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The Ising model: from elliptic curves to modular forms and Calabi–Yau equations

A Bostan[1], S Boukraa[2,7], S Hassani[3,7], M van Hoeij[4,7], J-M Maillard[5], J-A Weil[6] and N Zenine[3]

[1] INRIA Paris-Rocquencourt, Domaine de Voluceau, B.P. 105 78153 Le Chesnay Cedex, France
[2] LPTHIRM and Département d'Aéronautique, Université de Blida, Blida, Algeria
[3] Centre de Recherche Nucléaire d'Alger, 2 Bd. Frantz Fanon, BP 399, 16000 Alger, Algeria
[4] Florida State University, Department of Mathematics, 1017 Academic Way, Tallahassee, FL 32306-4510, USA
[5] LPTMC, UMR 7600 CNRS, Université de Paris, Tour 23, 5ème étage, Case 121, 4 Place Jussieu, 75252 Paris Cedex 05, France
[6] XLIM, Université de Limoges, 123 Avenue Albert Thomas, 87060 Limoges Cedex, France

E-mail: alin.bostan@inria.fr, boukraa@mail.univ-blida.dz, hoeij@mail.math.fsu.edu, maillard@lptmc.jussieu.fr, jacques-arthur.weil@unilim.fr and njzenine@yahoo.com

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Abstract
We show that almost all the linear differential operators factors obtained in the analysis of the n-particle contributions $\tilde{\chi}^{(n)}$'s of the susceptibility of the Ising model for $n \leqslant 6$ are linear differential operators *associated with elliptic curves*. Beyond the simplest differential operators factors which are homomorphic to symmetric powers of the second order operator associated with the complete elliptic integral E, the second and third order differential operators Z_2, F_2, F_3, $\tilde{L}_3$ can actually be interpreted as *modular forms* of the elliptic curve of the Ising model. A last order-4 globally nilpotent linear differential operator is not reducible to this elliptic curve, modular form scheme. This operator is shown to actually correspond to a natural generalization of this elliptic curve, modular form scheme, with the emergence of a Calabi–Yau equation, corresponding to a selected ${}_4F_3$ hypergeometric function. This hypergeometric function can also be seen as a Hadamard product of the complete elliptic integral K, with a remarkably simple algebraic pull-back (square root extension), the corresponding Calabi–Yau fourth order differential operator having a symplectic differential Galois group $SP(4, \mathbb{C})$. The mirror maps and higher order Schwarzian ODEs, associated with this Calabi–Yau ODE, present all the nice physical and mathematical ingredients we had with elliptic curves and modular forms, in particular an exact (isogenies) representation of the

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generators of the renormalization group, extending the modular group $SL(2, \mathbb{Z})$ to a $GL(2, \mathbb{Z})$ symmetry group.

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1. Introduction

In a previous paper [1] some massive computer calculations have been performed on the susceptibility of the square Ising model and on the n-particle (n-fold integrals) contributions $\tilde{\chi}^{(n)}$ of the susceptibility [2–4]. In three more recent papers [5–7] the linear differential operators for $\tilde{\chi}^{(5)}$ and $\tilde{\chi}^{(6)}$ were carefully analyzed. In particular, it was found that the minimal order linear differential operator for $\tilde{\chi}^{(5)}$ can be reduced to a minimal order linear differential operator L_{29} of order 29 for the linear combination

$$\Phi^{(5)} = \tilde{\chi}^{(5)} - \tfrac{1}{2}\tilde{\chi}^{(3)} + \tfrac{1}{120}\tilde{\chi}^{(1)}. \qquad (1)$$

This specific linear combination series being annihilated by an ODE of lower order, one has, thus, the occurrence of a *direct sum* structure. It was found [5, 6] that this linear differential operator L_{29} can be factorized as a product of an order-5, an order-12, an order-1, and an order-11 linear differential operators

$$L_{29} = L_5 \cdot L_{12} \cdot \tilde{L}_1 \cdot L_{11}, \qquad (2)$$

where the order-1 linear differential operator $\tilde{L}_1$ has a rational solution and where the order-11 linear differential operator has a direct-sum decomposition

$$L_{11} = (Z_2 \cdot N_1) \oplus V_2 \oplus \left(F_3 \cdot F_2 \cdot L_1^s\right), \qquad (3)$$

where L_1^s and N_1 are order-1 (globally nilpotent[8]) operators, Z_2 is the second order operator occurring in the factorization of the linear differential operator [8] associated with $\tilde{\chi}^{(3)}$ and seen to correspond[9] to a *modular form of weight one* [9], V_2 is a second order operator equivalent[10] to the second order operator associated with $\tilde{\chi}^{(2)}$ (or equivalently to the complete elliptic integral E), and F_2 and F_3 are remarkable second order and third order *globally nilpotent* linear differential operators [5, 9].

The order-5 linear differential operator L_5 was shown to be equivalent to the *symmetric fourth power* of (the second order operator[11]) L_E corresponding to the complete elliptic integral E. These operators were actually obtained in *exact arithmetics* [5, 6]. The order-12 linear differential operator L_{12} has been shown to be *irreducible* and has been proved to not be a symmetric product of differential operators of smaller orders (see [6] for details).

Similar calculations were actually achieved [7] on $\tilde{\chi}^{(6)}$. For the linear combination

$$\Phi^{(6)} = \tilde{\chi}^{(6)} - \tfrac{2}{3}\tilde{\chi}^{(4)} + \tfrac{2}{45}\tilde{\chi}^{(2)}, \qquad (4)$$

[8] That is of the form (7) for order-1 operators (see below). L_1^s has the simple rational solution $w^2/(1-4w)^2$.
[9] As well as the second order operator occurring in Apéry's proof of the irrationality of $\zeta(3)$.
[10] In the sense of the equivalence of linear differential operators [10, 11] (corresponding to the 'Homomorphisms' command in Maple). We refer to this (classical) notion of equivalence of linear differential operators everywhere in the paper. In the literature the wording 'operators of the same type' is also used [12].
[11] This second order operator does play a central role in the Gauss–Manin, or Picard–Fuchs, origin of the (sigma form of the) Painlevé equations occurring for the two-point correlation functions of the Ising model [13, 14].

one obtains a linear differential operator of (minimal) order 46 which has the following factorization:

$$L_{46} = L_6 \cdot L_{23} \cdot L_{17}, \tag{5}$$

with

$$L_{17} = \tilde{L}_5 \oplus L_3 \oplus (L_4 \cdot \tilde{L}_3 \cdot L_2), \qquad \tilde{L}_5 = \left(D_x - \frac{1}{x}\right) \oplus (L_{1,3} \cdot (L_{1,2} \oplus L_{1,1} \oplus D_x)), \tag{6}$$

where D_x denotes[12] $\mathrm{d}/\mathrm{d}x$, and where the $L_{1,n}$'s ($n = 1, 2, 3$) are order-1 linear differential operators (see appendix A in [7]) and L_2, L_3 and L_6 are [7] respectively equivalent (homomorphic) to L_E, to the symmetric square of L_E and to the *symmetric fifth power* of L_E.

The factorization (if any) of the order-23 linear differential operator L_{23} is beyond our current computational resources (see [7] for details).

• *Understanding the elementary factors: the modular form challenges*

Among these various globally nilpotent factors [9], of large order operators, one discovers order-1 linear differential operators which, because they are globally nilpotent, are all of the form

$$D_x - \frac{1}{N} \cdot \frac{\mathrm{d}\ln(R(x))}{\mathrm{d}x}, \tag{7}$$

thus having Nth root of rational functions solutions. One also discovers operators of various orders which are equivalent to symmetric powers of L_E, the second order operator corresponding to the complete elliptic integral of the first (or second kind), E or K, like V_2 in (3), or L_5 in (2), or L_3 in (6), or L_6 in (5), a remarkable second order operator Z_2 having a *modular form interpretation* [9], and a miscellaneous set of operators F_2, F_3, $\tilde{L}_3$, L_4, L_{12} and L_{23}.

These last linear differential operators are *not* equivalent to symmetric powers of L_E and are still waiting for a *modular form* interpretation, if we think that all the globally nilpotent factors of these operators of the 'Ising class' [15] *should have an interpretation in terms of elliptic curves in a modern sense*[13]. These linear differential operators, emerging in the analysis of $\tilde{\chi}^{(5)}$, namely the second order operator F_2 and the third order operator F_3, are waiting for such a modular form interpretation, as well as $\tilde{L}_3$ and L_4 emerging in the analysis of $\tilde{\chi}^{(6)}$. The two remaining operators L_{12} and L_{23} are too involved, and of too large order, for seeking a possible modular form interpretation (up to equivalence).

The purpose of this paper is to provide a mathematical interpretation of the F_2, F_3, $\tilde{L}_3$, L_4 elementary 'bricks' of the n-fold integrals $\tilde{\chi}^{(5)}$ and $\tilde{\chi}^{(6)}$, in order to *mathematically understand the Ising model*. In this paper we will, as much as possible, use the same notations as in our previous papers [1, 5, 7–9]. The linear differential operators, or equations, are in the $x = w$ variable for the high temperature differential operators and in the $x = w^2$ variable for the low temperature differential operators, where $w = (1 + s^2)/s/2$ and where $s = \sinh(2K)$ with the standard notations for the Ising model.

[12] In this paper we will use the notations D_t (resp. D_z) for $\mathrm{d}/\mathrm{d}t$ (resp. $\mathrm{d}/\mathrm{d}z$).

[13] Before Wiles' recent results, only elliptic curves with the property known as '*complex multiplication*' had been shown to be parametrized by modular functions (by Shimura in 1961).

2. Modular form recalls

2.1. Modular form recalls: Z_2 and Apéry modular linear differential operator

Let us introduce as in [9] the order-2 Heunian operator which has the following solution $Heun(8/9, 2/3, 1, 1, 1, 1; t)$:

$$\mathcal{H} = D_t^2 + \left(\frac{1}{t} + \frac{1}{t-1} + \frac{9}{9t-8}\right) \cdot D_t + 3\frac{3t-2}{(9t-8)(t-1)t}.$$

A simple change of variable (see equation (43) in [9]),

$$t = \frac{-8x}{(1-4x)(1-x)}, \tag{8}$$

transforms $\mathcal{H}$ into the order-2 linear differential operator[14]:

$$\mathcal{H}_x = D_x^2 + \frac{1+10x-19x^2-92x^3+12x^4+224x^5-64x^6}{(1+3x+4x^2)(1-2x)(1+2x)(1-4x)(1-x)\cdot x} \cdot D_x$$
$$+6\frac{(1+7x+4x^2)(1-2x)^2}{(1+3x+4x^2)(1-4x)^2(1-x)^2\cdot x}. \tag{9}$$

We found [9] that the second order linear differential operator Z_2, occurring as a factor of the linear differential operator annihilating $\tilde{\chi}^{(3)}$, is homomorphic [10, 11] to the operator (9). Recall [16] that the *fundamental weight-1 modular form*[15] h_N for the modular group $\Gamma_0(N)$ for $N = 6$, can be expressed as a simple *Heun* function, $Heun(9/8, 3/4, 1, 1, 1, 1, -t/8)$, or as a hypergeometric function:

$$\frac{2\sqrt{3}}{((t+6)^3(t^3+18t^2+84t+24)^3)^{1/12}}$$
$$\times\, {}_2F_1\left(\left[\frac{1}{12}, \frac{5}{12}\right]; [1]; 1728\frac{(t+9)^2(t+8)^3t}{(t+6)^3(t^3+18t^2+84t+24)^3}\right), \tag{10}$$

which is a solution of the order-2 linear differential operator:

$$D_t^2 + \left(\frac{1}{t+8} + \frac{1}{t} + \frac{1}{t+9}\right) \cdot D_t + \frac{t+6}{(t+8)(t+9)\cdot t}, \tag{11}$$

simply related to $\mathcal{H}$. Therefore, after some change of variables, one can see the (selected) solution of Z_2 as a hypergeometric function (*up to a Hauptmodul pull-back*) corresponding to a *weight-1 modular form*[16] (namely h_6 in [19]).

[14] Do note two minus sign misprints in the numerator of the D_x coefficient equation (44) of [9]: $-10x + 19x^2$ must be replaced by $+10x - 19x^2$, see (9).
[15] The modular form h_6 is also combinatorially significant: the perimeter generating function of the three-dimensional staircase polygons [17] can be expressed [9] in terms [16] of h_6. The modular form h_6 also occurs [9] in Apéry's study of $\zeta(3)$.
[16] The simplest weight-1 modular form is ${}_2F_1([1/12, 5/12], [1], \hat{J}) = 12^{1/4}\eta(\tau)^2\hat{J}^{-1/12}$ where $\hat{J}$ is the Hauptmodul, η is the Dedekind eta function and τ is the ratio of periods (see (4.6) in [18]).

Summing-up $\mathcal{H}_x$, given by (9), has the following solution:

$$\mathcal{S} = (\Omega \cdot \mathcal{M}_x)^{1/12} \times {}_2F_1\left(\left[\frac{1}{12}, \frac{5}{12}\right]; [1]; \mathcal{M}_x\right), \qquad \text{where}$$

$$\Omega = \frac{1}{1728} \frac{(1-4x)^6(1-x)^6}{x \cdot (1+3x+4x^2)^2(1+2x)^6},$$

$$\mathcal{M}_x = 1728 \frac{x \cdot (1+3x+4x^2)^2(1+2x)^6(1-4x)^6(1-x)^6}{(1+7x+4x^2)^3 \cdot P^3}, \tag{12}$$

$$P = 1 + 237x + 1455x^2 + 4183x^3 + 5820x^4 + 3792x^5 + 64x^6.$$

The solution of the operator Z_2, in terms of hypergeometric functions, can now be understood from this hypergeometric function (*up to a modular invariant pull-back*) structure.

2.2. *Recall on modular forms: Hauptmoduls and fundamental modular curves*

In several papers [20, 21] we underlined, for Yang–Baxter integrable models with a canonical *genus-one* parametrization [22, 23] (elliptic functions of modulus k), that the *exact* generators of the *renormalization group* must necessarily identify with various isogenies [24] which amounts to multiplying, or dividing, τ the ratio of the two periods of the elliptic curves, by an integer [25]. The simplest example is the *Landen transformation* [21] which corresponds to multiplying (*or dividing* because of the modular group symmetry $\tau \longleftrightarrow 1/\tau$, i.e. the exchange of the two periods of the elliptic curve), the ratio of the two periods:

$$k \quad \longleftrightarrow \quad k_L = \frac{2\sqrt{k}}{1+k}, \qquad \tau \longleftrightarrow 2\tau. \tag{13}$$

The other transformations[17] correspond to $\tau \leftrightarrow N \cdot \tau$ for various integers N. However, in the natural variables of the model (as $e^K, \tanh(K), k = s^2 = \sinh^2(2K)$, but not the 'transcendental variables' like τ or the nome q), these transformations are *algebraic* transformations corresponding, in fact, to the *fundamental modular curves*. For instance, the Landen transformation (13) corresponds to the *genus zero fundamental modular curve*

$$j^2 \cdot j'^2 - (j + j') \cdot (j^2 + 1487 \cdot jj' + j'^2) + 3 \cdot 15^3 \cdot (16j^2 - 4027jj' + 16j'^2)$$
$$- 12 \cdot 30^6 \cdot (j + j') + 8 \cdot 30^9 = 0, \tag{14}$$

which relates the two j-functions

$$j(k) = 256 \cdot \frac{(1-k^2+k^4)^3}{(1-k^2)^2 \cdot k^4}, \qquad j(k_L) = 16 \cdot \frac{(1+14k^2+k^4)^3}{(1-k^2)^4 \cdot k^2}$$

or to the fundamental modular curve

$$5^9v^3u^3 - 12 \cdot 5^6u^2v^2 \cdot (u+v) + 375uv \cdot (16u^2 + 16v^2 - 4027vu)$$
$$- 64(v+u) \cdot (v^2 + 1487vu + u^2) + 2^{12} \cdot 3^3 \cdot uv = 0, \tag{15}$$

which relates the two Hauptmoduls $u = 12^3/j(k)$, $v = 12^3/j(k_L)$.

The fact that such transformations correspond to either multiplying, or dividing, τ is associated with the *reversibility*[18] of these exact representations of the *renormalization*

[17] See for instance (2.18) in [26].

[18] The renormalization group (RG) is introduced, in the textbooks, as a semi-group, the transformations of the RG being *non-invertible*. We do not try here to address the embarrassing question of providing a mathematically well-defined definition of the RG. It is known that there are many problems with the RG, for instance for the (two-parameter) non-zero-temperature Ising model *with a magnetic field* [27–29]. We just remark here, for a Yang–Baxter integrable model like the Ising model without the magnetic field, that the RG generators, seen as transformations in the parameter space, identify with the isogenies of elliptic curves and are thus reversible in the τ-space.

group [25]. The 'price to pay' is that these algebraic transformations are not one-to-one transformations; they are sometimes called 'correspondences' by some authors.

A simple rational parametrization[19] of the genus zero modular curve (15) reads

$$u = u(z) = \frac{1728z}{(z+16)^3}, \qquad v = \frac{1728z^2}{(z+256)^3} = u\left(\frac{2^{12}}{z}\right). \qquad (16)$$

Note that the previously mentioned reversibility is also associated with the fact that the modular curve (15) is invariant by the permutation $u \leftrightarrow v$, which, within the previous rational parametrization (16), corresponds[20] to the *Atkin–Lehner involution* [30] $z \leftrightarrow 2^{12}/z$.

It has also been underlined in [20, 21] that seeing (13) as a transformation on *complex variables* (instead of real variables) provides, beyond $k = 0, 1$ (the infinite temperature fixed point and the critical temperature fixed point), two other *complex* fixed points which actually correspond to *complex multiplication* for the elliptic curve, and are, actually, fundamental new singularities[21] discovered on the $\chi^{(3)}$ linear ODE [8, 33, 34]. In general, within the theory of elliptic curves, this underlines the interpretation of (the generators of) the renormalization group as *isogenies of elliptic curves* [25]. *Hauptmoduls*[22], *modular curves and modular forms* play here a fundamental role.

