

$\longrightarrow$ it becomes interesting when M is structured or sparse

Transposed polynomial multiplication = middle product (MP)



Example:

$$\mathrm{mul}\,(2+3x, a+bx) \qquad \mathrm{mul}^{\mathbf{T}}\left(3+2x, a+bx+cx^2\right)$$

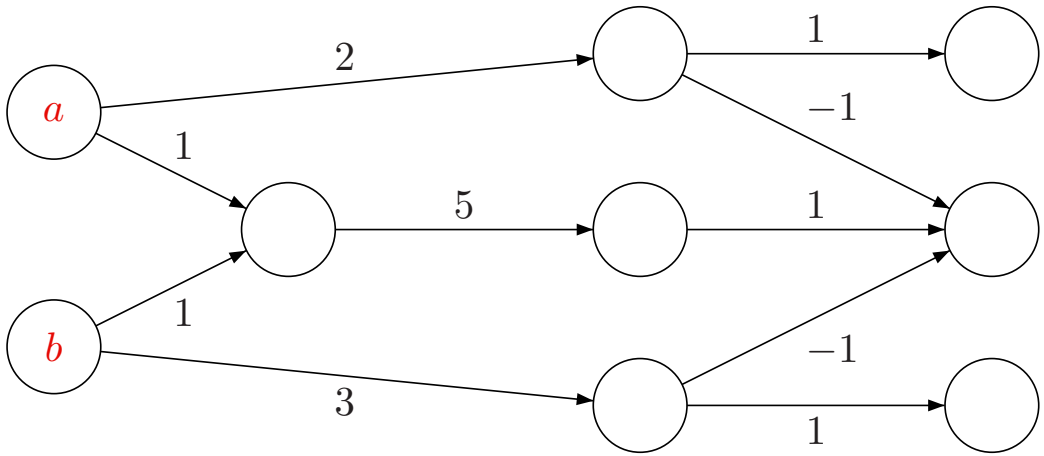
$$\begin{bmatrix} 2 & 0 \\ 3 & 2 \\ 0 & 3 \end{bmatrix} \times \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 2a \\ 3a+2b \\ 3b \end{bmatrix} \qquad \begin{bmatrix} 2 & 3 & 0 \\ 0 & 2 & 3 \end{bmatrix} \times \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 2a+3b \\ 2b+3c \end{bmatrix}$$

Theorem [Hanrot-Quercia-Zimmerman, 2002]
From any linear algorithm for multiplication in degree n of cost $\mathsf{M}(\mathbf{n})$, one can derive a linear algorithm for the $(n, 2n)$ MP, of cost $\mathsf{M}(\mathbf{n}) + \mathbf{O}(\mathbf{n})$.

▷ Particular instance of Tellegen's theorem, for Toeplitz-band matrices

A DAG computing $(2+3x)(a+bx)$ *à la Karatsuba*

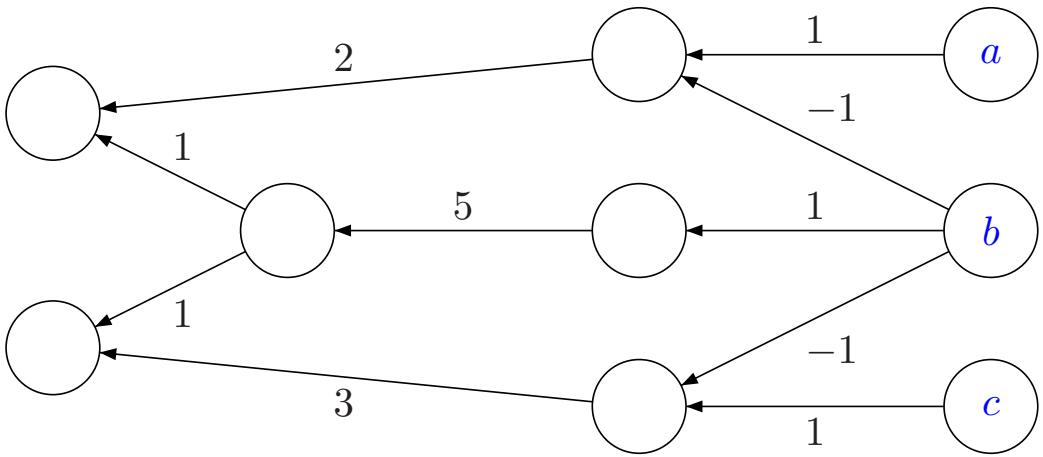
$$\mathrm{mul}\,(2+3x, a+bx)$$



$$\begin{bmatrix} 2 & 0 \\ 3 & 2 \\ 0 & 3 \end{bmatrix} \times \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 2a \\ 3a+2b \\ 3b \end{bmatrix} = \begin{bmatrix} 2a \\ 5(a+b)-2a-3b \\ 3b \end{bmatrix}$$

