Exercises on the chapter "Dense Linear Algebra"

To prepare for 2025-10-01

In what follows, $\mathbb{K}$ denotes an arbitrary field.

Exercise 1. Let $\mathsf{T}(n)$ be the complexity of multiplication of $n \times n$ lower triangular matrices with entries in $\mathbb{K}$. Show that one can multiply arbitrary $n \times n$ matrices in $\mathcal{M}_n(\mathbb{K})$ using $O(\mathsf{T}(n))$ arithmetic operations in $\mathbb{K}$.

Exercise 2. Let θ be a feasible exponent for matrix multiplication in $\mathcal{M}_n(\mathbb{K})$, and $P \in \mathbb{K}[x]$ with $\deg(P) < n$.

(a) Find an algorithm for the simultaneous evaluation of P at $\lceil \sqrt{n} \, \rceil$ elements of $\mathbb{K}$ using $O(n^{\theta/2})$ operations in $\mathbb{K}$.

(b) If Q is another polynomial in $\mathbb{K}[X]$ of degree less than n, show how to compute the first n coefficients of $P \circ Q := P(Q(x))$ using $O(n^{\frac{\theta+1}{2}})$ operations in $\mathbb{K}$.

Hint: Write $P(x)$ as $\sum_i P_i(x)(x^d)^i$, where d is well-chosen and the P_i's have degrees less than d.