Dwork's congruences

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May 16, 2019

p-adic cycles

Consider $E_t : y^2 = x(x-1)(x-t)$. A period,

$$\frac{1}{\pi}\int_1^\infty \frac{dx}{\sqrt{x(x-1)(x-t)}} = \sum_{k=0}^\infty \binom{2k}{k}^2 (t/16)^k.$$

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Theorem (B.Dwork, 1969)

Let p be a prime and $t_0 \in \mathbb{Z}_p$. Suppose $F_p(t_0)$ is not divisible by p. Then the p-adic limit

$$\lambda = (-1)^{(p-1)/2} \lim_{s\to\infty} F_{p^{s+1}}(t_0)/F_{p^s}(t_0)$$

exists and equals a root of the zeta-function of $E(t_0) (\mathrm{mod}\ p)$.

A variation

Define $f(x,y) = y^2 - x(x-1)(x-t_0)$. Define for every positive integer m,

$$\beta_m = \text{coefficient of } (xy)^{m-1} \text{ of } f(x,y)^{m-1}.$$

Explicitly (for those interested),

$$\beta_m = \binom{m-1}{(m-1)/2} \sum_{k=0}^{(m-1)/2} \binom{(m-1)/2}{k}^2 t_0^k \text{ when } m \text{ odd.}$$

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Theorem (formal group theory)

Suppose p does not divide β_p. Then

$$\beta_{p^{s+1}} \equiv \lambda \beta_{p^s} \pmod{p^{s+1}}$$

for all $s \geq 0$.

Newton polytope

Let $f(\mathbf{x}) = \sum_{i=1}^{N} f_i \mathbf{x}^{\mathbf{a}_i}$ be a Laurent polynomial in $\mathbf{x} = x_1, \ldots, x_n$ with coefficients $f_i \in \mathbb{Z}_p$.

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Let Δ be the Newton polytope of f, i.e convex hull of $\{\mathbf{a}_1, \ldots, \mathbf{a}_N\}$
Let Δ° be its interior and $\Delta^\circ_{\mathbb{Z}} = \Delta^\circ \cap \mathbb{Z}$.

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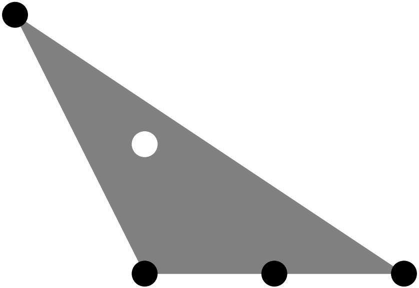
Example for $f(x, y) = y^2 - x(x-1)(x-t)$,



with $\Delta^\circ_{\mathbb{Z}} = \{(1, 1)\}$.

A generalization

Let $g = |\Delta^\circ_{\mathbb{Z}}|$. Define the $g \times g$-matrix β_m by

$$(\beta_m)_{\mathbf{u},\mathbf{v}} = \text{coefficient of } \mathbf{x}^{m\mathbf{v}-\mathbf{u}} \text{ in } f(\mathbf{x})^{m-1}$$

indexed by $\mathbf{u}, \mathbf{v} \in \Delta^\circ_{\mathbb{Z}}$.

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Theorem (M.Vlasenko, 2016)

Let p be a prime and suppose that $\det(\beta_p)$ is not divisible by p. Then there exists a $g \times g$-matrix Λ such that

$$\beta_{p^{s+1}} \equiv \Lambda \beta_{p^s} (\text{mod } p^{s+1})$$

for all $s \geq 0$.

An example

Let us take $f(x) = x^3 - x + 2$ (discriminant is -104).
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We get $\beta_{147} \equiv \begin{pmatrix} 52 & 132 \\ 32 & 96 \end{pmatrix} \pmod{147}$ and

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Similarly,

- $\det(\beta_{163} - \lambda) \equiv \lambda^2 - 1 \pmod{163}$
- $\det(\beta_{151} - \lambda) \equiv \lambda^2 + \lambda + 1 \pmod{151}$

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We get

$$\beta_{47^2} \equiv \begin{pmatrix} 476 & 194 \\ 341 & 1782 \end{pmatrix} \beta_{47} (\mathrm{mod}\ 47^2)$$

and characteristic polynomial: $Q(\lambda) = \lambda^2 + 2160\lambda + 92 (\mathrm{mod}\ 47^2)$.