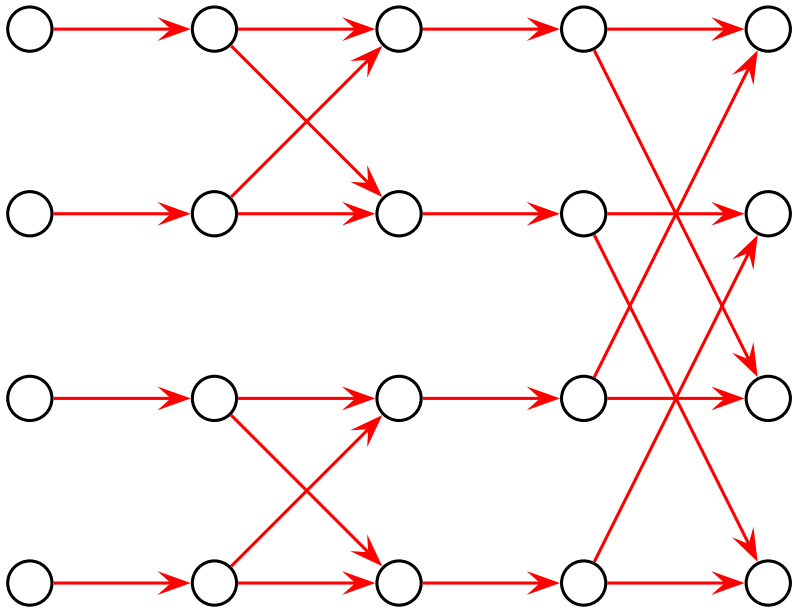
The transposed DAG computes the middle product of $3+2x$ and $a+bx+cx^2$ *à la Karatsuba*

$$\mathrm{mul}^{\mathbf{T}}\left(3+2x, a+bx+cx^2\right)$$



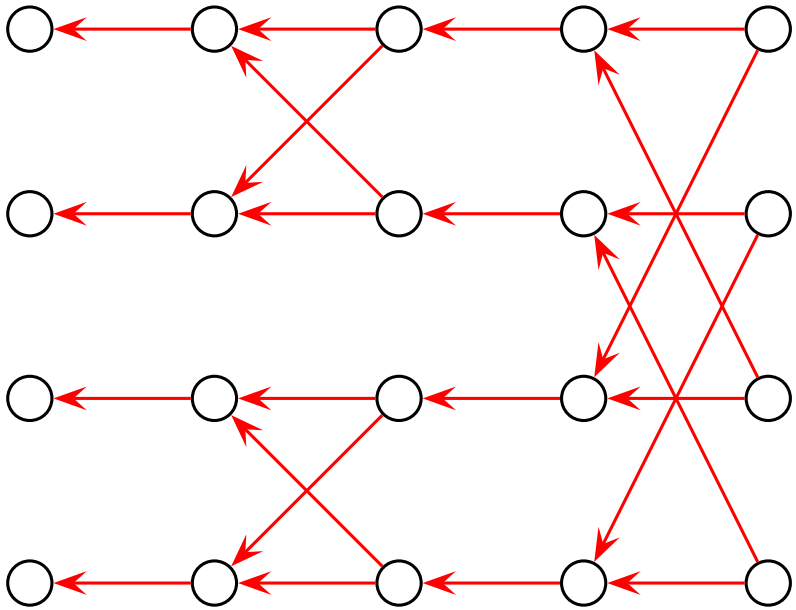
$$\begin{bmatrix} 2 & 3 & 0 \\ 0 & 2 & 3 \end{bmatrix} \times \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 2a+3b \\ 2b+3c \end{bmatrix} = \begin{bmatrix} 2(a-b)+5b \\ 3(b-c)+5b \end{bmatrix}$$

Duality between two classes of FFT algorithms



The Cooley-Tukey decimation-in-time DFT, on 4 points

Duality between two classes of FFT algorithms



The Gentleman-Sande decimation-in-frequency DFT, on 4 points

Automatic discovery of Horner's rule

$$[p_0, \ldots, p_n] \mapsto \sum_{i=0}^{n} p_i a^i$$

Automatic discovery of Horner's rule

$$\mathbf{M} = [1, a, \ldots, a^n]$$
$$[p_0, \ldots, p_n] \mapsto \sum_{i=0}^{n} p_i a^i$$

Automatic discovery of Horner's rule

$$\mathbf{M} = [1, a, \ldots, a^n]$$

$$x_0 \mapsto [x_0, ax_0, \ldots, a^n x_0] \qquad [p_0, \ldots, p_n] \mapsto \sum_{i=0}^{n} p_i a^i$$

Automatic discovery of Horner's rule

$$\mathbf{M} = [1, a, \ldots, a^n]$$

$$x_0 \mapsto [x_0, ax_0, \ldots, a^n x_0] \qquad [p_0, \ldots, p_n] \mapsto \sum_{i=0}^{n} p_i a^i$$

Input x_0.
$p_0 \leftarrow x_0$;
for j **from** 1 **to** n **do**
  $p_j \leftarrow p_{j-1}$;
  $p_j \leftarrow a p_j$;
Output $p = [p_0, \ldots, p_n]$.