Along this modular form line, let us consider the second order linear differential operator

$$\alpha = D_z^2 + \frac{\left(z^2+56z+1024\right)}{z\cdot(z+16)(z+64)}\cdot D_z - \frac{240}{z\cdot(z+16)^2(z+64)},$$

which has the (modular form) solution:

$$ {}_2F_1\left(\left[\frac{1}{12}, \frac{5}{12}\right], [1]; 1728\frac{z}{(z+16)^3}\right) = 2\cdot\left(\frac{z+256}{z+16}\right)^{-1/4}\cdot {}_2F_1\left(\left[\frac{1}{12}, \frac{5}{12}\right], [1]; 1728\frac{z^2}{(z+256)^3}\right). \qquad (17)$$

In fact the *two pull-backs* in the arguments of the *same* hypergeometric function are *actually related by the fundamental modular curve* (15) (see (16)). Do note that, generically, the existence of *several pull-backs* for a hypergeometric function is a *quite rare situation*. The covariance (17) is thus the very expression of a modular form property with respect to transformation $\tau \leftrightarrow 2\tau$, corresponding to the modular curve (15).

This example (see (17)) is a simple illustration of the special role played by selected hypergeometric functions *having several possible pull-backs* (in fact an infinite number). This phenomenon is *linked to elliptic curves in a deep and fundamental sense*, namely the occurrence of *modular forms* and of isogenies represented by algebraic transformations, called by some authors 'correspondence', between the pull-backs. These algebraic transformations correspond to *modular curves* (and to *exact algebraic representations of the generators of the renormalization group* [25]). We will say in short that such hypergeometric functions

[19] Corresponding to Atkin–Lehner polynomials and Weber's functions.

[20] Conversely, and more precisely, writing $1728z^2/(z+256)^3 = 1728z'/(z'+16)^3$ gives the Fricke–Atkin–Lehner [30, 31] involution $z\cdot z' = 2^{12}$, together with the quadratic relation $z - zz' - 48zz' - 4096z' = 0$.

[21] Suggesting an understanding [21, 32] of the quite rich structure [21] of the infinite number of singularities of the $\chi^{(n)}$'s in the complex plane, from a Hauptmodul approach [21, 32]. Furthermore, the notion of *Heegner numbers* is closely linked to the *isogenies* mentioned here [21]. An exact value of the j-function $j(\tau)$ corresponding to one of the first Heegner number is, for instance, $j = 12^3$.

[22] It should be recalled that the *mirror symmetry* found with *Calabi–Yau manifolds* [35–40] *can be seen as higher order generalizations of Hauptmoduls*. We can, thus, expect generalizations of this identification of the renormalization and modular structure when one is not restricted to elliptic curves anymore.

are 'associated with elliptic curves'. Simpler examples of *isogenies* associated with rational transformations, instead of 'correspondence' like (15), are displayed in [25].

3. Modular form solution of F_2 and the corresponding fundamental modular curve $X_0(2)$

The exact expressions of the selected linear differential operators Z_2, F_2 and F_3 which emerged [5] in $\tilde{\chi}^{(5)}$, can be found in [41], and the exact expressions of the selected linear differential operators $\tilde{L}_3$, and L_4 which emerged [7] in $\tilde{\chi}^{(6)}$, can be found in [42].

In the following we will not detail how we have been able to get the solutions of the linear differential operators F_2, and F_3, and the solutions of $\tilde{L}_3$, and L_4. These details will be displayed in forthcoming publications. These (slightly involved) solutions[23] are displayed in [43]. We focus, here, on the structures, and mathematical meaning, associated with these solutions.

Actually, the series expansion of the solutions of (the globally nilpotent) operator F_2 gives more than G-series [44, 45]: it yields series with *integer* coefficients, suggesting that F_2 could also have, like the previous Z_2, a modular form interpretation.

The solutions of the second order linear differential operator [5] F_2 can, actually, be written in terms of hypergeometric functions (the $\rho_i(x)$'s are two rational functions)

$$\rho_1(x)^{1/4} \cdot {}_2F_1\big(\big[\tfrac{1}{4}, \tfrac{1}{4}\big], \big[\tfrac{1}{2}\big]; p(x)\big) + \rho_2(x)^{1/4} \cdot {}_2F_1\big(\big[\tfrac{5}{4}, \tfrac{5}{4}\big], \big[\tfrac{3}{2}\big]; p(x)\big),$$

with the pull-back:

$$p(x) = -\frac{1}{64}\frac{(1-4x)(1+6x+13x^2+4x^3)^2}{(1+2x)^3 \cdot x^3}. \qquad (18)$$

A well-known symmetry of many hypergeometric functions amounts to changing the pull-back $p(x)$ into[24] $q(x) = 1 - p(x)$:

$$q(x) = 1 - p(x) = \frac{1}{64}\frac{(1+4x)^2(1-x)^3(1+3x+4x^2)}{(1+2x)^3 \cdot x^3}, \qquad (19)$$

where the selected[25] singularities $1 + 3x + 4x^2 = 0$, seen [8] in $\tilde{\chi}^{(3)}$, and more specifically [9] in Z_2, clearly occur.

Alternatively, the solutions of F_2 can also be written as (the R_i's are rational functions)

$$R_1(x)^{1/12} \cdot {}_2F_1\big(\big[\tfrac{1}{12}, \tfrac{1}{12}\big], \big[\tfrac{2}{3}\big]; p_i(x)\big) + R_2(x)^{1/12} \cdot {}_2F_1\big(\big[\tfrac{13}{12}, \tfrac{13}{12}\big], \big[\tfrac{5}{3}\big]; p_i(x)\big),$$

with *two different possible pull-backs* $p_1(x)$ and $p_2(x)$ reading, respectively,

$$\frac{(1+8x+14x^2-36x^3-151x^4-188x^5-16x^6-64x^7)^3}{1728 \cdot (1+2x)^6 \cdot (1+4x)^2 \cdot (1-x)^3 \cdot (1+3x+4x^2) \cdot x^6},$$

$$-\frac{(1+8x+14x^2-276x^3-1591x^4-3068x^5-1936x^6-64x^7)^3}{1728 \cdot (1+3x+4x^2)^2 \cdot (1-x)^6 \cdot (1+4x)^4 \cdot (1+2x)^3 \cdot x^3}.$$

Note that these two pull-backs *are not related by a simple Atkin–Lehner involution* [30] ($p_2(x) \neq p_1(N/x)$, with N being some integer). However, introducing the rational expression

$$R(U) = -\frac{1}{27}\frac{(U-4)^3}{U^2}, \qquad (20)$$

[23] The reader can view all these linear differential equations, as well as their explicit ${}_2F_1$-type solutions, in [43].
[24] In addition to $p(x) \to q(x) = 1/p(x)$.
[25] Complex fixed points of the Landen transformation, Heegner numbers with complex multiplication of the elliptic curve [21].

one actually finds that

$$p_1(x) = R\left(\frac{1}{q(x)}\right), \qquad p_2(x) = R(q(x)). \tag{21}$$

The relation between these two pull-backs corresponds to the (genus zero) curve

$$110\,592 \cdot \alpha^2\beta^2 - 64 \cdot (\alpha^3 + \beta^3) - 95232 \cdot (\alpha^2\beta + \alpha\beta^2) + 6000 \cdot (\alpha^2 + \beta^2) - 1510125 \cdot \alpha\beta \\ - 187\,500 \cdot (\alpha + \beta) + 1953\,125 = 0, \tag{22}$$

with the simple rational parametrization deduced from (21): $(\alpha(z), \beta(z)) = (R(U), R(1/U))$. One immediately finds that (22) is nothing but (15) where (u, v) have been changed into $(1/u, 1/v)$. Actually, up to a rescaling[26] of U by a factor -64, the parametrization of the rational curve (22) can be rewritten in a form where a $z \leftrightarrow 2^{12}/z$ Atkin–Lehner involution is made explicit:

$$(\alpha(z), \beta(z)) = \left(\frac{1}{1728}\frac{(z+16)^3}{z}, \frac{1}{1728}\frac{(z+256)^3}{z^2}\right), \tag{23}$$

$$\beta(z) = \alpha\left(\frac{2^{12}}{z}\right). \tag{24}$$

One recognizes (up to a 1728 normalization factor) the *fundamental modular curve* $X_0(2)$

$$A^2B^2 - (A+B)\cdot(A^2 + 1487AB + B^2) - 40773\,375 \cdot AB + 162\,000 \cdot (A^2 + B^2) \\ - 8748\,000\,000 \cdot (A+B) + 157\,464\,000\,000\,000 = 0, \tag{25}$$

and its well-known rational parametrization

$$A = A(j_2) = \frac{(256 + j_2)^3}{{j_2}^2}, \qquad B = A\left(\frac{2^{12}}{j_2}\right). \tag{26}$$

3.1. Dedekind eta function parametrization

It is well known that the *Dedekind eta function* [46, 47], in power 24, is a cusp automorphic form of weight 12, related to the *discriminant of the elliptic curve*. Recalling the Weierstrass' modular discriminant [48], defined as

$$\Delta(\tau) = (2\pi)^{12} \cdot \eta(\tau)^{24}, \tag{27}$$

which is this modular form of weight 12, we get rid of this $(2\pi)^{12}$ factor and define

$$\Delta(q) = q \cdot \prod_{n=1}^{\infty} (1 - q^n)^{24}, \tag{28}$$

where q is the nome of the elliptic curve, that is, the exponential of the ratio of the two periods[27] of the elliptic curve: $q = \exp(\tau)$.

One can now introduce a 'second layer' of parametrization writing the j-function as a ratio of the Dedekind eta function

$$j_2 = j_2(q) = \frac{\Delta(q)}{\Delta(q^2)}, \qquad A(j_2) = \frac{(256 + j_2)^3}{{j_2}^2}. \tag{29}$$

[26] $R(U) = j_2(-64U)/1728$ with $j_2(t) = (t + 256)^3/1728/t^2$.

[27] In the literature the ratio of the two periods of the elliptic curve often encapsulates a $2i\pi$ (or $i\pi$) factor and one defines $q = \exp(2i\pi\tau)$. For some Calabi–Yau reason (see (125) below) we prefer to write $q = \exp(\tau)$.

One deduces the alternative parametrization of (25)

$$(A, B) = (A(j_2(q)), A(j_2(q^2)) = \left(A\left(\frac{\Delta(q)}{\Delta(q^2)}\right), A\left(\frac{\Delta(q^2)}{\Delta(q^4)}\right)\right), \tag{30}$$

making crystal clear that the fundamental modular curve (25) is a representation of $\tau \to 2 \cdot \tau$, or $q \to q^2$ (and in the same time[28] $\tau \to \tau/2$).

The Atkin–Lehner involutive transformation $j_2 \to 2^{12}/j_2$ and transformation $q \to q^2$ are *actually compatible* thanks to the remarkable 'Ramanujan-like' functional identity on Dedekind η functions

$$4096 \cdot \Delta(q) \cdot \Delta(q^4)^2 - \Delta(q^2)^3 + (\Delta(q) + 48 \cdot \Delta(q^2)) \cdot \Delta(q) \cdot \Delta(q^4) = 0, \tag{31}$$

yielding

$$A\left(\frac{\Delta(q^2)}{\Delta(q^4)}\right) = A\left(2^{12} \Big/ \left(\frac{\Delta(q)}{\Delta(q^2)}\right)\right), \tag{32}$$

making (26) and (30) compatible.

There are many other nice functional and differential relations on the Dedekind eta functions that are shown below.

(i) The modular functions (see (19) in [49] page 16)

$$t = \left(\frac{\eta(6\tau)\eta(\tau)}{\eta(2\tau)\eta(3\tau)}\right)^{12}, \qquad g = \frac{\eta(6\tau)^8\eta(\tau)^4}{\eta(2\tau)^8\eta(3\tau)^4}, \tag{33}$$

have the following relation:

$$t = g \cdot \frac{1 - 9g}{1 - g}. \tag{34}$$

This is *exactly* the covering necessary to see Z_2 as a modular form (see equation (A.3) in [9]).

(ii) Differential equations are actually satisfied by modular forms [49, 50]. Introducing the same t as in (33) and the following function $F(t)$:

$$t = \left(\frac{\eta(\tau)\eta(6\tau)}{\eta(2\tau)\eta(3\tau)}\right)^{12}, \qquad F = \frac{(\eta(2\tau)\eta(3\tau))^7}{(\eta(\tau)\eta(6\tau))^5}, \tag{35}$$

one has an Apéry's third order ODE [9] on the modular form[29] $F(t)$. This ODE corresponds to the linear differential operator[30]:

$$(t^2 - 34t + 1) \cdot t^2 \cdot D_t^3 + (6t^2 - 153t + 3) \cdot t \cdot D_t^2 + (7t^2 - 112t + 1) \cdot D_t + (t - 5) \tag{36}$$

that reads in terms of the homogeneous derivative $\theta = t \cdot \mathrm{d}/\mathrm{d}t$:

$$(t^2 - 34t + 1) \cdot \theta^3 + (3t^2 - 51t) \cdot \theta^2 + (3t^2 - 27t) \cdot \theta + (t^2 - 5t), \tag{37}$$

this operator[31] being linked to the *modularity of the algebraic variety*

$$x + \frac{1}{x} + y + \frac{1}{y} + z + \frac{1}{z} + w + \frac{1}{w} = 0,$$

[28] Thanks to the $\tau \leftrightarrow 1/\tau$ symmetry of the modular group corresponding to the exchange the two periods of the elliptic curve.

[29] $F(t)$ and t are modular forms on $\Gamma_0(6)$ which has four inequivalent cusps ∞, 0, 1/2, 1/3. $F(t)$ is a weight 2 modular form.

[30] In Apéry's proof of the irrationality of $\zeta(3)$ a crucial role is played by the linear differential operator (36).

[31] Introducing the inhomogeneous order-2 ODE corresponding to (37) with the very simple rhs $6t$, and considering the ratio of solution of (37) and of this inhomogeneous order-2 ODE, one can build [49] a modular form of weight 4, by performing the third order derivative with respect to τ, the ratio of two solutions of (36).

that is, to the one-parameter family of K3-surfaces[32] [53]:

$$1 - (1 - XY) \cdot Z - z \cdot XYZ \cdot (1 - X) \cdot (1 - Y) \cdot (1 - Z) = 0.$$

(iii) Another example of linear differential equations, satisfied by modular forms, can be found in page 18 of [49]:

$$t = \left(\frac{\eta(2\tau)\eta(6\tau)}{\eta(\tau)\eta(3\tau)}\right)^6, \qquad F = \left(\frac{\eta(\tau)^2\eta(3\tau)^2}{\eta(2\tau)\eta(6\tau)}\right)^2, \tag{38}$$

the third order ODE on $F(t)$ corresponding to the linear differential operator

$$(64t^2 + 20t + 1) \cdot \theta^3 + (192t^2 + 30t) \cdot \theta^2 + (192t^2 + 18t) \cdot \theta + (64t^2 + 4t). \tag{39}$$

(iv) A third example [49] is

$$t = \left(\frac{\eta(3\tau)\eta(6\tau)}{\eta(\tau)\eta(2\tau)}\right)^4, \qquad F = \frac{(\eta(\tau)\eta(2\tau))^3}{\eta(3\tau)\eta(6\tau)}, \tag{40}$$

with a third order ODE on the modular form $F(t)$. This ODE corresponds to the linear differential operator

$$(81t^2 + 14t + 1) \cdot \theta^3 + (243t^2 + 21t) \cdot \theta^2 + (243t^2 + 13t) \cdot \theta + (81t^2 + 3t), \tag{41}$$

If one chooses two linearly independent solutions (F_1, F_2) of these last order-3 linear differential operators appropriately [49], then t is a modular form of τ the ratio of these two solutions. The function $t(\tau)$ satisfies a well-known third order *nonlinear* ODE known as the *Schwarzian equation* in the literature[33]:

$$2Q(t) \cdot \left(\frac{dt}{d\tau}\right)^2 + \{t, \tau\} = 0, \tag{42}$$

where $\{z, t\}$ denotes the *Schwarzian derivative* with respect to τ:

$$\frac{d}{d\tau} = q \cdot \frac{d}{dq}, \qquad \{z, \tau\} = \frac{z^{(3)}}{z'} - \frac{3}{2} \cdot \left(\frac{z''}{z'}\right)^2, \tag{43}$$

and where $Q(t)$ is a rational function that can be simply deduced [49] from the coefficients of the kth order (here $k = 3$) linear ODE on $F(t)$.

[32] The simplest example of Calabi–Yau manifolds are K3 surfaces. This Apéry operator (37) was seen in [51] (see also [9]) to correspond to a *symmetric square* of a second order operator associated with a modular form (see also section (2.1)). Along this line it is worth recalling that some one-parameter families of K3 surfaces can be obtained from the *square of families of elliptic curves* (see the so-called *Shioda–Inose structures* and their Picard–Fuchs differential equations [52] and see also relations (1.9) in [40]).

[33] The simplest example of Schwarzian equation, associated with the complete elliptic integral K, corresponds to the rhs in (42) reading $-\frac{1}{2} \cdot \frac{t^2-t+1}{t^2 \cdot (t-1)^2}$. The study of the Schwarzian equation is, in general, complicated, but Halphen found a simpler equivalent system of differential equations [54, 55].

4. Modular form solution of F_3, related to h_6, Apéry and Z_2

The order-3 linear differential operator F_3, occurring as a factor of the differential operator annihilating $\tilde{\chi}^{(5)}$, was rationally reconstructed in [5]. It can be seen to be homomorphic to the *symmetric square* of a second order operator. Similarly to what we had for F_2, the (analytical at $x = 0$) solution of F_3 corresponds to a series with *integer* coefficients, suggesting, again, a modular form interpretation.

