Exercise on the chapter
"Resultants and Newton iteration"

To prepare for 4 November 2024

Exercise 1. The aim of this exercise is to prove algorithmically the identity

$$\sqrt[3]{\sqrt[3]{2}-1} = \sqrt[3]{\frac{1}{9}} - \sqrt[3]{\frac{2}{9}} + \sqrt[3]{\frac{4}{9}}\,. \tag{E}$$

Let $a = \sqrt[3]{2}$ and $b = \sqrt[3]{\frac{1}{9}}$.

(a) Determine $P_c \in \mathbb{Q}[x]$ annihilating $c = 1 - a + a^2$, by using a resultant.

(b) Deduce $P_R \in \mathbb{Q}[x]$ annihilating the RHS of (E), by another resultant.

(c) Show that the polynomial computed in (b) also annihilates the LHS of (E).

(d) Conclude.

Exercise 2. Let $\mathbb{K}$ be an effective field of characteristic zero, and assume that $F \in \mathbb{K}[[x]]$ with $F(0) = 1$.

(a) What is the complexity of computing $\sqrt{F}$, by using $\sqrt{F} = \exp(\frac{1}{2}\log F)$?

(b) Describe a Newton iteration that directly computes $\sqrt{F}$, without appealing to successive logarithm and exponential computations.

(c) Estimate the complexity of the algorithm in (b).