Automatic discovery of Horner's rule

$$\mathbf{M} = [1, a, \ldots, a^n]$$

$$x_0 \mapsto [x_0, ax_0, \ldots, a^n x_0] \qquad [p_0, \ldots, p_n] \mapsto \sum_{i=0}^{n} p_i a^i$$

Input x_0.	**Input** $p = [p_0, \ldots p_n]$.
$p_0 \leftarrow x_0$;	
for j **from** 1 **to** n **do**	
$p_j \leftarrow p_{j-1}$;	
$p_j \leftarrow a p_j$;	
Output $p = [p_0, \ldots, p_n]$.	**Output** x_0.

Automatic discovery of Horner's rule

$$\mathbf{M} = [1, a, \ldots, a^n]$$

$$x_0 \mapsto [x_0, ax_0, \ldots, a^n x_0] \qquad [p_0, \ldots, p_n] \mapsto \sum_{i=0}^{n} p_i a^i$$

Input x_0.
$p_0 \leftarrow x_0$;
for j **from** 1 **to** n **do**
  $p_j \leftarrow p_{j-1}$;
  $p_j \leftarrow a p_j$;
Output $p = [p_0, \ldots, p_n]$.

Input $p = [p_0, \ldots p_n]$.
for j **from** n **downto** 1 **do**

Output x_0.

Automatic discovery of Horner's rule

$$\mathbf{M} = [1, a, \ldots, a^n]$$

$$x_0 \mapsto [x_0, ax_0, \ldots, a^n x_0] \qquad [p_0, \ldots, p_n] \mapsto \sum_{i=0}^{n} p_i a^i$$

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Input $p = [p_0, \ldots p_n]$.
for j **from** n **downto** 1 **do**
  $p_j \leftarrow a p_j$;

Output x_0.

Automatic discovery of Horner's rule

$$\mathbf{M} = [1, a, \ldots, a^n]$$

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Input x_0.
$p_0 \leftarrow x_0$;
for j **from** 1 **to** n **do**
  $p_j \leftarrow p_{j-1}$;
  $p_j \leftarrow a p_j$;
Output $p = [p_0, \ldots, p_n]$.

Input $p = [p_0, \ldots p_n]$.
for j **from** n **downto** 1 **do**
  $p_j \leftarrow a p_j$;
  $p_{j-1} \leftarrow p_j + p_{j-1}$;

Output x_0.

Automatic discovery of Horner's rule

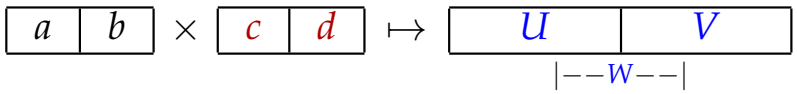
$$\mathbf{M} = [1, a, \ldots, a^n]$$

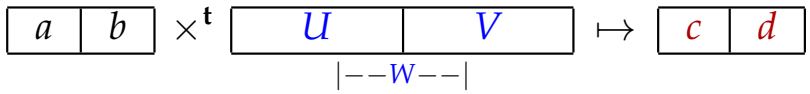
$$x_0 \mapsto [x_0, ax_0, \ldots, a^n x_0] \qquad [p_0, \ldots, p_n] \mapsto \sum_{i=0}^{n} p_i a^i$$

Input x_0.
$p_0 \leftarrow x_0$;
for j **from** 1 **to** n **do**
$p_j \leftarrow p_{j-1}$;
$p_j \leftarrow a p_j$;
Output $p = [p_0, \ldots, p_n]$.

Input $p = [p_0, \ldots p_n]$.
for j **from** n **downto** 1 **do**
$p_j \leftarrow a p_j$;
$p_{j-1} \leftarrow p_j + p_{j-1}$;
$x_0 \leftarrow p_0$;
Output x_0.

Karatsuba's algorithm and its transpose





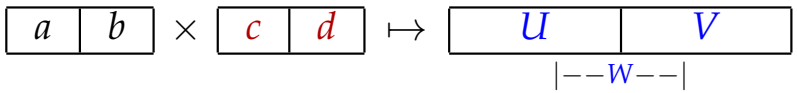
mul

Input (c, d).