Actually the solutions of this second order operator can be expressed in terms of quadratic expressions of Legendre or ${}_2F_1$ hypergeometric functions *with a rational pull-back*. The three solutions of F_3 can be expressed in terms of Legendre functions where the pull-back P_1, in these Legendre functions, reads

$$P_1(x) = \frac{1}{108} \frac{(1-2x)(1+2x)(1+32x^2)^2}{x^2} \tag{44}$$

$$= \frac{1}{108} \frac{(1-4x)^3 \cdot (1+4x)^3}{x^2} + 1. \tag{45}$$

The solutions can also be expressed as quadratic expressions of hypergeometric functions:

$${}_2F_1\left(\left[\tfrac{1}{6}, \tfrac{1}{6}\right], \left[\tfrac{1}{2}\right]; P_1\right), \qquad {}_2F_1\left(\left[\tfrac{2}{3}, \tfrac{2}{3}\right], \left[\tfrac{3}{2}\right]; P_1\right). \tag{46}$$

Do note that the pull-back (44) is not unique. Another (Atkin–Lehner involution related) pull-back works equally well:

$$P_2(x) = P_1\left(\frac{1}{8x}\right) = -\frac{1}{108} \frac{(1-4x)(1+4x)(1+2x^2)^2}{x^4} \tag{47}$$

$$= \frac{-1}{108} \cdot \frac{(1-2x)^3(1+2x)^3}{x^4} + 1. \tag{48}$$

Note that these two rational pull-backs are functions of x^2. These two pull-backs can be seen as a rational parametrization $(a, b) = (P_1(x), P_2(x))$ of the (a, b)-symmetric genus zero curve:

$$-625 + 525 \cdot (a+b) + 3ba + 96 \cdot (a^2 + b^2) - 528 \cdot (ba^2 + b^2 a) + 4 \cdot (a^3 + b^3) + 432 \cdot a^2 b^2 = 0. \tag{49}$$

Keeping in mind[34] (45) and (48), we could have considered the algebraic curve relating $(A, B) = (1 - P_1(x), 1 - P_2(x))$, which reads

$$- 432 A^2 B^2 + 4 \cdot (A^3 + B^3) + 336 \cdot (A^2 B + B^2 A) + 381 \cdot AB - 12 \cdot (A^2 + B^2) + 12 \cdot (A + B) - 4 = 0, \tag{50}$$

which is rationally parametrized as $(A, B) = (A(z), B(z))$, where $A(z)$ and $B(z)$ read, respectively,

$$A(z) = \frac{1}{1728} \frac{(z+16)^3}{z}, \qquad B(z) = \frac{1}{432} \frac{(z+64)^3}{z^2}, \tag{51}$$

where $A(z)$ and $B(z)$ are related by a *Atkin–Lehner involution* $B(z) = A(2^{10}/z)$. This rational parametrization is *extremely similar* to the parametrization (23), (26) of the *fundamental modular curve* (25). One deduces, from (51), the rational parametrization for the curve (49)

$$a = -\frac{(z+64)(z-8)^2}{1728 \cdot z}, \qquad b = -\frac{(z+16)(z-128)^2}{432 \cdot z^2}, \tag{52}$$

[34] And keeping in mind the well-known symmetry of many hypergeometric functions changing the pull-back $p(x)$ into $q(x) = 1 - p(x)$. The other well-known symmetry $P_1(x) \leftrightarrow 1/P_1(x)$ corresponds to $x^2 \longleftrightarrow (1-4x^2)/(1+32x^2)/4$.

where, again, $b(z) = a(2^{10}/z)$. Within parametrization (52), expressions $(P_1(x), P_2(x))$ (see (44), (47)) correspond to $z = -256 \cdot x^2$ or $z = -4/x^2$.

From (51) it is thus tempting to interpret the *new* genus zero algebraic curve (49), or (50), as a *modular curve*[35] relating two *Hauptmoduls* corresponding to the two pull-backs (44) and (47), similarly to what was found (see subsection 2.1) for the second order operator Z_2 and its weight-1 modular form solutions. It was seen to be related [9] to a second order linear differential operator occurring in Apery's analysis of $\zeta(3)$:

$$4x \cdot (x^2 - 34x + 1) \cdot {D_x}^2 + 4 \cdot (1 - 51x + 2x^2) \cdot D_x + x - 10. \qquad (53)$$

From its rational parametrization this *new curve* (50) is extremely similar to (25), the *fundamental modular curve* $X_0(2)$. One has, of course, the well-known (and slightly tautological) algebraic geometry statement that all the genus zero curves of the plane are birationaly equivalent. But refereing to the 'second layer' of parametrization (see (30) above) can we say that this new curve is *also* a representation of $\tau \longleftrightarrow N\tau$ and thus 'truly' a modular curve?

4.1. The new curve (50) and the modular group $\Gamma_0(6)$

Seeking for hypergeometric functions with pull-backs that *are not rational* functions anymore, but algebraic extensions, we actually found another description of the solutions. The second order operator (53) can be solved in terms of hypergeometric functions ${}_2F_1\left(\left[\frac{2}{3}, \frac{2}{3}\right], \left[\frac{3}{2}\right]; P_\pm\right)$ with the two possible algebraic (Galois conjugate) pull backs:

$$P_\pm = -\frac{1}{216}\frac{x^4 - 23x^3 - 156x^2 - 23x + 1}{x^2} \pm \frac{1}{216}\frac{(x-1)(x^2 - 7x + 1)}{x^2}\sqrt{1 - 34x + x^2}. \qquad (54)$$

To get rid of the square root in (54), we introduce a parametrization of the rational curve $y^2 = 1 - 34x + x^2$, namely

$$x = \frac{1}{2}\frac{(u+18)(u+16)}{u}, \qquad y = \frac{1}{2}\frac{288 - u^2}{u}. \qquad (55)$$

In terms of this rational parametrization (55) the two possible pull-backs (54) read, respectively,

$$P_+(u) = -\frac{1}{432}\frac{(u+24)^2\left(u^2 + 12u - 72\right)^2}{u \cdot (u+16)(u+18)^2},$$
$$P_-(u) = -\frac{1}{216}\frac{(u+12)^2(u^2 - 48u - 1152)^2}{u^2(u+16)^2(u+18)} = P_+\left(\frac{288}{u}\right), \qquad (56)$$

or, more simply on $(Q_+(u), Q_-(u)) = (1 - P_+(u), 1 - P_-(u))$

$$Q_+(u) = \frac{1}{432}\frac{(u+12)^6}{u \cdot (u+16)(u+18)^2},$$
$$Q_-(u) = Q_+\left(\frac{288}{u}\right) = \frac{1}{216}\frac{(u+24)^6}{u^2 \cdot (u+16)^2(u+18)}. \qquad (57)$$

[35] Modular curves of genus 0, which are quite rare, turned out to be of major importance in relation with the monstrous moonshine conjectures [56, 57]. In general, a modular function field is a function field of a modular curve (or, occasionally, of some other moduli space that turns out to be an irreducible variety). Genus 0 means that such a function field has a single transcendental function as the generator: for example the j-function. The traditional name for such a generator, which is unique up to a Möbius transformation and can be appropriately normalized, is a Hauptmodul (main or principal modular function).

Noticeably, these two pull-backs (56) *can actually be seen as another rational parametrization* $(a, b) = (P_+(u), P_-(u))$ *of the genus zero curve* (49). The two pull-backs (57) are another rational parametrization of the 'new' algebraic curve (50), with $(A, B) = (Q_+(u), Q_-(u))$.

Recalling the fact that the two pull-backs $P_1(x)$ and $P_2(x)$ (see (44), (47)) were functions of x^2, the correspondence between $P_1(x)$ and $P_2(x)$, and the two pull-backs (56), corresponds to the following change of variables:

$$x^2 = -\frac{1}{128}\frac{(u+16)\cdot u}{u+18}. \tag{58}$$

Results (56) could have been obtained, alternatively, recalling the change of variable (34), namely $x = v \cdot (1-9v)/(1-v)$, which transforms the linear differential operator (53) into (after rescaling by $(1-v)^{1/2}$):

$$D_v^2 + \frac{1-20v+27v^2}{(1-9v)(1-v)v}\cdot D_v - 3\cdot\frac{1-3v}{(1-9v)(1-v)v}, \tag{59}$$

which corresponds to the (staircase-polygon [17]) second order operator $\mathcal{Z}_3$ seen in appendix A of [9] (see equation after (A.4)). The variable u is related to the previous variable v by $u = 18\cdot(v-1)$.

However, recalling the weight-1 modular form interpretation of[36] operator Z_2 in [9], we had the occurrence of Hauptmoduls $\mathcal{M}_z = \frac{12^3}{j}$ (resp. $\frac{12^3}{j'}$) corresponding to the (genus zero [16]) modular curve[37]

$$\Phi_6(j, j') = \Phi_6(j', j) = 0, \tag{60}$$

obtained from the elimination of z between[38]

$$\begin{aligned} j = j_6(z) &= \frac{(z+6)^3(z^3+18z^2+84z+24)^3}{z\cdot(z+9)^2(z+8)^3}, \\ &= j_2\left(\frac{z\cdot(z+8)^3}{z+9}\right) = \frac{(z+16)^3}{z}\circ\frac{z\cdot(z+8)^3}{z+9}, \\ &= j_3\left(\frac{z\cdot(z+9)^2}{z+8}\right) = \frac{(z+27)(z+3)^3}{z}\circ\frac{z\cdot(z+9)^2}{z+8} \end{aligned} \tag{61}$$

and

$$\begin{aligned} j' = j_6\left(\frac{2^3\cdot 3^2}{z}\right) &= \frac{(15\,552+3888z+252z^2+z^3)^3(z+12)^3}{z^6(z+8)^2(z+9)^3} \\ &= j_2'\left(\frac{z^3(z+8)}{(z+9)^3}\right) = \frac{(z+256)^3}{z^2}\circ\frac{z^3\cdot(z+8)}{(z+9)^3} \\ &= j_3'\left(\frac{z^2\cdot(z+9)}{(z+8)^2}\right) = \frac{(z+27)(z+243)^3}{z^3}\circ\frac{z^2\cdot(z+9)}{(z+8)^2} \end{aligned} \tag{62}$$

with the covering [9]

$$z = \frac{72x}{(1-x)(1-4x)}, \tag{63}$$

[36] Or of the order-2 operator $\mathcal{Z}_3$ corresponding to staircase polygons [17].
[37] Which amounts to multiplying, or dividing, the ratio of the two periods of the elliptic curve by 6.
[38] Here '$\circ$' denotes the composition of two rational functions i.e. $f(z)\circ g(z) = f(g(z))$.

which is a slight modification of (8). Let us introduce the alternative covering $u = 2z$; the two Hauptmoduls $\mathcal{M}_z = \frac{12^3}{j}$ (resp. $\frac{12^3}{j'}$) read

$$P_1^{(6)}(u) = \frac{110\,592 \cdot u \cdot (u+16)^3\,(u+18)^2}{(12+u)^3 \cdot (192+336u+36u^2+u^3)^3} = \left(12^3 \Big/ \left(\frac{(u+32)^3}{4u}\right)\right) \circ \left(\frac{u \cdot (u+16)^3}{4 \cdot (u+18)}\right), \tag{64}$$

$$P_2^{(6)}(u) = P_1^{(6)}\left(\frac{288}{u}\right) = \frac{3456 \cdot u^6 \cdot (u+16)^2 (u+18)^3}{(u+24)^3 (124\,416+15\,552u+504u^2+u^3)^3} = \left(12^3 \Big/ \left(\frac{(u+512)^3}{2u^2}\right)\right) \circ \frac{u^3\,(u+16)}{(u+18)^3}. \tag{65}$$

The relation between the Apéry operator (53) and the Z_2 weight-1 modular forms associated with (60) seems to say that there should be some (at first sight totally unexpected) relation between hypergeometric functions with the pull-backs (56) and the hypergeometric functions with the pull-backs (64). *We have actually been able to find such a 'quite non-trivial' relation*

$$C_6(u) \cdot {}_2F_1\left(\left[\frac{1}{12}, \frac{5}{12}\right], [1]\,;\, P_1^{(6)}(u)\right) = 2^{1/2} \cdot \rho \cdot C_+(u) \cdot {}_2F_1\left(\left[\frac{2}{3}, \frac{2}{3}\right], \left[\frac{3}{2}\right];\, P_+(u)\right) - \rho \cdot C_-(u) \cdot {}_2F_1\left(\left[\frac{2}{3}, \frac{2}{3}\right], \left[\frac{3}{2}\right];\, P_-(u)\right), \tag{66}$$

where $C_+(u)$, $C_-(u)$, $C_6(u)$ and ρ read, respectively,

$$\left(\frac{(u+24)^2}{u}\right)^{1/2} \cdot \left(\frac{16u^2}{(u+16)(u+18)^2}\right)^{2/3} \cdot \left(\frac{u^2+12u-72}{64u}\right),$$

$$\left(\frac{2 \cdot (u+12)^2}{u}\right)^{1/2} \cdot \left(\frac{4u}{(u+18)(u+16)^2}\right)^{2/3} \cdot \left(\frac{u^2-48u-1152}{16u}\right),$$

$$\left(\frac{144u^2}{18^2 \cdot (12+u)(192+336u+36u^2+u^3)}\right)^{1/4}, \qquad \frac{2}{3}\,\frac{(\Gamma\,(2/3))^3}{\pi^2}.$$

Note, however, that $P_1^{(6)}(u)$ (or $P_2^{(6)}(u)$) *cannot be expressed* as a rational function of the two pull-backs $P_\pm(u)$ (see (56)). The relation between $a_6 = P_1^{(6)}(u)$ and $a = P_+(u)$ (resp. $b_6 = P_2^{(6)}(u)$ and $b = P_-(u)$) reads a (necessarily genus zero) algebraic curve

$$(a-1) \cdot (9a-25)^3 \cdot a_6^2 + 8(a-1) \cdot (1458a^2 - 1215a + 125) \cdot a_6 + 16 = 0, \tag{67}$$

and the same genus zero algebraic curve where one replaces $(a_6, a) \rightarrow (b_6, b)$. Note that the relation between $P_1^{(6)}(u)$ and $P_-(u)$ (resp. $P_2^{(6)}(u)$ and $P_+(u)$) is much more involved.

Recalling (60) one deduces, from the previous calculations, that the (genus zero) modular curve

$$\Phi_6\left(\frac{12^3}{a_6}, \frac{12^3}{b_6}\right) = 0 \tag{68}$$

is *actually 'equivalent' to the genus zero algebraic curve* (49) *up to the algebraic covering* (67).

4.2. Dedekind parametrization of the new curve

Let us revisit the rational curve (50) that we rewrite (with a rescaling of A and B by 1728) as

$$-y^2z^2 + 16(y+z)(z^2+83yz+y^2) - 82\,944\cdot(z^2+y^2) \\ +2633\,472\cdot yz + 143\,327\,232\cdot(y+z) - 82\,556\,485\,632 = 0. \tag{69}$$

The curve (69) is rationally parametrized by

$$z = z(j_2) = \frac{(256+j_2)^3}{{j_2}^2}, \qquad y = z\left(\frac{2^{14}}{j_2}\right) = \frac{(64+j_2)^3}{16\cdot j_2}.$$

The correspondence with the previous rational parametrization is $j_2 = 4\cdot z$. Similar to what was done for the fundamental modular curve (25), we can introduce a second 'layer' of parametrization, writing j_2 as a ratio of the Dedekind eta function (28)

$$j_2 = j_2(q) = 4\frac{\Delta(q)}{\Delta(q^2)}, \qquad A(j_2) = \frac{(256+j_2)^3}{{j_2}^2},$$

which yields the following parametrization for (69) ($A(j_2)$ is the same function as in (29)):

$$z = A(j_2(q)) = A\left(4\frac{\Delta(q)}{\Delta(q^2)}\right), \qquad y = A\left(\frac{j_2(q^2)}{4}\right) = A\left(\frac{\Delta(q^2)}{\Delta(q^4)}\right),$$

which makes clear that the algebraic curve (69) is a representation of $q \to q^2$. The compatibility between the Atkin–Lehner involution $j_2 \leftrightarrow 2^{14}/j_2$ and the $q \to q^2$ transformation corresponds to

$$A\left(\frac{\Delta(q^2)}{\Delta(q^4)}\right) = A\left(2^{14}\Big/\left(4\frac{\Delta(q)}{\Delta(q^2)}\right)\right) \tag{70}$$

which is nothing but (32) corresponding to the functional equation (31).

Matching $y/1728$ or $z/1728$ (see (44), (47)) with $1-P_1$ or $1-P_2$, one gets

$$j_2 = -1024x^2 \qquad \text{or} \qquad j_2 = -\frac{16}{x^2}, \tag{71}$$

yielding a *straight interpretation of the* x variable in the n-fold integrals of the Ising model *in terms of the Dedekind eta function*, and more precisely, of the discriminant Δ:

$$x^2 = -\frac{1}{256}\cdot\frac{\Delta(q)}{\Delta(q^2)}, \qquad \text{or} \qquad x^2 = -4\cdot\frac{\Delta(q^2)}{\Delta(q)}. \tag{72}$$

5. Modular form solution of $\tilde{L}_3$: is it associated with the new curve (50) or with the fundamental modular curve $X_0(2)$?

In [7] an order-3 linear differential operator $\tilde{L}_3$ was found as a factor of the minimal order operator for $\tilde{\chi}^{(6)}$. This order-3 operator is (as it should) globally nilpotent, and one can see that it is reducible to an order-2 operator in the sense that it is homomorphic to the *symmetric square* of an order-2 linear differential operator:

$$x\cdot(1-16x)^2\cdot(1-4x)^2\cdot D_x^2 + (1-24x)(1-16x)(1-4x)^2\cdot D_x + 108x^2, \tag{73}$$

yielding solutions in terms of hypergeometric functions like ${}_2F_1([1/8, 3/8], [1/2]; \mathcal{P}_1(x))$, with the pull-back $\mathcal{P}_1(x)$ reading

$$\mathcal{P}_1(x) = \frac{(1-12x)^2}{(1-16x)(1-4x)^2}, \tag{74}$$

or more simply

$$1 - \mathcal{P}_1(x) = -\frac{256 \cdot x^3}{(1-16x)(1-4x)^2}. \tag{75}$$

5.1. An 'uneducated' guess

A simple calculation shows that some 'Atkin–Lehner' transform of (74)

$$\mathcal{P}_2(x) = \mathcal{P}_1\left(\frac{1}{64x}\right) = -\frac{4 \cdot (3-16x)^2 \cdot x}{(1-4x)(1-16x)^2}, \tag{76}$$

or more simply

$$1 - \mathcal{P}_2(x) = \frac{1}{(1-4x)(1-16x)^2}, \tag{77}$$

provides *another rational parametrization* $(a, b) = (\mathcal{P}_1(x), \mathcal{P}_2(x))$ *of the new genus zero curve* (49).