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Times its reciprocal $Q(47/\lambda)$ gives

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Similarly, for $p = 59$ we get

$$\lambda^4 + 2\lambda^3 + 2\lambda^2 + 59 \cdot 2\lambda (\mathrm{mod}\ 59^2).$$

Regular functions

Sketch of a proof of Vlasenko's result for $f(x,y) = y^2 - x(x-1)(x-t)$ and β_m the 1×1-matrix with element the coefficient of $(xy)^{m-1}$ in $f(x,y)^{m-1}$.

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Consider the $\mathbb{Z}_p$-module Ω_f generated by the functions

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$$\Omega_f/d\Omega_f \cong H^2_{\mathrm{DR}}(\mathbb{T}^2 \setminus E) \cong H^1_{\mathrm{DR}}(E) \cong \mathbb{C}\frac{dx}{y} + \mathbb{C}\frac{xdx}{y}.$$

Formal expansions

We expand

$$\frac{x^r y^s}{(y^2 - x(x-1)(x-16t))^k}$$

formally as

$$\frac{x^r y^s}{y^{2k}} \left(1 - \frac{x(x-1)(x-16t)}{y^2}\right)^{-k}$$

and then as geometric expansion

$$\sum_{m \geq 0} \binom{m+k-1}{m} \times \frac{x^r y^s}{y^{2k}} \times \frac{x^m (x-1)^m (x-16t)^m}{y^{2m}}.$$

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This is contained in set of Laurent series Ω_{formal} of the form

$$\sum_{n/2 < m < 3n/2} a_{mn} \frac{x^m}{y^n}.$$

It gives embedding of Ω_f into Ω_{formal}.

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Lemma (Katz)

$$\sum_{m,n} a_{m,n}x^m y^n \in d\Omega_{\text{formal}} \iff p^{\min(\text{ord}_p(m),\text{ord}_p(n))} | a_{m,n} \text{ for all } m, n.$$

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Indication of proof:

$$x\frac{\partial}{\partial x}\sum a_{m,n}x^m y^n = \sum m a_{m,n} x^m y^n.$$

Clearly $ma_{m,n}$ is divisible by $p^{\mathrm{ord}_p(m)}$.

Finiteness

Theorem (Be-Vlasenko, 2018)

Suppose β_p is not divisible by p. Then the quotient module $\Omega_f / d\Omega_{\mathrm{formal}}$ is generated over $\mathbb{Z}_p$ by

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So, for any $(k-1)!\frac{x^r y^s}{f^k} \in \Omega_f$ there exists $\alpha \in \mathbb{Z}_p$ such that

$$(k-1)!\frac{x^r y^s}{f^k} - \alpha\frac{xy}{f} \in d\Omega_f.$$

Cartier operator

We define the Cartier operator $\mathscr{C}_p : \Omega_{\mathrm{formal}} \to \Omega_{\mathrm{formal}}$ by

$$\mathscr{C}_p : \sum_{m,n} a_{m,n} x^m y^n \mapsto \sum_{m,n} a_{pm,pn} x^m y^n.$$

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Lemma

We have

- $\mathscr{C}_p \circ x\frac{\partial}{\partial x} = px\frac{\partial}{\partial x} \circ \mathscr{C}_p$ and similar for $y\frac{\partial}{\partial y}$.
- $\mathscr{C}_p : d\Omega_{\mathrm{formal}} \to pd\Omega_{\mathrm{formal}}$.

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- $\mathscr{C}_p : d\Omega_{\text{formal}} \to pd\Omega_{\text{formal}}$.
- $\mathscr{C}_p(g(x^p, y^p)h(x,y)) = g(x,y)\mathscr{C}_p(h(x,y))$.

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where $pG(x,y) = f(x^p,y^p) - f(x,y)^p$.

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where $pG(x,y)=f(x^p,y^p)-f(x,y)^p$. Expand in geometric series

$$\begin{aligned}&\mathscr{C}_p\left(\sum_{r=0}^{\infty}p^r\frac{xyf(x,y)^{p-1}G(x,y)^r}{f(x^p,y^p)^{r+1}}\right)\\ =&\sum_{r=0}^{\infty}\frac{p^r}{r!}\frac{r!}{f(x,y)^{r+1}}\mathscr{C}_p(xyf(x,y)^{p-1}G(x,y)^r)\end{aligned}$$

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$$= \sum_{r=0}^{\infty} \frac{p^r}{r!}\frac{r!}{f(x,y)^{r+1}}\mathscr{C}_p(xyf(x,y)^{p-1}G(x,y)^r)$$

The latter sum is in $\widehat{\Omega}_f = \varprojlim \Omega_f/p^s\Omega_f$, the p-adic completion of Ω_f.