$e \leftarrow c + d$;
$U \leftarrow \mathrm{mul}(a, c)$;
$V \leftarrow \mathrm{mul}(b, d)$;
$W \leftarrow \mathrm{mul}(a + b, e)$;
$W \leftarrow W - U - V$;

Output (U, V, W).

Karatsuba's algorithm and its transpose

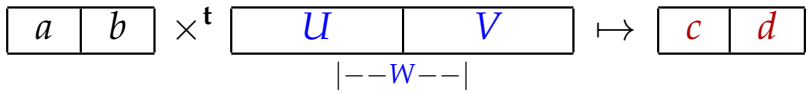


mul

Input (c, d).

$e \leftarrow c + d$;
$U \leftarrow \mathrm{mul}(a, c)$;
$V \leftarrow \mathrm{mul}(b, d)$;
$W \leftarrow \mathrm{mul}(a + b, e)$;
$W \leftarrow W - U - V$;

Output (U, V, W).



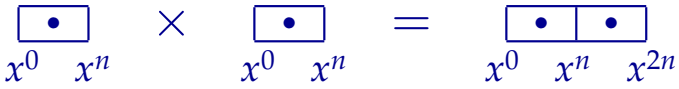
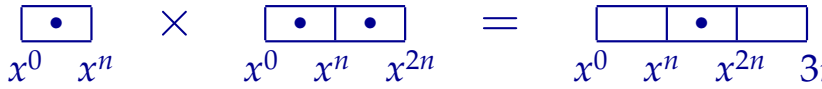
$\mathrm{mul}^{\mathbf{T}}$

Input (U, V, W).

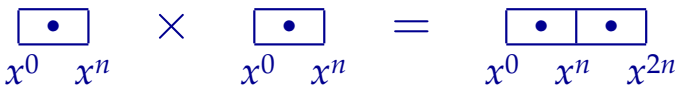
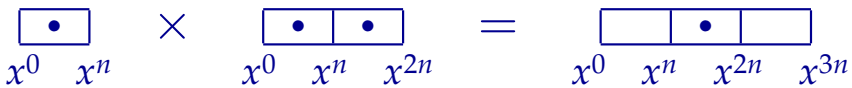
$V \leftarrow V - W$;
$U \leftarrow U - W$;
$e \leftarrow \mathrm{mul}^{\mathbf{T}}(a + b, W)$;
$d \leftarrow \mathrm{mul}^{\mathbf{T}}(b, V)$;
$c \leftarrow \mathrm{mul}^{\mathbf{T}}(a, U)$;
$c \leftarrow c + e$;
$d \leftarrow d + e$;

Output (c, d).

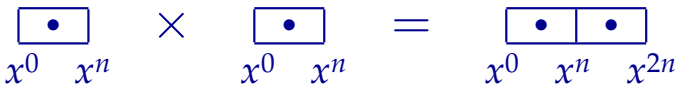
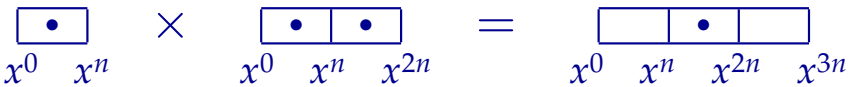
Tellegen's Polynomial Dictionary

direct problem	transposed problem
multiplication	middle product
	

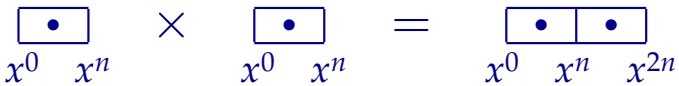
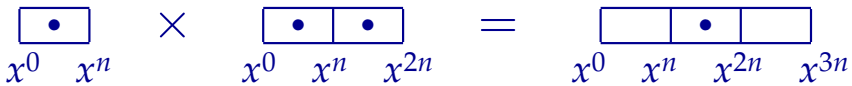
Tellegen's Polynomial Dictionary

direct problem	transposed problem
multiplication	middle product
	
division with remainder	extension of recurrences
$A \mapsto A \bmod P$	$(a_0, \ldots, a_{n-1}) \mapsto (a_0, \ldots, a_{2n-1})$

Tellegen's Polynomial Dictionary

direct problem	transposed problem
multiplication	middle product
	
division with remainder	extension of recurrences
$A \mapsto A \bmod P$	$(a_0, \ldots, a_{n-1}) \mapsto (a_0, \ldots, a_{2n-1})$
multipoint evaluation	generalized power sums
$P \mapsto (P(a_0), \ldots, P(a_{n-1}))$	$(p_0, \ldots, p_{n-1}) \mapsto (\sum p_i, \ldots, \sum p_i a_i^{n-1})$