Let us introduce the x^2-dependent transformation

$$x \quad \longrightarrow \quad R(x) = \frac{1}{16} \cdot \frac{1-16x^2}{1-4x^2}; \tag{78}$$

one actually finds a nice relation between the two pull-backs $\mathcal{P}_1(x)$ and this 'guessed candidate' $\mathcal{P}_2(x)$ for the other pull-back of $\tilde{L}_3$ (if any):

$$\mathcal{P}_1(R(x)) = P_1(x), \qquad \mathcal{P}_2(R(x)) = P_2(x). \tag{79}$$

These two equalities are actually compatible with the two Atkin–Lehner involutions for $(\mathcal{P}_1, \mathcal{P}_2)$ and (P_1, P_2), because of the nice functional relation on $R(x)$:

$$R\left(\frac{1}{8x}\right) = \frac{1}{64 \cdot R(x)}. \tag{80}$$

It is, thus, extremely tempting to imagine that $\tilde{L}_3$ is, like F_3, related to the *new modular curve* (49) or (50). Furthermore, this would yield some (deep) relation between the singularities of the $\chi^{(2n)}$ and singularities of the $\chi^{(2n+1)}$, that is to say, between the low and high temperature singularities of the susceptibility of the Ising model. *This is not the case*: $\tilde{L}_3$ is, in fact, related to the fundamental modular curve $X_0(2)$.

5.2. Modular form solution of $\tilde{L}_3$: the fundamental modular curve $X_0(2)$

Actually, leaving, again, the framework of *rational* pull-backs, one gets the two (Galois-conjugate) *algebraic pull-backs* $\mathcal{P}_\pm[\tilde{L}_3]$:

$$3456 \cdot \mathcal{P}_\pm[\tilde{L}_3] = \frac{(40x^2-17x+1)(400x^4-928x^3+297x^2-31x+1)}{x^6}$$
$$\pm \frac{(1-12x)(1-4x)(1-7x)(25x^2-17x+1)}{x^6} \cdot \sqrt{1-16x}. \tag{81}$$

The relation between these two Galois-conjugate pull-backs actually corresponds to (22) which is *nothing* (up to a 1728 rescaling factor, see (25)) but the *fundamental modular curve* $X_0(2)$.

To get rid of the square root singularity, we introduce the variable y:

$$y^2 = 1 - 16x, \qquad \text{i.e.} \quad x = -\frac{1}{16}(y^2 - 1). \tag{82}$$

The two previous algebraic pull-backs become respectively

$$\frac{1}{27}\frac{\left(5y^3 - 9y^2 + 15y - 3\right)^3}{(y+1)^6(y-1)^3}, \qquad \frac{1}{27}\frac{\left(5y^3 + 9y^2 + 15y + 3\right)^3}{(y+1)^3\,(y-1)^6}, \tag{83}$$

which can be seen to be a rational parametrization of (22). Recalling the previous parametrization (23) one finds that the z variable in (23), must be equal to $z = 64 \cdot (y+1)^3/(y-1)^3$ or $z = 64 \cdot (y-1)^3/(y+1)^3$.

6. The puzzling L_4: preliminary results on L_4

Let us now focus on the order-4 linear differential operator L_4, discovered as a factor of $\chi^{(6)}$, and that we were fortunate enough to get exactly by rational reconstruction[39] [7]. Let us display a few results on L_4.

6.1. Negative results on L_4

Suppose that L_4 is equivalent (in the sense of equivalence of linear differential operators [10, 11]) to a symmetric cube of a second order linear differential operator L_2. Take a point $x = a$, and suppose that the highest exponent of $\ln(x-a)$ that appears in the formal solutions of L_2, at $x = a$, equals ρ. Then the highest exponent of $\ln(x-a)$ that appears in the formal solutions of L_4, at $x = a$, must be 3ρ. Now look at the formal solutions of L_4; at $x = 1/8$, one gets a contradiction. Similar reasoning (using both $x = 1/8$ and $x = 0$) shows that L_4 *can also not be related to the symmetric product of two second order operators*.

Our preliminary calculations show that L_4 is not ${}_4F_3$-solvable *if one restricts to rational pull-backs*, in the sense that there is no ${}_4F_3$ differential operator that can be sent to L_4 (with a change of variables $x \to$ rational function in x, followed by multiplying by $\exp(\int$(rational function in x)), followed by homomorphisms). In essence, if we allow the following functions in $\mathbb{C}(x)$: exp, log and any ${}_pF_q$ function, composed with any rational functions, and anything one can form from those functions using addition, multiplication, derivatives, indefinite integral, then we believe that L_4 is *not solvable in that class of functions*.

6.2. Positive results on L_4

The order-4 linear differential operator L_4 exhibits, however, a set of nice properties. Let us display some of these nice properties.

(i) An irreducible linear differential equation (resp. irreducible linear differential operator) is said to be of maximal unipotent monodromy (MUM) if all the indicial exponents at $x = 0$ are zero (one Jordan block to be maximal). The formal solutions of an order-4

[39] For details on the rational reconstruction see [58].

MUM linear differential operator can be written as

$$\begin{aligned} y_0 &= y_0, \\ y_1 &= y_0 \cdot \ln(x) + \tilde{y}_1, \\ y_2 &= \tfrac{1}{2} y_0 \cdot \ln(x)^2 + \tilde{y}_1 \cdot \ln(x) + \tilde{y}_2, \\ y_3 &= \tfrac{1}{6} y_0 \cdot \ln(x)^3 + \tfrac{1}{2} \tilde{y}_1 \cdot \ln(x)^2 + \tilde{y}_2 \ln(x) + \tilde{y}_3. \end{aligned} \tag{84}$$

The indicial exponents of L_4 at $x = 0$ read $-6, -4, -4$ and 0. Therefore, the order-4 operator L_4 is *not* MUM; however, the formal solutions *can be cast exactly as for a MUM linear differential operator*. The formal solutions of L_4 can be written as

$$\begin{aligned} y_0 &= y_0, \\ y_1 &= y_0 \cdot \ln(x) + \tilde{y}_{10}, \\ y_2 &= \tfrac{1}{2} y_0 \cdot \ln(x)^2 + \tilde{y}_{21} \cdot \ln(x) + \tilde{y}_{20}, \\ y_3 &= \tfrac{1}{6} y_0 \cdot \ln(x)^3 + \tilde{y}_{32} \cdot \ln(x)^2 + \tilde{y}_{31} \cdot \ln(x) + \tilde{y}_{30}. \end{aligned} \tag{85}$$

Such particular form for the set of series solutions is often obtained for irreducible operators.

There are four independent series that can be chosen as y_0, $\tilde{y}_{10}$, $\tilde{y}_{20}$ and $\tilde{y}_{30}$. The other series in front of the logs should depend on these four chosen series. For L_4, these series read

$$\begin{aligned} \tilde{y}_{21} &= \tilde{y}_{10} - \frac{2854\,486\,264\,697}{459\,375 \cdot 10^6} y_0, \\ \tilde{y}_{32} &= \frac{1}{2} \tilde{y}_{10} + \frac{38\,100\,003\,421\,933}{11\,484\,375 \cdot 10^4} y_0, \\ \tilde{y}_{31} &= \tilde{y}_{20} + \frac{85\,327}{128} \tilde{y}_{10} - \frac{103\,266\,884\,399\,422\,867}{504 \cdot 10^{11}} y_0, \end{aligned}$$

where we see that it is a matter of combination to cast the formal solutions (85) in the form (84). Making the combination $y_2 - c_{21} y_1$ and $y_3 - 2c_{32} y_2 - (c_{312} - 2c_{21} c_{32}) y_1$, (where c_{21}, resp. c_{32} and c_{312}, are the coefficients in front of y_0 in $\tilde{y}_{21}$, resp. $\tilde{y}_{32}$ and $\tilde{y}_{31}$, in (N)), one obtains the new set

$$\begin{aligned} y_0 &= y_0, \\ y_1 &= y_0 \cdot \ln(x) + \tilde{y}_{10}, \\ y_2 &= \tfrac{1}{2} y_0 \cdot \ln(x)^2 + \tilde{y}_{10} \cdot \ln(x) + \tilde{y}_{20} - c_{21} \tilde{y}_{10}, \\ y_3 &= \tfrac{1}{6} y_0 \cdot \ln(x)^3 + \tfrac{1}{2} \tilde{y}_{10} \cdot \ln(x)^2 + (\tilde{y}_{20} + c_{311} \tilde{y}_{10} - 2c_{32} c_{10} \tilde{y}_{10}) \cdot \ln(x) \\ &\qquad + \tilde{y}_{30} - 2c_{32} \cdot \tilde{y}_{20} + (2c_{32} c_{21} - c_{312}) \cdot \tilde{y}_{10}, \end{aligned} \tag{86}$$

where $(\tilde{y}_{20} - c_{21} \tilde{y}_{10})$ identifies with $(\tilde{y}_{20} + c_{311} \tilde{y}_{10} - 2c_{32} c_{10} \tilde{y}_{10})$ since $c_{21} = -c_{311} + 2c_{32} c_{10}$ for the actual values of the combination coefficients.

The remaining difference between a MUM linear differential operator and the formal solutions of L_4 is the beginning of the series of the nonleading logs. For instance the local exponent -4 being twice, one should have a series starting as x^{-4} in front of the logs like $\tilde{y}_{10}$

$$\tilde{y}_{10} = \frac{1}{84\,000 x^4} + \frac{11}{16\,800 x^3} + \frac{9329}{336\,000 x^2} + \frac{8023}{8400 x} + \frac{99\,922\,803\,261\,913}{3675\,000\,000\,000} + \cdots.$$

We may imagine that by acting by an intertwiner on the formal solutions, one ends up with the series starting at x, i.e. L_4 may be equivalent to a linear differential operator which is MUM.

(ii) The order-4 linear differential operator L_4 is of course *globally nilpotent* [9]. The p-curvature [59] of the order-4 globally nilpotent differential operator L_4 can be put in a remarkably simple Jordan form:

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \qquad (87)$$

Its characteristic and minimal polynomial is T^4. Such an operator *cannot be a symmetric cube of a second order* operator in $\mathbb{C}(x)[D_x]$. We, however, determined the differential Galois group [60, 61] of this linear differential operator L_4 and *actually found that it is*[40] *the symplectic group* $SP(4, \mathbb{C})$. A crucial step to exhibit this symplectic structure, amounts to calculating the *exterior square* of L_4, and verify that this, at first sight order-6, exterior square either reduces to an order-5 operator, or is a direct sum of an order-5 operator and an order-1 operator with a rational solution. L_4 corresponds to this last scenario.

Denoting y_i the four solutions of an order-4 linear differential operator, the exterior square of that operator is a linear differential operator that annihilates the expressions

$$w_{i,j} = y_i \frac{\mathrm{d}y_j}{\mathrm{d}x} - y_j \frac{\mathrm{d}y_i}{\mathrm{d}x}, \qquad i \neq j = 0, 1, 2, 3. \qquad (88)$$

It should, then, be (at first sight) of order 6.

The six solutions (88) of the exterior square of a MUM order-4 operator, contain logs with degrees (at most) 1, 2, 3, 3, 4 and 5. There are then two solutions (with the same degree in the logs) that can be equal:

$$y_0 \frac{\mathrm{d}}{\mathrm{d}x} y_3 - y_3 \frac{\mathrm{d}}{\mathrm{d}x} y_0 = y_1 \frac{\mathrm{d}}{\mathrm{d}x} y_2 - y_2 \frac{\mathrm{d}}{\mathrm{d}x} y_1. \qquad (89)$$

When this occurs the exterior square, annihilating five independent solutions, will be of order 5. This is equivalent (see proposition 2 in [62]), for the coefficients of the order-4 linear differential operator, to verify condition (122) below.

Here, computing the exterior square of the linear differential operator L_4, one finds an order-6 linear differential operator with the *direct sum decomposition*

$$\mathrm{ext}^{(2)}(L_4) = \tilde{N}_1 \oplus N_5, \qquad (90)$$

with

$$sol(\tilde{N}_1) = \frac{N(x)}{D(x)},$$

$$\begin{aligned} N(x) = {} & -12 + 2548x - 502\,593x^2 + 43\,407\,720x^3 - 1959\,091\,320x^4 \\ & + 52\,738\,591\,890x^5 - 904\,049\,598\,675x^6 + 10\,126\,459\,925\,120x^7 \\ & - 74\,115\,473\,257\,440x^8 + 350\,453\,101\,085\,400x^9 - 113\,358\,908\,9074\,624x^{10} \\ & + 4059\,589\,860\,750\,336x^{11} - 25\,595\,376\,023\,494\,656x^{12} \qquad (91) \\ & + 141\,123\,001\,405\,931\,520x^{13} - 440\,315\,308\,230\,574\,080x^{14} \end{aligned}$$

[40] $SP(4, \mathbb{C})$*is not the monodromy group: it is equal to the Zariski closure of the (countable) monodromy group, i.e. the differential Galois group.*

$$+\ 705\,909\,942\,330\,064\,896x^{15} - 496\,507\,256\,028\,790\,784x^{16}$$
$$+\ 140\,082\,179\,425\,173\,504x^{17},$$
$$D(x) = x^9 \cdot (1-16x)^{13} \cdot (1-4x)^2 (8 - 252x + 1678x^2 - 3607x^3 - 4352x^4).$$

This decomposition induces on the six solutions (88) of the exterior square, the following relation (up to a constant in $sol(\tilde{N}_1)$):

$$9701\,589\,902\,493 w_{0,1} + 609\,600\,054\,750\,928 w_{0,2} + 91\,875 \cdot 10^7 \cdot (w_{1,2} - w_{0,3}) = sol(\tilde{N}_1),$$

which shows the occurrence of a relation between the four solutions y_0, y_1, y_2 and y_3 of L_4 and their first derivative.

(iii) Along this globally nilpotent line, the L_4 operator is 'more' than a *G-operator* [44, 45], with its associated G-series. The series solution (analytical at $x = 0$) $sol(L_4)$ is a *series with integer coefficients* in the variable $y = x/2$:

$$\begin{aligned}
sol(L_4) = {} & 175 + 34\,398y + 4017\,125y^2 + 362\,935\,156y^3 \\
& + 28\,020\,752\,579y^4 + 1943\,802\,285\,620y^5 + 124\,761\,498\,220\,195y^6 \\
& + 7549\,851\,868\,859\,190y^7 + 436\,341\,703\,365\,296\,321y^8 \\
& + 24\,309\,515\,324\,321\,362\,986y^9 + 1314\,618\,756\,208\,478\,845\,353y^{10} \\
& + 69\,377\,289\,961\,823\,319\,909\,960y^{11} + 3588\,051\,829\,563\,766\,082\,490\,527y^{12} \\
& + 182\,471\,551\,181\,260\,556\,637\,299\,032y^{13} \\
& + 9150\,139\,649\,421\,210\,256\,395\,488\,775y^{14} \\
& + 453\,470\,079\,520\,701\,103\,056\,020\,155\,546y^{15} \\
& + 22\,252\,827\,613\,097\,363\,700\,809\,754\,930\,653y^{16} \\
& + 1083\,008\,337\,798\,028\,206\,538\,475\,233\,669\,454y^{17} \\
& + 52\,344\,841\,647\,844\,780\,032\,111\,214\,432\,202\,429y^{18} \\
& + 2515\,396\,349\,437\,801\,867\,561\,610\,046\,658\,122\,428y^{19} \\
& + 120\,295\,197\,044\,047\,707\,889\,910\,797\,105\,191\,140\,059y^{20} \\
& + 5729\,990\,034\,986\,443\,499\,765\,238\,359\,785\,037\,134\,524y^{21} \\
& + 272\,033\,605\,883\,471\,055\,363\,216\,581\,302\,378\,024\,952\,171y^{22} \\
& + 12\,879\,727\,903\,873\,470\,148\,364\,481\,804\,530\,391\,226\,578\,654y^{23} + \cdots. \qquad (92)
\end{aligned}$$

7. Various scenarios for L_4

The order-4 linear differential operator L_4 is thus slightly puzzling. It has clearly a lot of remarkable properties but *cannot be reduced* in a simple, or even in an involved way (up to rational pull-backs, up to operator equivalence [10, 11] i.e. homomorphisms and up to symmetric powers or products), to *elliptic curves or modular forms*. Is this operator going to be a counterexample to our favorite 'mantra' that the Ising model is nothing but the theory of elliptic curves and other modular forms ?

Let us display a few possible scenarios[41].

[41] Taking into account the previous known results on L_4, and consequently having some overlap.

7.1. Hypergeometric functions, modular forms and mirror maps

Because of the globally nilpotent character of L_4, we have some 'hypergeometric functions' prejudice[42]. Furthermore, we would also like to see some 'renormalization group' symmetries acting on the solutions, may be in a more involved way than what was described, in subsection 2.2 and displayed in [25], as isogenies of elliptic curves. We seem to have obstructions with rational pull-backs on hypergeometric functions. We should therefore consider generalizations to *algebraic pull-backs*, but we expect the pull-backs to be 'special' possibly corresponding to modular curves. Furthermore the *integrality* property of the solution series (92) (integer coefficients for the series) suggests to remain close to concepts, and structures, like modular forms, theta functions (which are modular forms) and mirror maps [39, 54].

Along these lines, let us recall a selected hypergeometric function closely linked with isogenies of elliptic curves, modular form and mirror maps. Let us recall (see comments after formula (6.2) in [62]) that[43]

$$ {}_3F_2\left(\left[\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right], [1, 1]; z\right) = (1-z)^{1/2} \cdot {}_3F_2\left(\left[\frac{1}{4}, \frac{3}{4}, \frac{1}{2}\right], [1, 1]; \frac{-4z}{(1-z)^2}\right) \qquad (93) $$

is a *modular map* [62]. More generally, one has remarkable quadratic relations (see (6.3) in [62]), where the *Landen transformation*

$$ z \quad \longrightarrow \quad \frac{4z}{(1+z)^2} \qquad (94) $$

clearly pops out.

