Exercises on the chapter “Dense Linear Algebra”

To prepare for 2023-10-12

In what follows, \( \mathbb{K} \) denotes an arbitrary field.

**Exercise 1.** Let \( T(n) \) be the complexity of multiplication of \( n \times n \) lower triangular matrices with entries in \( \mathbb{K} \). Show that one can multiply arbitrary \( n \times n \) matrices in \( \mathcal{M}_n(\mathbb{K}) \) using \( O(T(n)) \) arithmetic operations in \( \mathbb{K} \).

**Exercise 2.** Let \( \theta \) be a feasible exponent for matrix multiplication in \( \mathcal{M}_n(\mathbb{K}) \), and \( P \in \mathbb{K}[x] \) with \( \deg(P) < n \).

(a) Find an algorithm for the simultaneous evaluation of \( P \) at \( \lceil \sqrt{n} \rceil \) elements of \( \mathbb{K} \) using \( O(n^{\theta/2}) \) operations in \( \mathbb{K} \).

(b) If \( Q \) is another polynomial in \( \mathbb{K}[X] \) of degree less than \( n \), show how to compute the first \( n \) coefficients of \( P \circ Q := P(Q(x)) \) using \( O(n^{\frac{\theta+1}{2}}) \) operations in \( \mathbb{K} \).

Hint: Write \( P(x) \) as \( \sum_i P_i(x)(x^d)^i \), where \( d \) is well-chosen and the \( P_i \)'s have degrees less than \( d \).