Tellegen's Polynomial Dictionary

direct problem	transposed problem
multiplication	middle product
	
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$A \mapsto A \bmod P$	$(a_0, \ldots, a_{n-1}) \mapsto (a_0, \ldots, a_{2n-1})$
multipoint evaluation	generalized power sums
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shift of polynomials	evaluation in falling factorial basis
$P(x) \mapsto P(x+1)$	$P = \sum a_i x^{\underline{i}} \mapsto (P(0), \ldots, P(n-1))$

Tellegen's Polynomial Dictionary

direct problem	transposed problem
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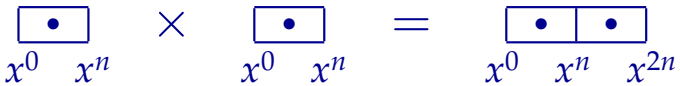	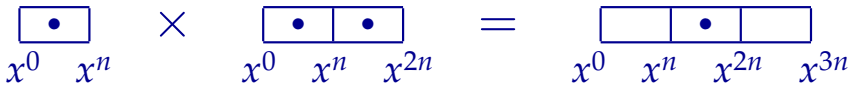
division with remainder	extension of recurrences
$A \mapsto A \bmod P$	$(a_0, \ldots, a_{n-1}) \mapsto (a_0, \ldots, a_{2n-1})$
multipoint evaluation	generalized power sums
$P \mapsto (P(a_0), \ldots, P(a_{n-1}))$	$(p_0, \ldots, p_{n-1}) \mapsto (\sum p_i, \ldots, \sum p_i a_i^{n-1})$
shift of polynomials	evaluation in falling factorial basis
$P(x) \mapsto P(x+1)$	$P = \sum a_i x^{\underline{i}} \mapsto (P(0), \ldots, P(n-1))$
modular composition	power projection
$\ldots$	$\ldots$

POWER SERIES COMPOSITION

Composition of series, first remarks

Pb: Given $N \in \mathbb{N}$, $f \in \mathbb{K}[[x]]$ and $g \in x\mathbb{K}[[x]]$ compute $f(g(x)) \bmod x^N$

▷ Naive approach (by Horner scheme) $O(N\,\mathsf{M}(N))$

▷ [Paterson and Stockmeyer, 1973] $O(\sqrt{N}(\mathsf{M}(N) + \mathsf{MM}(\sqrt{N})))$

By Shanks' 1969 baby-steps giant-steps technique: split polynomials in chunks of length $\sqrt{N}$, matrices in blocks of size $\sqrt{N} \times \sqrt{N}$.

▷ [Brent and Kung, 1978] $O(\sqrt{N \log N}\,\mathsf{M}(N))$

Similar splitting + Taylor formula.

Composition of series, special cases

Pb: Given $N \in \mathbb{N}$, $f \in \mathbb{K}[[x]]$ and $g \in x\mathbb{K}[[x]]$ compute $f(g(x)) \bmod x^N$

- $f = 1/(1-x)$, $f = \exp(x)$, $f = \log(1-x)$ (by Newton iteration) $O(\mathsf{M}(N))$
- $g \in x\mathbb{K}[x]$ [Brent and Kung, 1978] $O(\mathsf{M}(N)\log N)$
- More generally g is algebraic [van der Hoeven, 2002] $O(\mathsf{M}(N)\log N)$
- If $g = \exp(x) - 1$ or $g = \log(1+x)$ [Gerhard, 2000] $O(\mathsf{M}(N)\log N)$
- Many other cases [B., Salvy, Schost, 2008] $O(\mathsf{M}(N)\log N)$
  (via Tellegen's transposition principle)

A more general problem: Modular Composition

Problem: Given $f, g, h \in \mathbb{K}[x]_n$ compute $f(g(x)) \bmod h(x)$

Faster Modular Composition

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BRUNO SALVY, Inria, France
ÉRIC SCHOST, University of Waterloo, Canada
GILLES VILLARD, CNRS, France

A new Las Vegas algorithm is presented for the composition of two polynomials modulo a third one, over an arbitrary field. When the degrees of these polynomials are bounded by n, the algorithm uses $O(n^{1.43})$ field operations, breaking through the 3/2 barrier in the exponent for the first time. The previous fastest algebraic algorithms, due to Brent and Kung in 1978, require $O(n^{1.63})$ field operations in general, and $n^{3/2+o(1)}$ field operations in the particular case of power series over a field of large enough characteristic. If using cubic-time matrix multiplication, the new algorithm runs in $n^{5/3+o(1)}$ operations, while previous ones run in $O(n^2)$ operations.

Our approach relies on the computation of a matrix of algebraic relations that is typically of small size. Randomization is used to reduce arbitrary input to this favorable situation.

CCS Concepts: • **Computing methodologies** → **Algebraic algorithms**; • **Theory of computation** → **Algebraic complexity theory**.