This selected hypergeometric function satisfies the quadratic relation[44]

$$ {}_3F_2\big(\big[\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}\big], [1, 1]; 4t \cdot (1-t)\big) = \big({}_2F_1\big(\big[\tfrac{1}{2}, \tfrac{1}{2}\big], [1]; t\big)\big)^2 \qquad (95) $$

which yields the relation [49, 67] (q denotes the nome)

$$ \theta_3^4(q) = {}_3F_2\left(\left[\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right], [1, 1]; 4\frac{\theta_2^4(q)}{\theta_3^4(q)} \cdot \frac{\theta_4^4(q)}{\theta_3^4(q)}\right), \qquad (96) $$

which is naturally associated with a very simple *mirror map* involving theta functions[45]:

$$ z = z(q) = 4\frac{\theta_2^4(q)}{\theta_3^4(q)} \cdot \frac{\theta_4^4(q)}{\theta_3^4(q)}, \qquad \theta_3^4(q) = \frac{q}{z \cdot \sqrt{1-z}} \cdot \frac{\mathrm{d}z}{\mathrm{d}q}. \qquad (97) $$

Generalizations of this kind of relations are certainly the kind of relations we are seeking, for the solutions of L_4, but, unfortunately, this requires a lot of 'guessing' (of the selected hypergeometric function, of the mirror map, of some well-suited ratio of theta functions, etc).

7.2. Hadamard product

In our 'negative' result section (6.1) we saw that L_4, which is a *globally nilpotent* operator, cannot be reduced to ${}_2F_1$ hypergeometric functions associated with elliptic curves (modular

[42] It has been conjectured by Dwork [63] that globally nilpotent second order operators are necessarily associated with hypergeometric functions. This conjecture was ruled out by Krammer [64, 65] for some examples corresponding to periods of Abelian surfaces over a Shimura curve.

[43] We have studied in some detail the renormalization transformation $z \to -4z/(1-z)^2$ in [25].

[44] Compare this relation with the Bailey theorem of products [66]: $({}_2F_1([\frac{1}{2}, \frac{1}{2}], [1]; t))^2 = {}_4F_3([\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1], [1, 1, 1]; 4t \cdot (1-t))$.

[45] Which are, as we know, modular forms of half-integer weight.

forms), up to transformations *compatible with the global nilpotence*, namely equivalence (homomorphisms) of linear differential operators, symmetric powers or symmetric products of linear differential operators (even up to *rational* pull-backs). Having in mind this idea of compatibility with the global nilpotence, one more operation, a 'product' operation, can still be introduced, namely the *Hadamard product* which *quite canonically*[46] *builds globally nilpotent differential operators from globally nilpotent elementary 'bricks'*.

It has been seen in G Almkvist and W Zudilin [62] that one can build many (Calabi–Yau) order-4 operators from the order-2 elliptic curves operators, using the *Hadamard product* of series expansions. The Hadamard product is[47], just a convolution [69] ($|z| < |w| < 1$),

$$f * g(z) = \frac{1}{2i\pi} \cdot \int_\gamma f(w) \cdot g(z/w) \cdot \frac{\mathrm{d}w}{w}. \qquad (98)$$

Let us recall the order-2 (elliptic curve associated) operator

$$L_E = x \cdot (1-x) \cdot D_x^2 + (1-x) \cdot D_x + \tfrac{1}{4}, \qquad (99)$$

which has $EllipticE(x^{1/2})$ as a solution, and let us consider the *Hadamard product* of (the series expansion, at $x = 0$, of) $EllipticE(x^{1/2})$ with itself. This Hadamard square of (the series of) EllipticE is actually (the series of) a selected ${}_4F_3$ hypergeometric function

$$\frac{2 \cdot EllipticE}{\pi} \star \frac{2 \cdot EllipticE}{\pi} = {}_4F_3\big([-1/2, 1/2, 1/2, -1/2], [1, 1, 1]; x\big), \qquad (100)$$

which is a solution of the *globally nilpotent*[48] fourth order linear operator that we will write as

$$\begin{aligned} Had^2(L_E) = & -1 - 8 \cdot (x-2) \cdot D_x + 8 \cdot (14 - 13x) \cdot x \cdot D_x^2 \\ & + 96 \cdot (1-x) \cdot x^2 \cdot D_x^3 + 16 \cdot (1-x) \cdot x^3 \cdot D_x^4. \end{aligned} \qquad (101)$$

The Jordan form of the p-curvature [9] of the globally nilpotent fourth order linear operator (101) *actually identifies* with the 4×4 matrix (87), of characteristic and minimal polynomial T^4. Such a linear differential operator *cannot be a symmetric cube of a second order* operator in $\mathbb{C}(x)[D_x]$. We can, however, certainly say that the globally nilpotent fourth order linear differential operator (101) is *'associated with elliptic curves'*, and we will also say, by abuse of language[49], that the linear differential operator (101) is the Hadamard product (at $x = 0$) of (99) with itself, or the Hadamard square of the linear differential operator (99), and we will write $Had^2(L_E) = L_E \star L_E$. Several examples of 'Hadamard powers' of the complete elliptic integral K are given in (appendix A.1).

In our miscellaneous analysis of various (large order globally nilpotent) linear differential operators, we try to decompose these (large) operators into products, and ideally direct-sums [1, 5, 7, 70], of factors of smaller orders. We then try, in order to understand their 'very nature', to see if these *irreducible* factors are, up to equivalence of linear differential operators, and up to pull-backs, symmetric products of operators[50] of smaller orders. Since the Hadamard product quite naturally builds globally nilpotent operators from globally nilpotent ones, and since it already provided examples [71] of (Calabi–Yau) order-4 operators for which the corresponding *mirror symmetries are generalizations of Hauptmoduls* (basically products of

[46] *The fact that the global nilpotence is preserved by the Hadamard product is a consequence of the stability of the notion of G-connection under higher direct images for smooth morphisms* [68].

[47] Deligne's formula (simple application of the residue formula).

[48] The Hadamard product of two hypergeometric series is of course a hypergeometric series. The minimal operator of a G-series is globally nilpotent, and the Hadamard product of two G-series is a G-series.

[49] The Maple command gfun[hadamardproduct](eq_1, eq_2) returns the ODE that annihilates the termwise product of two holonomic power series of ODEs, eq_1 and eq_2.

[50] That is, simple products of the solutions.

elliptic curves Hauptmoduls [50]), we can see the Hadamard product as a quite *canonical transformation to add to the symmetric product of linear differential operators*[51].

We will say that an *irreducible* differential operator is 'associated with an elliptic curve' if it can be shown to be equivalent, *up to pull-backs*, to a symmetric product, or a Hadamard product, of second order hypergeometric differential operators corresponding to elliptic curves (see [25]). If the differential operator is *factorizable*, we will say that it is 'associated with an elliptic curve', if each factor in the factorization is.

Is L_4 in [7] an operator 'associated with an elliptic curve'? This looks like a quite systematic (almost algorithmic) approach. In practice, it remains, unfortunately, (computationally) very difficult[52] to recognize Hadamard products *up to homomorphism* transformations.

7.3. Calabi–Yau and $SP(4, \mathbb{C})$. Recalling threefold Calabi–Yau manifolds

We have discovered a symplectic $SP(4, \mathbb{C})$ differential Galois group for L_4. Many order-4 operators (often obtained by the Hadamard product of second order operators and corresponding to Calabi–Yau ODEs) were found to exhibit a symplectic $SP(4, \mathbb{C})$ differential Galois group to such a large extent that it may be tempting, for order-4 operators, to see the occurrence of a $SP(4, \mathbb{C})$ differential Galois group as a strong [72] indication[53] in favor of a Calabi–Yau ODE [72]. On the other hand, one may have the prejudice that Calabi–Yau ODEs and manifolds, which are well known in string theory, have no reason to occur in (integrable) lattice statistical mechanics. This is no longer true after Guttmann's paper[54] which showed very clearly [74] the emergence of Calabi–Yau ODEs in lattice statistical mechanics.

At this step, let us recall the famous non-trivial[55] example [35] of Candelas *et al* of the (threefold) *Calabi–Yau manifold*. The order-4 linear differential operator (in terms of the homogeneous derivative $\theta = z \cdot \mathrm{d}/\mathrm{d}z$)

$$\theta^4 - 5z \cdot (5\theta + 1) \cdot (5\theta + 2) \cdot (5\theta + 3) \cdot (5\theta + 4) \tag{102}$$

has the simple hypergeometric solution

$${}_4F_3\big(\big[\tfrac{1}{5}, \tfrac{2}{5}, \tfrac{3}{5}, \tfrac{4}{5}\big], [1, 1, 1]; 5^5 z\big), \tag{103}$$

which is associated with the *threefold* Calabi–Yau manifold [35, 75]:

$$x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_5^5 - z^{-1/5} \cdot x_1 x_2 x_3 x_4 x_5 = 0. \tag{104}$$

Actually the hypergeometric solution (103) can be written as a multiple integral with an algebraic integrand having (104) as a divisor. Being a multiple integral with an *algebraic integrand* it is in mathematical language [9] 'a *period*' and, consequently, the associated order-4 linear differential operator (102) is necessarily [9] globally nilpotent.

The differential Galois group of (102) is actually the symplectic [72, 76] group $SP(4, \mathbb{C})$. More precisely, the Picard–Fuchs linear differential operator (102), with its solution (103) $(x = z)$, reads

$$x^4 \cdot (1 - 3125x) \cdot D_x^4 + 2x^3 \cdot (3 - 12\,500x) \cdot D_x^3 \\ + x^2 \cdot (7 - 45\,000x) \cdot D_x^2 + x \cdot (1 - 15\,000x) \cdot D_x + 120x,$$

[51] For operators, not necessarily irreducible, this amounts to considering five 'Grothendieckian' operations: three products, the products of the operators, the products of the solutions of the operators (symmetric product), the Hadamard product (convolution, Fourier transform), as well as the operator equivalence, and the substitution (pull-back) [44, 45, 59].

[52] The transformations 'Homomorphisms' and 'Hadamard product' mess up with each other quite badly so a Hadamard product becomes difficult to recognize after homomorphism (i.e. gauge) transformations.

[53] In fact Calabi–Yau ODEs *do not* reduce to the $SP(4, \mathbb{C})$ differential Galois group.

[54] See [73] for the fast track communication.

[55] Elliptic curves can be seen as the simplest examples of Calabi–Yau manifolds.

which can be written in the form (3.9) in [35], when rescaling $z = 5^5 \cdot x$:

$$\frac{\mathrm{d}^4 F(z)}{dz^4} - 2\frac{(4z-3)}{(1-z)\cdot z} \cdot \frac{\mathrm{d}^3 F(z)}{\mathrm{d}z^3} - \frac{1}{5}\frac{(72z-35)}{(1-z)\cdot z^2} \cdot \frac{\mathrm{d}^2 F(z)}{\mathrm{d}z^2}$$
$$- \frac{1}{5}\frac{(24z-5)}{(1-z)\cdot z^3} \cdot \frac{\mathrm{d}F(z)}{\mathrm{d}z} - \frac{24}{625}\frac{F(z)}{(1-z)\cdot z^3}.$$

Along this line, some list of Calabi–Yau ODEs and Calabi–Yau linear differential operators have been obtained [71] by G Almkvist *et al* seeking systematically for order-4 differential operators obtained from Hadamard product constructions of second order operators, often within a symplectic and MUM framework. Such long, and detailed, list of Calabi–Yau differential operators are precious, but, again, it is not straightforward to see if an order-4 operator, like L_4, reduces to one of the Calabi–Yau differential operators in such lists, up to homomorphisms and *up to pull-backs*.

7.4. The ${}_4F_3$ scenario

As far as order-4 operators that cannot be simply reduced to elliptic curves are concerned, we already saw [33], in the Ising model, an example corresponding [77] to the *form factors* $C^{(1)}(k, n)$, expressed in terms of a ${}_4F_3$ hypergeometric function:

$$b(k, n) = {}_4F_3\left(\left[\frac{1+k+n}{2}, \frac{1+k+n}{2}, \frac{2+k+n}{2}, \frac{2+k+n}{2}\right],\right.$$
$$\left.[1+k, 1+n, 1+k+n];\, 16x\right). \qquad (105)$$

It is a solution of an order-4 linear differential operator which can be written in terms of the homogeneous derivative θ (in the usual quasi-factorized form for ${}_nF_{n-1}$ hypergeometric function):

$$J_{k,n} = 16 \cdot x \cdot \left(\theta + \frac{1+k+n}{2}\right)^2 \cdot \left(\theta + \frac{2+k+n}{2}\right)^2 - (\theta+k)\cdot(\theta+n)\cdot(\theta+k+n)\cdot\theta. \qquad (106)$$

All these operators (106) are, in fact, homomorphic (see (appendix B)). The linear differential operator (106) is not MUM (except for $k = n = 0$); however, $b(k, n)$ is clearly a *Hadamard product* (see (G.1) in [33]) of two algebraic functions for k and n integers (or an algebraic function and a ${}_3F_2$ function otherwise).

The *exterior square* of $J_{k,n}$ is a sixth order operator which is invariant by $k \leftrightarrow n$. Noticeably, this *exterior square* of $J_{k,n}$ has a very simple *rational solution*:

$$\frac{1}{P} \qquad \text{where} \quad P = (1-16x)\cdot x^{k+n+1}, \qquad (107)$$

which shows that one actually has a *symplectic structure when* $k + n$ is an integer number. Actually, performing the direct-sum factorization[56] of the *exterior square* of $J_{k,n}$ gives (when $k \neq \pm n$), with P given by (107),

$$Ext^2(J_{k,n}) = \Omega_1^{(R)} \oplus \Omega_1^{(2)} \oplus \left(Q_2^{(1)} \cdot Q_2^{(2)}\right), \qquad (108)$$

with

$$\Omega_1^{(R)} = D_x + \frac{\mathrm{d}}{\mathrm{d}x}\ln(P), \qquad \Omega_1^{(2)} = D_x + \frac{\mathrm{d}}{\mathrm{d}x}\ln\left(\frac{x^N\cdot(1-16x)^M}{P_{k,n}}\right).$$

[56] DFactorLCLM in Maple.

Here $P_{k,n}$ is a polynomial, N and M are integers depending of k and n, and $Q_2^{(1)}$ and $Q_2^{(2)}$ are equivalent, and homomorphic to $Q_2^{(1)}(k = 1, n = 0)$:

$$D_x^2 + 2\frac{(3-64x)}{(1-16x)\cdot x} \cdot D_x + 2\frac{3-98x}{(1-16x)\cdot x^2}, \tag{109}$$

whose solutions can be expressed in terms of hypergeometric functions:

$$\frac{1}{x^2} \cdot {}_2F_1\left(\left[\frac{3}{2}, \frac{3}{2}\right], [2]; 1-16x\right),$$
$$\frac{1}{x^2\cdot(1-32x)^{3/2}} \cdot {}_2F_1\left(\left[\frac{3}{4}, \frac{5}{4}\right], [1]; \frac{1}{(1-32x)^2}\right). \tag{110}$$

The integer M is equal to 0 if $k - n$ is even and is equal to 1 if $k - n$ is odd, the integer N and the degree of the polynomial reading, respectively,

$$\frac{3}{2}\cdot(n+k) + \frac{1}{2}\cdot|n-k| + 1, \qquad \frac{n+k}{2} + \frac{|n+k|}{2} - \frac{3}{2} - \frac{1}{2}\cdot(-1)^{n-k}.$$

The symplectic form of the *exterior square* of $J_{k,n}$ is singular if, and only if, $k = \pm n$. The exterior square $Ext^2(J_{k,n})$ has no direct sum factorization for $k = \pm n$. It factorizes in the product of an order-1, two order-2 and an order-1 operators.

Furthermore, the function

$$a(k,n) = \binom{k+n}{k} \cdot b(k,n), \tag{111}$$

which *corresponds to the form factor* $C^{(1)}(k,n)$, has a series expansion with *integer coefficients*:

$$a(k,n) = \binom{k+n}{k} + \frac{(k+n+1)(k+n+2)^2}{(n+1)(k+1)} \cdot \binom{k+n}{k} \cdot x$$
$$+ \frac{\alpha_2(k,n)}{2} \cdot \binom{k+n}{k} \cdot x^2 + \frac{\alpha_3(k,n)}{6} \cdot \binom{k+n}{k} \cdot x^3 + \cdots$$

where $\alpha_2(k,n)$ and $\alpha_3(k,n)$ read, respectively,

$$\frac{(k+n+1)(k+n+2)(k+n+3)^2(k+n+4)^2}{(k+1)(k+2)(n+1)(n+2)},$$
$$\frac{(k+n+1)(k+n+2)(k+n+3)(k+n+4)^2(k+n+5)^2(k+n+6)^2}{(k+1)(k+2)(k+3)(n+1)(n+2)(n+3)}.$$

8. L_4 is Calabi–Yau

8.1. Warm-up: discovering the proper algebraic extension for the pull-backs

The ${}_4F_3$ function satisfies a linear differential operator $L_{4,3}$ with three singularities $0, 1, \infty$.

The singularities of L_4 at $x = 1/16$, and at $x = \infty$, have exponents: integer, integer, half-integer, half-integer, and have only one logarithm there. This configuration is not compatible with any of the singularities of $L_{4,3}$ under *rational* pullbacks.

The singularity at $x = 1$ of $L_{4,3}$ has exponents $0, 1, 2, \lambda$, where λ depends on the parameters of the ${}_4F_3$ function. The exponents $0, 1, 2$ correspond to solutions without logarithms. Thus, by choosing the ${}_4F_3$ parameters to set λ to an integer (we take $\lambda = 1$) we get one logarithm at $x = 1$. Then the $x = 1$ singularity of $L_{4,3}$ has the same number of logarithms as the $x = 1/16$ and $x = \infty$ singularities of L_4. However, under rational pullbacks there is still no match because the exponents of $L_{4,3}$ at $x = 1$, which are now 0, 1, 1, 2, do not match (modulo the integers) the exponents of L_4 at $x = 1/16$ and $x = \infty$.