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Hence

$$\mathscr{C}_p \left(\frac{xy}{f} \right) \equiv \lambda \frac{xy}{f} \pmod{d\Omega_{\rm formal}}$$

for some $\lambda \in \mathbb{Z}_p$.

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Hence

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for some $\lambda \in \mathbb{Z}_p$.
Some more careful analysis shows that

$$\mathscr{C}_p\left(\frac{xy}{f}\right) = \lambda \frac{xy}{f} + p d\eta$$

with $d\eta$ is a derivative in $d\Omega_{\text{formal}}$

Katz's theorem

From previous slide:

$$\mathscr{C}_p\left(\frac{xy}{f}\right) = \lambda\frac{xy}{f} + pd\eta.$$

Choose integers u, v such that $u/2 < v < 3u/2$ and $s \geq 0$. Take coefficient of $x^{up^s}y^{-vp^s}$ on both sides. Recall that

$$\frac{xy}{f} = \sum_{n/2<m<3n/2} a_{m,n}x^m y^{-n}.$$

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The up^s, vp^s coefficient of $d\eta$ is divisible by p^s.

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The up^s, vp^s coefficient of $d\eta$ is divisible by p^s.We get

Theorem (Katz,1985), case $g = 1$

$$a_{up^{s+1},vp^{s+1}} \equiv \lambda a_{up^s,vp^s} (\mathrm{mod}\ p^{s+1}).$$

Final step

$$\mathscr{C}_p\left(\frac{xy}{f}\right) = \lambda\frac{xy}{f} + pd\eta$$

with $d\eta$ is a derivative in $d\Omega_{\mathrm{formal}}$.
Multiply on both sides by $\frac{f^{p^s}}{(xy)^{p^s}}$ and take the constant term.

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Multiply on both sides by $\frac{f^{p^s}}{(xy)^{p^s}}$ and take the constant term.
Middle term: $\cdot\text{const}\ \frac{f^{p^s-1}}{(xy)^{p^s-1}} = \beta_{p^s}$.

Final step

$$\mathscr{C}_p\left(\frac{xy}{f}\right) = \lambda\frac{xy}{f} + pd\eta$$

with $d\eta$ is a derivative in $d\Omega_{\mathrm{formal}}$.
Multiply on both sides by $\frac{f^{p^s}}{(xy)^{p^s}}$ and take the constant term.
Middle term: $\cdot$const $\frac{f^{p^s-1}}{(xy)^{p^s-1}} = \beta_{p^s}$.
For the left hand term observe that

$$\text{const}\ \frac{f(x,y)^{p^s}}{(xy)^{p^s}}\mathscr{C}_p\left(\frac{xy}{f}\right) = \text{const}\ \mathscr{C}_p\left(\frac{f(x^p,y^p)^{p^s}}{(xy)^{p^{s+1}}}\frac{xy}{f(x,y)}\right).$$

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Modulo p^{s+1} this equals

$$\text{const}\left(\frac{f(x,y)^{p^{s+1}}}{(xy)^{p^{s+1}}}\frac{xy}{f(x,y)}\right) \equiv \beta_{p^{s+1}} (\mathrm{mod}\ p^{s+1}).$$

Final step ct'd

For the last term we get

$$p\frac{f^{p^s}}{(xy)^{p^s}}d\eta \equiv p \cdot d\left(\frac{f^{p^s}}{(xy)^{p^s}}\eta\right) \pmod{p^{s+1}}.$$

The constant term of a derivative is 0.

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Result,

$$\beta_{p^{s+1}} \equiv \lambda\beta_{p^s} \pmod{p^{s+1}}.$$

Conclusion

Recall

Theorem (M.Vlasenko, 2016)

Let p be a prime and suppose that $\det(\beta_p)$ is not divisible by p. Then there exists a $g \times g$-matrix Λ such that

$$\beta_{p^{s+1}} \equiv \Lambda \beta_{p^s} (\mathrm{mod}\ p^{s+1})$$

for all $s \geq 0$.

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Conclusion

With the analysis given above we conclude that Λ is the matrix of the action of $\mathscr{C}_p$ on the rank g module $\Omega_f / d\Omega_{\mathrm{formal}}$.