Additional Key Words and Phrases: composition of polynomials, complexity

▷ $O(n^\kappa)$, where $4/3 \leq \kappa < 1.43$ depends on (rectangular) matrix exponent

Modular Composition: a special case

Problem: Given $f, g, h \in \mathbb{F}_q[x]_n$ ($q = p^k$), compute $f(g(x)) \bmod h(x)$

2008 49th Annual IEEE Symposium on Foundations of Computer Science

Fast modular composition in any characteristic

Kiran S. Kedlaya*
Department of Mathematics
MIT

Christopher Umans†
Department of Computer Science
Caltech

Abstract

We give an algorithm for modular composition of degree n univariate polynomials over a finite field $\mathbb{F}_q$ requiring $n^{1+o(1)} \log^{1+o(1)} q$ bit operations; this had earlier been achieved in characteristic $n^{o(1)}$ by Umans (2008). As an application, we obtain a randomized algorithm for factoring degree n polynomials over $\mathbb{F}_q$ requiring $(n^{1.5+o(1)} + n^{1+o(1)} \log q) \log^{1+o(1)} q$ bit operations, improving upon the methods of von zur Gathen & Shoup (1992) and Kaltofen & Shoup (1998). Our results also imply algorithms for irreducibility testing and computing minimal polynomials whose running times are best-possible, up to lower order terms.

backbone of numerous algorithms for computing with polynomials over finite fields, most notably the asymptotically fastest methods for polynomial factorization.

In contrast to other basic modular operations on polynomials (e.g modular multiplication), it is *not* possible to obtain an asymptotically fast algorithm for modular composition with fast algorithms for each step in the natural two step procedure (i.e., first compute $f(g(x))$, then reduce modulo $h(x)$). This is because $f(g(x))$ has n^2 terms, while we hope for a modular composition algorithm that uses only about $O(n)$ operations. Not surprisingly, it is by considering the overall operation (and beating n^2) that asymptotic gains are made in algorithms that employ modular composition.

Perhaps because nontrivial algorithms for modular com-

▷ $(n \cdot \log(q))^{1+o(1)}$ bit operations, but $(n \cdot p)^{1+o(1)}$ operations in $\mathbb{F}_q$

Quasi-optimal power series composition

Pb: Given $N \in \mathbb{N}$, $f \in \mathbb{K}[[x]]$ and $g \in x\mathbb{K}[[x]]$ compute $f(g(x)) \bmod x^N$

Power Series Composition in Near-Linear Time

Yasunori Kinoshita * Baitian Li †

Abstract

We present an algebraic algorithm that computes the composition of two power series in $\tilde{O}(n)$ time complexity. The previous best algorithms are $O(n^{1+o(1)})$ by Kedlaya and Umans (FOCS 2008) and an $O(n^{1.43})$ algebraic algorithm by Neiger, Salvy, Schost and Villard (JACM 2023).

Our algorithm builds upon the recent Graeffe iteration approach to manipulate rational power series introduced by Bostan and Mori (SOSA 2021).

▷ $O(\mathsf{M}(N)\log N)$ operations in $\mathbb{K}$, without any restrictions on f, g or $\mathbb{K}$
▷ The algorithm is amazingly "simple" and beautiful
▷ It relies on (a bivariate extension of) [B., Mori, 2021] and on Tellegen's transposition principle

Quasi-optimal power series composition: proof strategy

Pb: Given $N \in \mathbb{N}$, $f \in \mathbb{K}[[x]]$ and $g \in x\mathbb{K}[[x]]$ compute $f(g(x)) \bmod x^N$

$O(\mathsf{M}(N) \log N)$ [Kinoshita, Li, 2024]

• Idea 1: Composition = (Power Projection)$^{\mathsf{T}}$; therefore, by Tellegen's principle, it is enough to solve PowerProjection in $O(\mathsf{M}(N) \log N)$

• Idea 2: Solving PowerProjection is equivalent to solving

$$[x^{N-1}] \frac{P(x)}{1 - yg(x)}$$

• Idea 3: Last problem can be solved using [B., Mori, 2021] and fast bivariate multiplication (e.g. via Kronecker substitution, or evaluation/interpolation)

Quasi-optimal power series composition: proof strategy

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$O(\mathsf{M}(N)\log N)$ [Kinoshita, Li, 2024]

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• Idea 3: Last problem can be solved using [B., Mori, 2021] and fast bivariate multiplication (e.g. via Kronecker substitution, or evaluation/interpolation)

▷ Open question: can this algorithm be generalized to modular composition?