A pullback $x \mapsto (x - a)^2$ doubles the exponents at $x = a$, and likewise, a field extension of degree 2 can divide the exponents into half. The exponents at $x = 1$ of $L_{4,3}$ must be divided into half to match (modulo the integers) the exponents of L_4 at $x = 1/16$ and $x = \infty$. So this field extension must ramify at $x = 1/16$ and $x = \infty$, and this tells us that the field extension must be $\mathbb{C}(x) \subset \mathbb{C}(x, \sqrt{1-16x})$. We can write this latter field as $\mathbb{C}(\xi)$ where $\xi = \sqrt{1-16x}$. With a pullback in $\mathbb{C}(\xi)$, the $x = 1$ singularity of $L_{4,3}$ can be matched with the $x = 1/16$ and $x = \infty$ singularities of L_4.

A necessary (but not sufficient) condition for a homomorphism between two operators[57] to exist is that the exponents of the singularities must match modulo the integers, and the number of logarithms must match at every singularity. But once one knows that the pullback for L_4 must be in $\mathbb{C}(\xi)$, it suddenly becomes easy to find a pullback that meets this necessary condition. Once the pullback is found, we can check if a homomorphism exists (and if so, find it) with DEtools[Homomorphisms] in Maple.

8.2. The ${}_4F_3$ result

Seeking for ${}_4F_3$ hypergeometric functions up to homomorphisms, and assuming an algebraic pull-back with the *square root extension* $(1 - 16 \cdot w^2)^{1/2}$, we actually found[58] that the solution of L_4 *can be expressed in terms of a selected* ${}_4F_3$ which is precisely the *Hadamard product of two elliptic functions*

$$ {}_4F_3\left(\left[\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}\right], [1, 1, 1]; z\right) = {}_2F_1\left(\left[\tfrac{1}{2}, \tfrac{1}{2}\right], [1]; z\right) \star {}_2F_1\left(\left[\tfrac{1}{2}, \tfrac{1}{2}\right], [1]; z\right), \tag{112} $$

where the pull-back z *is nothing but* s^8 with $x = w^2$, where w is the natural variable for the $\tilde{\chi}^{(n)}$'s n-fold integrals [8, 33], $w = s/(2(1 + s^2))$:

$$ z = \left(\frac{1 + (1 - 16 \cdot w^2)^{1/2}}{1 - (1 - 16 \cdot w^2)^{1/2}}\right)^4 = s^8. \tag{113} $$

Let us recall that $t = k^2 = s^4$ and that $EllipticK(k)$ in Maple is the integral with a k^2 in the square root, so $t^2 = k^4 = s^8$:

$$ \frac{2}{\pi} \cdot EllipticK(y) = {}_2F_1\left(\left[\frac{1}{2}, \frac{1}{2}\right], [1]; y^2\right); \tag{114} $$

therefore, the solution of L_4 is expressed in terms of the Hadamard square of $EllipticK$, yielding in Maple notations

$$ \begin{aligned} {}_4F_3\left(\left[\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right], [1, 1, 1]; t^2\right) &= {}_2F_1\left(\left[\frac{1}{2}, \frac{1}{2}\right], [1]; t^2\right) \star {}_2F_1\left(\left[\frac{1}{2}, \frac{1}{2}\right], [1]; t^2\right) \\ &= \frac{2}{\pi} \cdot EllipticK(t) \star \frac{2}{\pi} \cdot EllipticK(t), \end{aligned} \tag{115} $$

extremely similar to the previously seen Hadamard product (100).

[57] Here the two operators are L_4 and a pullback of $L_{4,3}$, both viewed as elements of $\mathbb{C}(\xi)[D_\xi]$.
[58] Details will be given in forthcoming publications.

8.3. The Calabi–Yau result

This result could be seen as already achieving the connection with *elliptic curves* we were seeking for. In fact, looking at the Calabi–Yau list of fourth order operators obtained by Almkvist *et al* [71], one discovers that this selected ${}_4F_3$ hypergeometric function *actually corresponds to a Calabi–Yau ODE. This is Calabi–Yau ODE number 3 in page 10 of the Almkvist et al list* (see table A of Calabi–Yau equations page 10 in [71]).

Remark 1. The hypergeometric function (112) also corresponds to the *hyper body-centered cubic lattice Green function* [73, 78]:

$$P(0, z) = \int_0^\pi \int_0^\pi \int_0^\pi \int_0^\pi \frac{\mathrm{d}k_1\, \mathrm{d}k_2\, \mathrm{d}k_3\, \mathrm{d}k_4}{1 - z \cdot \cos(k_1)\cos(k_2)\cos(k_3)\cos(k_4)}. \tag{116}$$

It may well be, following the ideas of Christoll [79, 80], that the Calabi–Yau threefold corresponding to (116), (112) similar to (104), *is nothing but the denominator of the integrand of* (116) $(1 - z \cdot \cos(k_1)\cos(k_2)\cos(k_3)\cos(k_4))$, *written in an algebraic way* ($z_i = \exp(\mathrm{i}k_i)$):

$$\begin{aligned} &{}_4F_3\left(\left[\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right], [1, 1, 1]; z\right) \\ &\qquad \simeq \int\!\!\int\!\!\int\!\!\int \frac{\mathrm{d}z_1\, \mathrm{d}z_2\, \mathrm{d}z_3\, \mathrm{d}z_4}{8z_1z_2z_3z_4 - \left(1 + z_1^2\right) \cdot \left(1 + z_2^2\right) \cdot \left(1 + z_3^2\right) \cdot \left(1 + z_4^2\right) \cdot z}. \end{aligned} \tag{117}$$

Along this line

$$8z_1z_2z_3z_4 - \left(1 + z_1^2\right) \cdot \left(1 + z_2^2\right) \cdot \left(1 + z_3^2\right) \cdot \left(1 + z_4^2\right) \cdot z = 0$$

is a genus-1 curve in (z_1, z_2) which has to be seen on the same footing as the threefold Calabi–Yau manifold (104).

Remark 2. Note that the series expansion of (112) for the inverse $1/z$ of the pull-back is a series with *integer* coefficients in[59] the variable w:

$$\begin{aligned} &{}_4F_3\left(\left[\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right], [1, 1, 1]; \left(\frac{1 - (1 - 16 \cdot w^2)^{1/2}}{1 + (1 - 16 \cdot w^2)^{1/2}}\right)^4\right) \\ &\qquad = 1 + 16w^8 + 512w^{10} + 11\,264w^{12} + 212\,992w^{14} \\ &\qquad\quad + 3728\,656w^{16} + 62\,473\,216w^{18} + 1019\,222\,016w^{20} \\ &\qquad\quad + 16\,350\,019\,584w^{22} + 259\,416\,207\,616w^{24} \\ &\qquad\quad + 4086\,140\,395\,520w^{26} + \cdots. \end{aligned} \tag{118}$$

We also have this *integrality property* for $256z$:

$$\begin{aligned} &{}_4F_3\left(\left[\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right], [1, 1, 1]; 256z\right) = 1 + 16z + 1296z^2 + 160\,000z^3 \\ &\qquad + 24\,010\,000z^4 + 4032\,758\,016z^5 + 728\,933\,458\,176z^6 \\ &\qquad + 138\,735\,983\,333\,376z^7 + 27\,435\,582\,641\,610\,000z^8 \\ &\qquad + 5588\,044\,012\,339\,360\,000z^9 + 1165\,183\,173\,971\,324\,375\,296z^{10} \\ &\qquad + 247\,639\,903\,129\,149\,250\,277\,376z^{11} \\ &\qquad + 53\,472\,066\,459\,540\,320\,483\,696\,896z^{12} + \cdots. \end{aligned} \tag{119}$$

[59] This is not true in z or s.

The solution of L_4 had been seen to be a series with *integer* coefficients (see (92)). Now that we know that L_4 has a *Calabi–Yau interpretation*; this *integrality property* can be seen as associated with *mirror maps and mirror symmetries* (see section (9) below), *as well as inherited from the Hadamard square structure* (see (appendix A.1) below).

8.4. Speculations: ${}_4F_3$ generalizations and beyond

Let us consider a few generalizations of (112), the selected ${}_4F_3$ we discovered for L_4.

More generally, the hypergeometric function

$${}_4F_3\left(\left[\frac{1}{2}+q,\,\frac{1}{2}+r,\,\frac{1}{2}+s,\,\frac{1}{2}+t\right],\,[n+1,\,m+1,\,p+1];\,x\right)$$

corresponds to the linear differential operator

$$\Omega_{n,m,p;q,r,s,t} = \left(\theta+\frac{1}{2}+q\right)\cdot\left(\theta+\frac{1}{2}+r\right)\cdot\left(\theta+\frac{1}{2}+s\right)\cdot\left(\theta+\frac{1}{2}+t\right) - \frac{1}{x}\cdot(\theta+n)\cdot(\theta+m)\cdot(\theta+p)\cdot\theta. \qquad (120)$$

For any integer n, m, p, q, r, s, t all these operators (120) are actually equivalent (homomorphic, see (appendix B)). Therefore, all these linear differential operators (120) are homomorphic to (120) for $n = m = p = q = r = s = t = 0$, which is actually a Calabi–Yau equation.

We have also encountered another kind of generalization of (112): the Hadamard powers generalizations (A.4) (see (appendix A.1)).

It is thus quite natural to consider the linear differential operators corresponding to generalizations like

$${}_nF_{n-1}\left(\left[\tfrac{1}{2}+p_1,\ldots,\tfrac{1}{2}+p_n\right],[1+q_1,\ldots,1+q_{n-1}];4^n\cdot x\right),$$

where the p_i's and q_i's are integers and see, if *up to homomorphisms and rational or algebraic*[60] *pull-backs* one can try to understand the remaining quite large order operators L_{12} and L_{23}, in such a large enough framework. There is no conceptual obstruction to such calculations. The obstruction is just the 'size' of the corresponding massive computer calculations necessary to achieve this goal.

9. Mirror maps for the Calabi–Yau ${}_4F_3([1/2, 1/2, 1/2, 1/2], [1, 1, 1]; 256\,x)$

An irreducible linear differential equation is said to be of MUM if all the exponents at 0 are zero (one Jordan block). This is the case for all the hypergeometric functions

$${}_nF_{n-1}\left(\left[\tfrac{1}{2},\tfrac{1}{2},\ldots,\tfrac{1}{2}\right],[1,1,\ldots,1];4^n x\right).$$

The hypergeometric function

$${}_4F_3\left(\left[\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}\right],[1,1,1];256x\right),$$

which is MUM, corresponds to the fourth order linear operator

$$x^4\cdot(1-256x)\cdot D_x^4 + 2x^3\cdot(3-1024x)\cdot D_x^3 + x^2\cdot(7-3712x)\cdot D_x^2 + x\cdot(1-1280x)\cdot D_x - 16x,$$

[60] Not too involved in a first approach, just square root extensions.

or, using the homogeneous derivative θ,

$$\theta^4 - 256 \cdot x \cdot \left(\theta + \frac{1}{2}\right)^4, \qquad \theta = x \cdot \frac{d}{dx}. \tag{121}$$

It verifies the symplectic condition [62]

$$a_1 = \frac{1}{2} \cdot a_2 \cdot a_3 - \frac{1}{8} \cdot a_3^3 + \frac{da_2}{dx} - \frac{3}{4} \cdot a_3 \cdot \frac{da_3}{dx} - \frac{1}{2} \cdot \frac{d^2 a_3}{dx^2}, \tag{122}$$

for the monic order-4 operator: $D_x^4 + a_3 \cdot D_x^3 + a_2 \cdot D_x^2 + a_1 \cdot D_x + a_0$.

Condition (122) is nothing but the condition for the *vanishing of the head coefficient* of D_x^6 of this exterior square (see proposition 3 of [62]). The exterior square of (121) is an *irreducible order-5* operator, instead of the order-6 operator one expects at first sight.

This opens room for a *non-degenerate alternate 2-form invariant by the (symplectic) group* $SP(4, \mathbb{C})$. Actually (121) has a $SP(4, \mathbb{C})$ differential Galois group.

Remark 3. Let us compare this situation with the one for a 'similar' order-4 operator homomorphic to (121):

$$\theta^4 - 256 \cdot x \cdot \left(\theta - \tfrac{1}{2}\right)^4. \tag{123}$$

The exterior square of (123) is an order-6 operator which is *the direct sum of an order-5 operator homomorphic to the order-5 exterior square of* (121) and an order-1 operator

$$D_x + \frac{1}{(1 - 256x) \cdot x} \tag{124}$$

which has the simple rational solution $(1 - 256x)/x$.

These two operators, (121) and (123), have the *exact same* $SP(4, \mathbb{C})$ *differential Galois group*, but, nevertheless, their corresponding exterior squares do not have the same order[61]. The situation for (121) can be thought as an 'evanescence' of the rational solution.

9.1. Mirror maps in a MUM framework

The solutions of (121) read

$$y_0 = 1 + 16x + 1296x^2 + 160\,000x^3 + 24\,010\,000x^4 + 4032\,758\,016x^5$$
$$+728\,933\,458\,176x^6 + \cdots,$$

$$y_1 = y_0 \cdot \ln(x) + \tilde{y}_1,$$

with

$$\tilde{y}_1 = 64x + 6048x^2 + \frac{2368\,000}{3}x^3 + \frac{365\,638\,000}{3}x^4 + \frac{104\,147\,576\,064}{5}x^5$$
$$+\frac{19\,045\,884\,743\,424}{5}x^6 + \frac{25\,588\,111\,188\,676\,608}{35}x^7 + \cdots$$

$$y_2 = y_0 \cdot \frac{\ln(x)^2}{2} + \tilde{y}_1 \cdot \ln(x) + \tilde{y}_2, \qquad \text{with:}$$

$$\tilde{y}_2 = 32x + 5832x^2 + \frac{8182\,400}{9}x^3 + \frac{1374\,099\,650}{9}x^4$$
$$+\frac{685\,097\,536\,032}{25}x^5 + \frac{129\,379\,065\,232\,032}{25}x^6 + \cdots$$

[61] Since the log degree of these operators is equal to 4, the order of these exterior squares is at least 5.

$$y_3 = y_0 \cdot \frac{\ln(x)^3}{6} + \tilde{y}_1 \cdot \frac{\ln(x)^2}{2} + \tilde{y}_2 \cdot \ln(x) + \tilde{y}_3,$$

$$\tilde{y}_3 = -64x - 4296x^2 - \frac{10\,334\,080}{27} \cdot x^3 - \frac{1110\,845\,155}{27} \cdot x^4 + \cdots$$

Introducing the nome [39, 60]

$$q = \exp\left(\frac{y_1}{y_0}\right) = x \cdot \exp\left(\frac{\tilde{y}_1}{y_0}\right), \tag{125}$$

one finds the expansion (with *integer* coefficients) of the nome as

$$\begin{aligned} q = {} & x + 64x^2 + 7072x^3 + 991\,232x^4 + 158\,784\,976x^5 \\ & + 27\,706\,373\,120x^6 + 64x^2 + 5130\,309\,889\,536x^7 + 992\,321\,852\,604\,416x^8 \\ & + 198\,452\,570\,147\,492\,456x^9 + 40\,747\,727\,123\,371\,117\,056x^{10} \\ & + 8546\,896\,113\,440\,681\,326\,848x^{11} + 1824\,550\,864\,289\,064\,534\,212\,608x^{12} \\ & + 395\,291\,475\,348\,616\,441\,757\,137\,536x^{13} \\ & + 86\,723\,581\,205\,125\,308\,226\,931\,367\,936x^{14} \\ & + 19\,233\,461\,618\,939\,530\,038\,756\,686\,458\,880x^{15} \\ & + 4305\,933\,457\,394\,032\,994\,320\,115\,176\,046\,592x^{16} \\ & + 972\,002\,126\,960\,220\,578\,680\,860\,300\,103\,711\,764x^{17} \\ & + 221\,026\,060\,926\,103\,071\,799\,983\,313\,019\,509\,871\,872x^{18} + \cdots, \end{aligned} \tag{126}$$

and, conversely, the *mirror map* [39, 40, 62] reads the following series with *integer* coefficients ($x = z(q(x))$):

$$\begin{aligned} z(q) = {} & q - 64q^2 + 1120q^3 - 38912q^4 - 1536\,464q^5 \\ & - 177\,833\,984q^6 - 19\,069\,001\,216q^7 - 2183\,489\,257\,472q^8 \\ & - 260\,277\,863\,245\,160q^9 - 32\,040\,256\,686\,713\,856q^{10} \\ & - 4047\,287\,910\,219\,320\,576q^{11} - 522\,186\,970\,689\,013\,088\,256q^{12} \\ & - 68\,573\,970\,045\,596\,462\,152\,576q^{13} \\ & - 9140\,875\,458\,960\,295\,169\,327\,104q^{14} \\ & - 1234\,198\,194\,801\,672\,701\,733\,531\,648q^{15} \\ & - 168\,503\,147\,864\,931\,724\,540\,942\,221\,312q^{16} \\ & - 23\,230\,205\,873\,245\,591\,254\,063\,032\,928\,212q^{17} \\ & - 3230\,146\,419\,442\,584\,387\,013\,916\,457\,526\,784q^{18} + \cdots. \end{aligned} \tag{127}$$

The *Yukawa coupling* [39, 62]

$$K(q) = \left(q \cdot \frac{\mathrm{d}}{\mathrm{d}q}\right)^2 \left(\frac{y_2}{y_0}\right) \tag{128}$$

has the following (integer coefficients) series expansion:

$$\begin{aligned} K(q) = {} & 1 + 32q + 4896q^2 + 702464q^3 \\ & + 102\,820\,640q^4 + 15\,296\,748\,032q^5 \\ & + 2302\,235\,670\,528q^6 + 349\,438\,855\,544\,832q^7 \\ & + 53\,378\,019\,187\,206\,944q^8 + 8194\,222\,260\,681\,725\,696q^9 \\ & + 1262\,906\,124\,008\,518\,928\,896q^{10} + 195\,269\,267\,971\,549\,608\,656\,896q^{11} \end{aligned}$$

$$+30\,273\,112\,887\,215\,918\,307\,768\,320q^{12}$$
$$+4703\,886\,698\,057\,200\,436\,126\,953\,472q^{13}$$
$$+732\,300\,206\,865\,552\,210\,649\,383\,895\,040q^{14}$$
$$+114\,192\,897\,568\,357\,606\,610\,746\,318\,782\,464q^{15}$$
$$+17\,832\,557\,144\,166\,657\,247\,747\,889\,907\,477\,280q^{16}$$
$$+2788\,280\,197\,510\,341\,680\,209\,147\,877\,101\,177\,216q^{17}$$
$$+436\,459\,641\,692\,984\,506\,336\,508\,940\,737\,030\,913\,792q^{18} + \cdots. \tag{129}$$

The nome series (126) has a radius of convergence $R = 1/256$, corresponding to the $z = 1/256$ singularity. *The mirror map* (127) *and the Yukawa series* (129) have a radius[62] of convergence $R \simeq 0.0062\,794\,754\ldots$, corresponding to the singularity $q_s \simeq 0.0062\,794\,754\ldots$ given by

$$q_s = \exp\left(\frac{x_0}{x_1}\right), \tag{130}$$

with

$$x_0 = {}_4F_3\left(\left[\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right], [1, 1, 1]; 1\right)$$

and

$$x_1 = \sum_{n=0}^{\infty} 4 \cdot \Gamma(n + 1/2)^4 \cdot (\Psi(n + 1/2) - \Psi(n + 1))/\,\Gamma(n + 1)^4/\pi^2.$$

Introducing the rational function[63]

$$q_2 = \frac{1}{2}\frac{327680z^2 - 1792z + 5}{z^2 \cdot (1 - 256z)^2}, \tag{131}$$

one finds that the mirror map (127) *actually verifies* a generalization of (42), namely the so-called *quantum deformation of the Schwarzian equation* (see (4.24) in [39]):

$$\frac{q_2}{5} \cdot z'^2 + \{z, \tau\} = \frac{2}{5} \cdot \frac{K''}{K} - \frac{1}{2} \cdot \left(\frac{K'}{K}\right)^2, \tag{132}$$

where the derivatives are with respect to τ, the ratio of the first two solutions $\tau = y_1/y_0$ (log of the nome (125)), and $\{z, \tau\}$ denotes the *Schwarzian derivative* (43).