Composition = (Power Projection)$^{\mathsf{T}}$ and Power Projection = N-th Coeff

Write $f(x) = u_0 + \cdots + u_{N-1}x^{N-1}$ and $\mathbf{u} = [u_0 \cdots u_{N-1}]^T$

Composition = (Power Projection)$^{\mathsf{T}}$ and Power Projection = N-th Coeff

Write $f(x) = u_0 + \cdots + u_{N-1}x^{N-1}$ and $\mathbf{u} = [u_0 \cdots u_{N-1}]^T$
Consider the $N \times N$ matrix A_g containing the coefficients of the powers of g:

$$A_g := \begin{bmatrix} [x^0]g(x)^0 & \cdots & [x^0]g(x)^{N-1} \\ \vdots & \ddots & \vdots \\ [x^{N-1}]g(x)^0 & \cdots & [x^{N-1}]g(x)^{N-1} \end{bmatrix}$$

Composition = (Power Projection)^T and Power Projection = N-th Coeff

Write $f(x) = u_0 + \cdots + u_{N-1}x^{N-1}$ and $\mathbf{u} = [u_0 \cdots u_{N-1}]^T$
Consider the $N \times N$ matrix A_g containing the coefficients of the powers of g:

$$A_g := \begin{bmatrix} [x^0]g(x)^0 & \cdots & [x^0]g(x)^{N-1} \\ \vdots & \ddots & \vdots \\ [x^{N-1}]g(x)^0 & \cdots & [x^{N-1}]g(x)^{N-1} \end{bmatrix}$$

▷ Let $\mathbf{v} = [v_0 \cdots v_{N-1}]^T$ be such that $\mathbf{v} = A_g \cdot \mathbf{u}$. Then

$$v_j = \sum_{i=0}^{N-1} u_i \cdot [x^j]g(x)^i = [x^j](f \circ g)$$

Composition = (Power Projection)^T and Power Projection = N-th Coeff

Write $f(x) = u_0 + \cdots + u_{N-1}x^{N-1}$ and $\mathbf{u} = [u_0 \cdots u_{N-1}]^T$
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$$A_g := \begin{bmatrix} [x^0]g(x)^0 & \cdots & [x^0]g(x)^{N-1} \\ \vdots & \ddots & \vdots \\ [x^{N-1}]g(x)^0 & \cdots & [x^{N-1}]g(x)^{N-1} \end{bmatrix}$$

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$$v_j = \sum_{i=0}^{N-1} u_i \cdot [x^j]g(x)^i = [x^j](f \circ g)$$

▷ Let's look at the transposed map $\mathbf{v} \mapsto \mathbf{u} := A_g^T \cdot \mathbf{v}$

$$u_i = \sum_{j=0}^{N-1} v_j \cdot [x^j]g(x)^i = [x^{N-1}]\, P(x) \cdot g(x)^i,$$

where $P(x) := \sum_{j=0}^{N-1} v_j x^{N-1-j}$.

Composition = (Power Projection)^T and Power Projection = N-th Coeff

Write $f(x) = u_0 + \cdots + u_{N-1}x^{N-1}$ and $\mathbf{u} = [u_0 \cdots u_{N-1}]^T$
Consider the $N \times N$ matrix A_g containing the coefficients of the powers of g:

$$A_g := \begin{bmatrix} [x^0]g(x)^0 & \cdots & [x^0]g(x)^{N-1} \\ \vdots & \ddots & \vdots \\ [x^{N-1}]g(x)^0 & \cdots & [x^{N-1}]g(x)^{N-1} \end{bmatrix}$$

▷ Let $\mathbf{v} = [v_0 \cdots v_{N-1}]^T$ be such that $\mathbf{v} = A_g \cdot \mathbf{u}$. Then

$$v_j = \sum_{i=0}^{N-1} u_i \cdot [x^j]g(x)^i = [x^j](f \circ g)$$

▷ Let's look at the transposed map $\mathbf{v} \mapsto \mathbf{u} := A_g^T \cdot \mathbf{v}$

$$u_i = \sum_{j=0}^{N-1} v_j \cdot [x^j]g(x)^i = [x^{N-1}]\, P(x) \cdot g(x)^i,$$

where $P(x) := \sum_{j=0}^{N-1} v_j x^{N-1-j}$. Therefore,

$$\sum_{i=0}^{N-1} u_i y^i = [x^{N-1}]\, P(x) \cdot \sum_{i=0}^{N-1} g(x)^i y^i = [x^{N-1}]\, \frac{P(x)}{1 - yg(x)} \bmod y^N$$

Problem: Given $n = 2^s < N$, $P \in \mathbb{K}[x]_{<N}$ and $g \in \mathbb{K}[x]_{<N}$, compute

$$[x^n] \frac{P(x)}{1 - yg(x)} \bmod y^N$$

▷ [B., Mori, 2021] compute

$$[x^n] \frac{P_0(x,y)}{Q_0(x,y)} \longleftarrow [x^{n/2}] \frac{P_1(x,y)}{Q_1(x,y)} \longleftarrow \cdots \longleftarrow [x^1] \frac{P_s(x,y)}{Q_s(x,y)},$$

where $P_0 := P(x)$, $Q_0 := 1 - yg(x)$ and

$$Q_i(x^2,y) = Q_{i-1}(x,y) \cdot Q_{i-1}(x,y), \quad P_i(x^2,y) + xR_i(x^2,y) = P_{i-1}(x,y) \cdot Q_{i-1}(-x,y)$$

▷ (P_0, Q_0) have bidegree $(n,1)$, then (P_1, Q_1) have bidegree $(n/2,2)$, etc

▷ Total cost: $C \cdot (\mathsf{M}(n,1) + \mathsf{M}(n/2,2) + \cdots + \mathsf{M}(1,n)) = O(\mathsf{M}(n) \log n)$, using e.g. Kronecker's substitution for bivariate polynomial products

Problem: Given $g \in \mathbb{K}[x]_{<N}$ and $\ell : \mathbb{K}[x]/(x^N) \to \mathbb{K}$, compute $\left(\ell(g^i)\right)_{i=0}^{N-1}$

Algorithm 1 (PowProj)
Input: n, m, $P(x,y), Q(x,y) \in \mathbb{A}[x,y]$
Output: $[x^{n-1}](P(x,y)/Q(x,y)) \bmod y^m$

Require: $[x^0y^0]Q(x,y) = 1$
1: **while** $n > 1$ **do**
2: $\quad U(x,y) \leftarrow P(x,y)Q(-x,y) \bmod x^n \bmod y^m$
3: $\quad$ **if** $n-1$ is even **then**
4: $\quad\quad P(x,y) \leftarrow \sum_{i=0}^{\lceil n/2 \rceil - 1} x^i [s^{2i}] U(s,y)$
5: $\quad$ **else**
6: $\quad\quad P(x,y) \leftarrow \sum_{i=0}^{\lceil n/2 \rceil - 1} x^i [s^{2i+1}] U(s,y)$
7: $\quad$ **end if**
8: $\quad A(x,y) \leftarrow Q(x,y)Q(-x,y) \bmod x^n \bmod y^m$
9: $\quad Q(x,y) \leftarrow \sum_{i=0}^{\lceil n/2 \rceil - 1} x^i [s^{2i}] A(s,y)$
10: $\quad n \leftarrow \lceil n/2 \rceil$
11: **end while**
12: **return** $(P(0,y)/Q(0,y)) \bmod y^m$

▷ $O(\mathsf{M}(N)\log N)$ operations in $\mathbb{K}$, without any restrictions on ℓ, g or $\mathbb{K}$

Problem: Given $f \in \mathbb{K}[x]_{<N}$ and $g \in \mathbb{K}[x]_{<N}$ compute $f(g(x)) \bmod x^N$

▷ Writing $\mathcal{F}_{a,b}(\sum_{i\geq 0} c_i(x)y^i) := \sum_{i=a}^{b-1} c_i(x)y^i$, we have

$$f(g(x)) \bmod x^N = \left[x^{N-1}\right] \frac{f(1/y)\cdot y^{N-1}}{1-y\cdot g(x)} = \mathcal{F}_{N-1,N}\left(\frac{f(1/y)\cdot y^{N-1}}{1-y\cdot g(x)}\right)$$

Algorithm 2 (Comp)
Input: n, d, m, $P(y) \in \mathbb{A}[y]$, $Q(x,y) \in \mathbb{A}[x,y]$
Output: $\mathcal{F}_{d,m}(P(y)/Q(x,y)) \bmod x^n$

Require: $[x^0y^0]Q(x,y) = 1$
1: **if** $n = 1$ **then**
2: $\quad C(y) \leftarrow (P(y)/Q(0,y)) \bmod y^m$
3: $\quad$ **return** $\sum_{i=d}^{m-1} y^{i-d}[t^i]C(t)$
4: **else**
5: $\quad A(x,y) \leftarrow Q(x,y)Q(-x,y) \bmod x^n \bmod y^m$
6: $\quad V(x,y) \leftarrow \sum_{i=0}^{\lceil n/2\rceil-1} x^i[s^{2i}]A(s,y)$
7: $\quad e \leftarrow \max\{0, d - \deg_y Q(x,y)\}$
8: $\quad W(x,y) \leftarrow \mathsf{Comp}(\lceil n/2\rceil, e, m, P(y), V(x,y))$
9: $\quad B(x,y) \leftarrow W(x^2,y)Q(-x,y)$
10: $\quad$ **return** $\sum_{i=d-e}^{m-e-1} y^{i-(d-e)}[t^i]B(x,t) \bmod x^n$
11: **end if**

▷ $O(\mathsf{M}(N)\log N)$ operations in $\mathbb{K}$, without any restrictions on f, g or $\mathbb{K}$

Schönhage's Golden Rules

RULE 8: *The development of fast algorithms is slow !*