The rhs of (132) generalizes the very simple rational function rhs we had on the Schwarzian equation (42) (see footnote 33). The rhs of (132) is basically a transcendental function depending on the Yukawa coupling function (128). It is natural to try to obtain a (nonlinear) ODE bearing *only* on the mirror map $z(q)$, and *not* the Yukawa coupling function (128) as well. This can be done (see (appendix C)) with a (complexity) price to pay, namely that these *higher order Schwarzian* (nonlinear) ODEs are *much more involved ODEs of much larger order*.

[62] Along this line of radius of convergence of the mirror map and related Schneider-Lang transcendence criteria, see [81].

[63] Which corresponds to take the values $r_0 = 16$, $r_2 = 384 = (3/2) \cdot 256$, $r_3 = 512$, $r_4 = 256$, in [39].

9.2. Higher order Schwarzian ODEs on the mirror map

Actually, we have also obtained the *higher order Schwarzian (nonlinear) ODE* (see (4.20) in [39]), verified by the *mirror map* (127). It is an order-7 nonlinear ODE given by the vanishing of a polynomial with integer coefficients in $z, z', z'', \ldots, z^{(7)}$, having 1211 monomials in $z, z', z'', \ldots, z^{(7)}$. This polynomial of degree 18 in z, 24 in z', 12 in z'', 6 in $z^{(3)}$, 4 in $z^{(4)}$, 3 in $z^{(5)}$, 2 in $z^{(6)}$ and 1 in $z^{(7)}$ can be downloaded in [83] to check that (127) *actually verifies* this higher order Schwarzian (nonlinear) ODE.

One can verify that these higher order Schwarzian ODEs, on the mirror map, are *actually compatible* with the (renormalization group, isogenies [25], etc) transformations $q \to q^n$, for any integer n. Changing $q \to q^n$ in the mirror map (127)

$$z(q^n) = q^n - 64q^{2n} + 1120q^{3n} - 38912q^{4n} - 1536\,464q^{5n} \\ - 177\,833\,984q^{6n} - 19\,069\,001\,216q^{7n} - 2183\,489\,257\,472q^{8n} \\ - 260\,277\,863\,245\,160q^{9n} + \cdots \tag{133}$$

one finds that this new function is *still a solution of the higher order Schwarzian ODE*.

Conversely, one can consider the *reciprocal* higher order Schwarzian ODE bearing on the log of the nome, $\tau = \ln(q) = \tau(z)$, seen as a function of z. It is an order-7 *nonlinear* ODE given by the sum of 602 monomial terms:

$$0 = z^6 \cdot (1 - 256 \cdot z)^2 \cdot P_\tau(z, \tau', \tau'', \cdots \tau^{(5)}) \cdot \tau^{(7)} + \cdots \\ + (1 - 256 \cdot z) \cdot Q_\tau(z, \tau, \cdots, \tau^{(6)}, \tau^{(7)}) + R_t(z, \tau, \tau', \cdots, \tau^{(5)}) \cdot \tau', \tag{134}$$

where the $\tau^{(m)}$'s are the mth z-derivative of $\tau(z)$, and where P_τ and R_τ are polynomials of the z and the $\tau^{(m)}$'s derivatives (see (D.3) below).

Let us consider the Moebius transformation (homographic transformation) on the log of the nome

$$\tau \quad \longrightarrow \quad \frac{a \cdot \tau + b}{c \cdot \tau + d}, \tag{135}$$

which transforms, as far as the z-derivatives are concerned, in an increasingly involved way with increasing orders of derivation:

$$\tau' \quad \longrightarrow \quad \frac{ad - cb}{(c \cdot \tau + d)^2} \cdot \tau', \\ \tau'' \quad \longrightarrow \quad \frac{(ad - cb)}{(c \cdot \tau + d)^2} \cdot \tau'' - 2\frac{(ad - cb) \cdot c}{(c \cdot \tau + d)^3} \cdot \tau'^2, \qquad \ldots. \tag{136}$$

It is a straightforward calculation to verify that the higher order Schwarzian ODE (134) is *actually invariant* by the Moebius transformation (135) and its deduced transformations on the derivatives (136). Do note that we do not impose $ad - cb = 1$: we are in $GL(2, \mathbb{Z})$ *not* in $SL(2, \mathbb{Z})$. The previous symmetry $q \to q^n$ (see (133)) of the higher order Schwarzian ODE corresponded to $\tau \to n \cdot \tau$. We have here a $GL(2, \mathbb{Z})$ symmetry group of the higher order Schwarzian ODE, corresponding to the extension of the well-known *modular group* $SL(2, \mathbb{Z})$ by the *isogenies* (exact representation of the *renormalization group* [25]) $\tau \to n \cdot \tau$, which extend quite naturally the modular group and isogenies symmetries encountered with elliptic curves [25]. Even leaving the elliptic curves or modular forms framework, for some natural generalizations (Calabi–Yau are natural generalizations of Hauptmoduls) it was crucial to get mathematical structures with *the canonical exact representation of the renormalization group* [25].

10. Late comments: the integrability behind the mirror

It is beyond the scope of this very down-to-earth paper to give a mathematical definition of mirror symmetries, since mathematicians are still seeking for the proper general framework to define them (mixed Hodge structures, flat connection underlying a variation of Hodge structures[64] in the Calabi–Yau case [84–86], T-duality [87], toric frameworks like in Batyrev's construction of mirror symmetry between hypersurfaces of toric Fano[65] varieties [89], *algebraic* Gauss–Manin connections [90], etc).

More familiar to physicists, in particular specialists of integrability, is the notion of the *Picard–Fuchs* [91] *equation*[66]. Along this Picard–Fuchs line, the occurrence of Painlevé VI equations is well known for the Ising model [13, 14] (see the Garnier or Schlesinger systems [93]). This Picard–Fuchs notion is central in any 'intuitive' understanding of mirror maps [94] and other Calabi–Yau manifolds (namely compact Kähler manifolds with Ricci-flat Kähler metrics).

As far as the 'proper integrable framework' of this paper is concerned let us underline the following comments. The 2D Ising model is a well-known free-fermion model with an elliptic parametrization. This elliptic parametrization is, of course, a straight consequence of the Yang–Baxter integrability of the model (here the star–triangle relation), and, therefore, it is not a surprise to see elliptic functions in the integrals of correlation functions (see for instance [95–97]). Along this line, even Painlevé VI equations can be seen as a Gauss–Manin deformation of an elliptic function second order ODE [13, 14, 98].

However, it is crucial to note that the elliptic parametrization is not one-to-one related with a Yang–Baxter integrability: the 16-vertex model, which, in general, is *not* Yang–Baxter integrable, has a *canonical* (compulsory!) *elliptic parametrization* [99], the elliptic parametrization being a consequence of the *integrability of the birational symmetries* [23] of the model, and of course not of a Yang–Baxter integrability that does not exist generically for that very model. The free-fermion character of the square Ising model is of course crucial in Wu *et al* [2] (Pfaffian, Toeplitz, etc) calculations to write explicitly the $\chi^{(n)}$'s as integrals of some integrand *algebraic* in some well-suited variables. However, Guttmann's paper [74] makes crystal clear with miscellaneous examples of Green's functions for many lattice statistical mechanics, or enumerative combinatorics, problems in *arbitrary* lattice dimensions, that Calabi–Yau ODEs emerge in a lattice statistical mechanics framework which is (at first sight) *quite remote from Yang–Baxter (tetrahedron, etc) integrability*, and even more *from free-fermion integrability*.

If the occurrence of linear differential operators *associated with elliptic curves* for square Ising correlation functions, or form factors, is not a surprise [77], the kind of linear differential operators that should emerge in quite involved *highly composite* objects like the n-particle components $\chi^{(n)}$ of the susceptibility of the square Ising model is far from clear. We just had a prejudice that they should be 'special' and could possibly be associated with elliptic curves. In fact, even if Yang–Baxter structures were the 'deus ex machina' behind these 'special' linear differential operators it seems impossible to use that property in any explicit calculation.

Moving away from Yang–Baxter integrability to other concepts of 'integrability', we are actually using the following ingredients: we have n-fold integrals of an integrand which is *algebraic* in the variables of integration and in the other remaining variables. *This algebraicity*

[64] See [82] for the introduction of the notion of variation of Hodge structures with their Gauss–Manin connections.
[65] Mirror symmetry, in a class of models of toric varieties with zero first Chern class Calabi–Yau manifolds and positive first Chern class (Fano varieties), was proven by K Hori and C Vafa [88].
[66] It is known that if a linear differential equation with coefficients in $\mathbb{Q}$ is of Picard–Fuchs type, then it also describes an abstract variation of $\mathbb{Q}$-Hodge structures, and it is globally nilpotent [92].

is the crucial point. As a consequence we know that these n-fold integrals can be interpreted as *'periods' of algebraic varieties* and verify globally nilpotent [9] linear differential equations: they are [44, 45, 59] '*derived from geometry*'. However, inside this 'geometry' framework [100] (in the sense of the mathematicians) theoretical physicists are exploring[67] 'special geometries'. These linear differential operators factorize in irreducible operators that are also necessarily globally nilpotent [9]. When one considers all the irreducible globally nilpotent linear differential operators of order N, that we have encountered (or the ones displayed by other authors in an enumerative combinatorics framework [73, 74], or in a more obvious Calabi–Yau framework [71, 89]), it appears that their differential Galois group are not the $SL(N, \mathbb{C})$ or extensions of $SL(N, \mathbb{C})$ groups one could expect generically, but *selected* $SO(N)$, $SP(N, \mathbb{C})$, $G_2, \ldots$ differential Galois group [101]. These are, typically, classification problems in algebraic geometry and/or[68] differential geometry. Our linear differential operators are, in fact, *'special' globally nilpotent* operators (G-operators). This paper can be seen as an attempt, through a fundamental model, the Ising model, to try to characterize the 'additional structures and properties' of these globally nilpotent operators. In the simple example of ${}_pF_q$ generalized hypergeometric functions, only operators with ${}_nF_{n-1}$ solutions [60] can be globally nilpotent. In a hypergeometric framework we are thus trying to see the emergence of 'special' ${}_nF_{n-1}$ hypergeometric functions.

The last results, displayed in this paper, show clearly, with the $SP(4, \mathbb{C})$ differential Galois group of L_4 for $\chi^{(6)}$, that these 'special geometries' *already emerge* on the Ising model which, therefore, *does not restrict to the theory of elliptic curves* [102] (and their associated elliptic functions and modular forms). Defining these 'special geometries' is still a work in progress[69] but it seems to be close to concepts like the concept of modularity (for instance, integrality of series) and other mirror symmetries [103].

11. Conclusion

All the massive calculations we have developed during several years [1, 5–8, 21] on the square lattice Ising model give coherent exact results that do show a lot of remarkable and deep (algebraico-differential) structures. These structures all underline the *deep connection between the analysis of the Ising model and the theory of elliptic functions* (modular forms, selected hypergeometric functions [25], modular curves, etc). In particular we have actually been able to understand almost all the factors obtained in the analysis of the $\tilde{\chi}^{(n)}$'s, as linear differential operators '*associated with elliptic curves*'. Some linear differential operators have a very straightforward relation with elliptic curves: they are homomorphic to symmetric powers of L_E or L_K, the second order operators corresponding to complete elliptic integrals E or K. We showed, in this paper, that the solutions of the second and third order operators Z_2, F_2, F_3, $\tilde{L}_3$ operators *can actually*[70] *be interpreted as modular forms* of the elliptic curve of the Ising model. These results are already quite a 'tour-de-force' and their generalization to the much larger (and involved) operators L_{12} and L_{23} seems out of reach for some time. The understanding of the 'very nature' of the globally nilpotent fourth order operator L_4 was, thus, clearly a very important challenge *to really understand the mathematical nature of the Ising*

[67] Sometimes, without knowing it, like Monsieur Jourdain (Le bougeois gentilhomme).

[68] Given a classification problem in algebraic geometry, using the mirror duality one can translate it into a problem in differential equations, solve this problem and translate the result back into geometry.

[69] For the next $\chi^{(n)}$'s, $n \geqslant 7$, we just know that the corresponding differential operator are globally nilpotent. We can only conjecture the emergence of these 'special geometries', as we already conjectured the integrality of the $\chi^{(n)}$'s series in well-suited variables (see equation (8) in [5]).

[70] At first sight it looks like a simple problem that could be solved using utilities like the 'kovacicsols' command [104] in Maple13: it is not even simple on a second order operator.

model. This has been achieved with the emergence of a *Calabi–Yau equation*, corresponding to a selected $_4F_3$ hypergeometric function, which can also be seen as a *Hadamard product* of the complete elliptic integral[71] K, with a remarkably simple algebraic pull-back (square root extension (113)), the corresponding Calabi–Yau fourth order operator having a *symplectic* differential Galois group $SP(4, \mathbb{C})$. The associated *mirror maps and higher order Schwarzian ODEs* present all the nice physical and mathematical ingredients we had with elliptic curves and modular forms, in particular an exact (isogenies) representation of the generators of the renormalization group, extending the modular group $SL(2, \mathbb{Z})$ to a $GL(2, \mathbb{Z})$ symmetry group.

We are extremely close to achieve our journey 'from Onsager to Wiles' (and now Calabi–Yau, etc), where we will, finally, be able to say that the *Ising model is nothing but the theory of elliptic curves, modular forms and other mirror maps and Calabi–Yau*. Do note that all the ideas, displayed here, *are not specific of the Ising model*[72] and can be generalized to most of the problems occurring in exact lattice statistical mechanics, enumerative combinatorics [73, 78], particle physics, etc (the elliptic curves being replaced by more general algebraic varieties, and the Hauptmoduls being replaced by the corresponding mirror symmetries generalizations [50]). We do hope that these ideas will, eventually yield the emergence of a new algebraic statistical mechanics classifying all the problems of theoretical physics[73] on a completely (effective) algebraic geometry basis.

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Appendix A. Hadamard product of operators depends on the expansion point

The Hadamard product of two operators depends on the point around which the series expansions are performed and, hence, the Hadamard product of the series solutions.

Let us consider the complete elliptic integral K which is already a Hadamard product

$$\frac{2}{\pi} \cdot EllipticK(4 \cdot x^{1/2}) = (1-4x)^{-1/2} \star (1-4x)^{-1/2}, \tag{A.1}$$

and the order-2 linear differential operator for $EllipticK(x^{1/2})$

$$D_x^2 + \frac{(1-2x)}{(1-x)\cdot x} \cdot D_x - \frac{1}{4}\frac{1}{(1-x)\cdot x}. \tag{A.2}$$

[71] Or the Hadamard product of $(1-16w^2)^{-1/2}$, the square root $(1-16w^2)^{1/2}$ playing a fundamental role in the algebraic extension necessary to discover the good pull-back (113), which is a crucial step to find the solution (see (8)).

[72] See [74]. We use the closed formulae for the $\chi^{(n)}$'s as n-fold integrals with *algebraic integrands*, derived from Pfaffian methods [2], but not directly the free-fermion character of the Ising model.

[73] Beyond lattice statistical mechanics [105–107].

The Hadamard square of a linear differential operator *does depend on the point around which the series are performed*. For $EllipticK$ with its three singularities $0, 1, \infty$, the Hadamard square yields the *same* operator of order-4 for the three expansion points $0, 1, \infty$; this order-4 operator corresponds to ${}_4F_3\big(\big[\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\big], [1, 1, 1]; z\big)$, namely

$$D_x^4 + 2\frac{(3-4x)}{(1-x)\cdot x}\cdot D_x^3 + \frac{1}{2}\frac{(14-29x)}{(1-x)\cdot x^2}\cdot D_x^2 + \frac{(1-5x)}{(1-x)\cdot x^3}\cdot D_x - \frac{1}{16}\frac{1}{(1-x)\cdot x^3}, \qquad \text{(A.3)}$$

for the three expansion points $0, 1, \infty$. However, for a generic expansion point $x = c$ (where $c \neq 0, 1/2, 1, \infty$), one gets an *order-6* linear differential operator.

A.1. Hadamard powers generalizations

Let us denote $Had^n(F) = F \star F \star \cdots \star F$, the Hadamard product of F, nth time with itself. Relation (112) can straightforwardly be generalized to arbitrary Hadamard powers of the complete elliptic integral K, which, as we know, plays a crucial role in our analysis of the Ising model [13, 14]:

$${}_{2n}F_{2n-1}\big(\big[\tfrac{1}{2}, \ldots, \tfrac{1}{2}\big], [1, \ldots, 1]; 16^n \cdot z\big) = Had^n\big({}_2F_1\big(\big[\tfrac{1}{2}, \tfrac{1}{2}\big], [1]; 16\cdot z\big)\big). \qquad \text{(A.4)}$$

These hypergeometric functions [108] are solutions of the $2n$th order linear differential operator:

$$\theta^{2n} - 16^n \cdot z \cdot \big(\theta + \tfrac{1}{2}\big)^{2n}. \qquad \text{(A.5)}$$

These relations are a subcase of the (slightly) more general Hadamard power relations

$${}_nF_{n-1}\left(\left[\frac{1}{2}, \ldots, \frac{1}{2}\right], [1, \ldots, 1]; 4^n \cdot z\right) = Had^n\left(\frac{1}{\sqrt{1-4z}}\right);$$

these last hypergeometric functions being solutions of the nth order linear differential operator:

$$\theta^n - 4^n \cdot z \cdot \big(\theta + \tfrac{1}{2}\big)^n. \qquad \text{(A.6)}$$

Do note that the corresponding ${}_{(n)}F_{(n-1)}$ hypergeometric series are, actually, series with *integer coefficients*:

$$\begin{aligned} Had^n\left(\frac{1}{\sqrt{1-4z}}\right) &= 1 + \sum_{k=1}^{\infty}\left(2\cdot\binom{2k-1}{k-1}\right)^n \cdot z^k \\ &= 1 + 2^n\cdot z + 6^n\cdot z^2 + 20^n\cdot z^3 + 70^n\cdot z^4 + 252^n\cdot z^5 + 924^n\cdot z^6 \\ &\quad + 3432^n\cdot z^7 + 12870^n\cdot z^8 + 48620^n\cdot z^9 + \cdots. \end{aligned} \qquad \text{(A.7)}$$

Appendix B. Equivalence of the $J_{k,n}$; equivalence of the $\Omega_{n,m,p;q,r,s,t}$

(i) All the operators $J_{k,n}$ defined by (106) are homomorphic. This can be seen recursively from the two operator equivalences

$$\mathcal{U}_1 \cdot J_{k,n} = J_{k,n+1}\cdot\mathcal{U}, \qquad \mathcal{V}_1\cdot J_{k,n} = J_{k+1,n}\cdot\mathcal{V},$$

where

$$\begin{aligned} \mathcal{U} = \frac{4n}{x}\cdot(1-16x)\cdot\theta^3 &- \frac{1}{x}\cdot(16(8n+6kn+k^2+5n^2)\cdot x - (k+5n)(k+n))\cdot\theta^2 \\ &- (16(k+n+1)(k^2+3kn+5n+2n^2)\cdot\theta \\ &+ \frac{k}{x}\cdot(5n^2+2kn+k^2))\cdot\theta - 4\cdot(k^2+2kn+4n+n^2)(k+n+1)^2, \end{aligned} \qquad \text{(B.1)}$$

$$\mathcal{U}_1 = \mathcal{U} + \frac{4n}{x} \cdot (1-32x) \cdot \theta^2 - \frac{4}{x} \cdot (8(4kn + 8n + 3n^2 + k^2) - kn) \cdot \theta$$
$$- 16 \cdot (k+n+2)(k^2 + 2kn + 4n + n^2) \qquad (B.2)$$

and where $\mathcal{V}$ (resp. $\mathcal{V}_1$) is $\mathcal{U}$ (resp. $\mathcal{U}_1$) where k and n have been permuted.

(ii) Let us now show that the order-4 linear differential operators $\Omega_{n,m,p;q,r,s,t}$, corresponding to the hypergeometric functions (120), are homomorphic.

One has

$$\Omega_{n,m,p;q,r,s,t} \cdot (\theta + n + 1) = (\theta + n + 1) \cdot \Omega_{n+1,m,p;q,r,s,t},$$
$$\Omega_{n,m,p;q,r,s,t} \cdot (\theta + m + 1) = (\theta + m + 1) \cdot \Omega_{n,m+1,p;q,r,s,t},$$
$$\Omega_{n,m,p;q,r,s,t} \cdot (\theta + p + 1) = (\theta + p + 1) \cdot \Omega_{n,m,p+1;q,r,s,t},$$

and

$$\left(\theta + q + \tfrac{3}{2}\right) \cdot \Omega_{n,m,p;q,r,s,t} = \Omega_{n,m,p;q+1,r,s,t} \cdot \left(\theta + q + \tfrac{1}{2}\right),$$
$$\left(\theta + r + \tfrac{3}{2}\right) \cdot \Omega_{n,m,p;q,r,s,t} = \Omega_{n,m,p;q,r+1,s,t} \cdot \left(\theta + r + \tfrac{1}{2}\right),$$
$$\left(\theta + s + \tfrac{3}{2}\right) \cdot \Omega_{n,m,p;q,r,s,t} = \Omega_{n,m,p;q,r,s+1,t} \cdot \left(\theta + s + \tfrac{1}{2}\right),$$
$$\left(\theta + t + \tfrac{3}{2}\right) \cdot \Omega_{n,m,p;q,r,s,t} = \Omega_{n,m,p;q,r,s,t+1} \cdot \left(\theta + t + \tfrac{1}{2}\right).$$

A simple composition of all these elementary relations show, by recursion, that the $\Omega_{n,m,p;q,r,s,t}$'s are all homomorphic.

Appendix C. Getting the higher order Schwarzian ODEs

Let us call $\mathcal{L}_4$ the order-4 linear differential operator corresponding to the Fuchsian ODE of the hypergeometric function:

$$_4F_3([1/2, 1/2, 1/2, 1/2], [1, 1, 1]; 256x). \qquad (C.1)$$

The formal solutions of $\mathcal{L}_4$ are denoted y_3, y_2, y_1 and y_0, where the subscript is for the higher exponent of the log. The nome map reads (see (126))

$$q(x) = \exp\left(\frac{y_1}{y_0}\right) = x + 64x^2 + 7072x^3 + 991232x^4 + \cdots$$

Our aim is to obtain the nonlinear ODE of $q(x)$.

Considering the log nome map as $y_1/y_0 = \ln(q)$ and, differentiating both sides, gives

$$y_1' y_0 - y_1 y_0' = y_0^2 \cdot \frac{q'}{q} \qquad (C.2)$$

where the left-hand side is the Wronskian of the solutions y_1 and y_0. This Wronskian is the solution of an order-5 linear differential operator $\mathcal{L}_5$, which is the exterior square of $\mathcal{L}_4$. We have then three equations

$$\mathcal{L}_5\left(y_0^2 \cdot \frac{q'}{q}\right) = 0, \qquad \mathcal{L}_4(y_0 \cdot \ln(q)) = 0, \qquad \mathcal{L}_4(y_0) = 0, \qquad (C.3)$$

to which we add the following two equations obtained by differentiating the last two equations:

$$D_x \cdot \mathcal{L}_4(y_0 \cdot \ln(q)) = 0, \qquad D_x \cdot \mathcal{L}_4(y_0) = 0. \qquad (C.4)$$

We solve this system of five equations in the unknowns $F^{(n)} = \mathrm{d}^n y_0/\mathrm{d}x^n$, to obtain

$$F^{(n)} = B_n \cdot F^{(0)}, \qquad n = 1, 2, \ldots, 5. \qquad (C.5)$$

The B_n's depend on x, $q(x)$ and its derivatives up to seven. All these relations should be compatible, by which it is meant that the derivative of one relation with n gives the relation with $n+1$. The result is then

$$\frac{\mathrm{d}}{\mathrm{d}x} B_n + B_n \cdot B_1 - B_{n+1} = 0, \qquad n = 1, 2, \ldots, 4. \tag{C.6}$$

The gcd of these four nonlinear differential equations is the ODE of $q(x)$. The nonlinear ODE of $q(x)$ contains the derivatives $q'(x), \ldots, q^{(7)}(x)$ with degrees, respectively, 12, 16, 8, 5, 4, 3, 2 and 1. The nonlinear differential equation of the mirror map (see (127))

$$X(q) = q - 64q^2 + 1120q^3 - 38912q^4 - 1536\,464q^5 + \cdots$$

can be obtained by using $X(q(x)) = x$ and the ODE of $q(x)$. The nonlinear ODE of $X(q)$ involves the derivatives $X^{(0)}(q), X^{(1)}(q), \ldots, X^{(7)}(q)$ with degrees, respectively, 18, 24, 12, 6, 4, 3, 2 and 1.

Appendix D. Higher order Schwarzian ODEs for the hypergeometric function ${}_4F_3([1/2,1/2,1/2,1/2],[1,1,1];256\,z)$

Actually the nonlinear ODE on the mirror map (127) is of the form

$$0 = (1 - 256 \cdot z) \cdot S(z, z', z'', \ldots, z^{(5)}, z^{(6)}, z^{(7)}) + z'^{12} \cdot P(z, z', \ldots, z^{(5)}), \tag{D.1}$$

$$\begin{aligned}
P(z', z, \ldots, z^{(5)}) &= -18z'^3 \cdot (188416z'^4 + 6z'z^{(3)} - 9z''^2) \cdot z^{(5)} \\
&\quad + 1909\,301\,941\,633\,024z'^{12} - 695\,516\,266\,496z'^9 z^{(3)} \\
&\quad + 12\,582\,912\left(82912z''^2 - 345z^{(4)}\right) \cdot z'^8 - 4(z^{(3)})^2)z'^3 \\
&\quad + 26\,046\,627\,840z'^7 z'' z^{(3)} + 2430z'z''^4 z^{(3)} \\
&\quad - 33\,914\,880(768z''^3 - (z^{(3)})^2 - z^{(4)}z'') \cdot z'^6 \\
&\quad - 203\,489\,280z'^5 z''^2 z^{(3)} + 135(1130\,496z''^4 + (z^{(4)})^2)z'^4 \\
&\quad - 180z^{(3)} \cdot (3z^{(4)}z'' - 1620z'^2 z''^2 (z^{(3)})^2 - 1215z''^6.
\end{aligned}$$

This is a nonlinear ODE where the derivatives $z^{(n)}$ are the nth derivatives in $\tau = \ln(q)$ of the mirror map $z(q)$.

Conversely, the ODE on the function $\tau = \ln(q) = \tau(z)$ is an order-7 nonlinear ODE given by the sum of 602 monomial terms:

$$\begin{aligned}
0 &= z^6 \cdot (1 - 256 \cdot z)^2 \cdot P_\tau(z, \tau', \tau'', \ldots, \tau^{(5)}) \cdot \tau^{(7)} + \cdots \\
&\quad + (1 - 256 \cdot z) \cdot Q_\tau(z, \tau, \ldots, \tau^{(6)}, \tau^{(7)}) + R_\tau(z, \tau, \tau', \ldots, \tau^{(5)}) \cdot \tau',
\end{aligned} \tag{D.2}$$

where

$$\begin{aligned}
R_\tau(\tau, \tau', \ldots, \tau^{(5)}) &= 18 \cdot (9\tau''^2 - 6\tau^{(3)}\tau' + 188416\tau'^2) \cdot \tau^{(5)} \\
&\quad - 1043\,274\,399\,744\tau''^2\tau' + 16\,957\,440 \cdot (256\tau' - \tau'') \cdot \tau' \cdot \tau^{(4)} \\
&\quad + 13\,023\,313\,920\tau''^3 + 1909\,301\,941\,633\,024\tau'^3 + 360(\tau^{(3)})^3 \\
&\quad + 135\tau' \cdot (\tau^{(4)})^2 + 4 \cdot (173\,879\,066\,624\tau'^2 - 135\tau''\tau^{(4)} \\
&\quad - 4341\,104\,640\tau''\tau' + 4239360\tau''^2) \cdot \tau^{(3)}.
\end{aligned}$$

The nonlinear ODE (D.2) is *a homogeneous polynomial expression of degree 4 in the seven derivatives* $(\tau', \tau'', \ldots, \tau^{(7)})$.

Rewritten in $q(z)$ the nonlinear ODE is an order-7 nonlinear ODE given by the sum of 2471 monomial terms in $(z, q, q'q'' \dots q^{(7)})$:

$$0 = z^6 \cdot q^6 \cdot (1-256\cdot z)^2 \cdot P_q(z, q', q'', \dots, q^{(5)}) \cdot q^{(7)} + \cdots \\ + (1-256\cdot z)\cdot Q_q(z, q, \dots, q^{(6)}, q^{(7)}) \\ + R_q(z, q, q', \dots, q^{(5)}) \cdot q^6 \cdot q', \tag{D.3}$$

where P_q and Q_q are polynomials of $z, q', q'', \dots, q^{(5)}$ and $z, q', q'', \dots, q^{(7)}$ respectively, and where

$$R_q(z, q, q', \dots, q^{(5)}) = 18q^4 \cdot (188\,416 q^2 q'^2 - 6q'q^{(3)}q^2 + 9q''^2q^2 - 3q'^4)\cdot q^{(5)} \\ + 4341\,104\,640 \cdot (q'^2 q^6 q^{(4)} + q^4 q'' q'^4 - q^3 q'^6) \\ - 1043\,274\,399\,744 q' q^6 q''^2 - 270 q'^5 q^{(4)} q^3 + 135 (q^{(4)})^2 q^6 q' \\ + 695\,516\,266\,496 q'^2 q^6 q^{(3)} + 13\,023\,313\,920 q^6 q''^3 \\ + 347\,758\,133\,248 q'^5 q^4 + 1909\,301\,941\,633\,024 q'^3 q^6 \\ + 13\,565\,952 q'^7 q^2 - 17\,364\,418\,560 q' q^6 q^{(3)} q'' - 36 q'^9 \\ + 1215 q''^4 q' q^4 + 648 q''^2 q'^5 q^2 - 432 q^{(3)} q'^6 q^2 + 360 {q^{(3)}}^3 q^6 \\ + 1620 (q^{(3)} q'' q^3 q'^4 - q''^3 q'^3 q^3 - q^{(3)} q''^2 q^4 q'^2) \\ + 540 \cdot (q'^3 q'' q^4 q^{(4)} - q^{(3)} q'' q^6 q^{(4)}) \\ + 16\,957\,440 \cdot (q^6 q''^2 q^{(3)} - q' q^6 q^{(4)} q'' \\ + q'^4 q^4 q^{(3)} - q'^5 q^3 q'' - q'^3 q^4 q''^2).$$

One can verify that $q = Constant$ is a solution of (D.3). Furthermore expansion (126) multiplied by an arbitrary constant is still a solution of (D.3):

$$q = C_0 \cdot (z + 64z^2 + 7072z^3 + 991\,232z^4 + 158\,784\,976z^5 + \cdots)$$

which is natural since (D.3) is a linear ODE on derivatives of $\ln(q)$.

Remark 4. Such nonlinear ODE is a 'machine' to build series with *integer* coefficients. For instance, if we explore the solutions of (D.3), of the form $q = z^2 + \cdots$, one gets

$$q = z^2 + 128z^3 + 18\,240z^4 + 2887\,680z^5 + 494\,460\,832z^6 \\ + 89\,757\,208\,576z^7 + 17\,035\,431\,116\,800z^8 + 3347\,987\,811\,139\,584z^9 \\ + 676\,624\,996\,390\,235\,600z^{10} + 139\,902\,149\,755\,519\,715\,328z^{11} \\ + 29\,480\,532\,176\,870\,291\,252\,224z^{12} + 6312\,281\,522\,697\,932\,105\,646\,080z^{13} \\ + 1370\,123\,593\,804\,106\,822\,389\,706\,240z^{14} \\ + 300\,913\,725\,420\,989\,219\,840\,662\,110\,208z^{15} \\ + 66\,768\,780\,541\,145\,654\,061\,810\,373\,951\,488z^{16} + \cdots. \tag{D.4}$$

In fact, this new series is *nothing but the square of* (126). Similarly one easily verifies that the cube of (126)

$$q = z^3 + 192z^4 + 33\,504z^5 + 5951\,488z^6 \\ + 1093\,928\,304z^7 + 207\,935\,296\,512z^8 \\ + 40\,712\,043\,902\,464z^9 + 8176\,029\,744\,758\,784z^{10} + \cdots \tag{D.5}$$

is also a solution of (D.3), and this is also true for negative powers of expansion (126), for instance the inverse (in the sense of the multiplication)

$$q = \frac{1}{z} - 64 - 2976z - 348\,160z^2 - 52\,017\,616z^3 - 8802\,913\,280z^4 \\ - 1608\,195\,557\,888z^5 - 309\,505\,032\,060\,928z^6 + \cdots \tag{D.6}$$

is also a solution of (D.3), and similarly, (126) to the power (-2)

$$q = \frac{1}{z^2} + \frac{128}{z} - 1856 - 315\,392z - 50\,614\,176z^2 - 8875\,323\,392z^3 - 1658\,793\,979\,904z^4 + \cdots \qquad \text{(D.7)}$$

is also a solution of (D.3).

This remarkable property, *expression of the renormalization group*[74], is in fact a straight consequence of the fact that the nonlinear ODE (D.2) is *a homogeneous polynomial of degree 4 in the seven derivatives* $(\tau', \tau'', \ldots, \tau^{(7)})$.

Conversely, in the 'mirror', in the nonlinear ODE (D.1), one can change q into $A \cdot q$ since the derivatives are all log derivatives of q:

$$z = A \cdot q - 64 \cdot A^2 \cdot q^2 + 1120 \cdot A^3 \cdot q^3 - 38\,912 \cdot A^4 \cdot q^4 - 1536\,464 \cdot A^5 \cdot q^5 + \cdots, \qquad \text{(D.8)}$$

which is also a solution of (D.1), and, of course, one can easily check the symmetry corresponding to change $q \rightarrow q^n$, the new mirror map series (133)

$$z(q^n) = q^{2n} - 64q^{4n} + 1120q^{6n} - 38912q^{8n} + \cdots \qquad \text{(D.9)}$$

being also a solution of (D.1).

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