The Bass-Quillen Conjecture, Swan's Question and the Zariski Uniformization Theorem

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Let R be a regular ring. The projective modules over polynomial algebras over regular local rings started with Serre's Conjecture extended by Bass and Quillen, to be an important subject in Commutative Algebra.

Bass-Quillen Conjecture Every finitely generated projective module P over a polynomial R-algebra, $R[T]$, $T = (T_1, \ldots, T_n)$ is extended from R, that is $P \cong R[T] \otimes_R (P/(T)P)$.

Positive answers were obtained if:

1. $\dim R \leq 1$ by Quillen and Suslin (1976),
2. R is essentially of finite type over a field by H. Lindel (1981),
3. $(R, \mathfrak{m}, k)$ is a regular local ring of unequal characteristic, essentially of finite type over $\mathbf{Z}$, with $p = \operatorname{char} k \notin \mathfrak{m}^2$ by Swan (1982).

Swan noticed that it will be useful for the general problem to have a positive answer to the following question:
Swan's Question, 1982 A regular local ring is a filtered inductive limit of regular local rings, essentially of finite type over $\mathbf{Z}$.

A partial positive answer to the above question is the following theorem.

Theorem 1

(P, 1989) Swan's question has a positive answer for a regular local ring $(R, \mathfrak{m}, k)$ in the following cases:

1. $p = \text{char } k \notin \mathfrak{m}^2$,
2. *R contains a field,*
3. *R is excellent Henselian.*

Using the results of Lindel and Swan one can get the following:

Corollary 2

(P, 1989) The BQ Conjecture is true in the following cases:

1. $p = \mathit{char}\ k \notin \mathfrak{m}^2$,
2. *R contains a field,*

Recently we gave a complete answer to Swan's Question and we expected to have some progress on the BQ Conjecture more precisely to the following question:

Question P Let $(R, \mathfrak{m})$ be a regular local ring, essentially smooth over $\mathbf{Z}_{(p)}$ and $b \in \mathfrak{m}^2$. Then is it true the BQ Conjecture for the regular local ring $R/(p-b)$?

Why this question? When R contained a field, Lindel showed that R is a regular local $\mathbf{Z}$-algebra, essentially smooth and it was easy to show that for those the BQ Conjecture holds. When $0 \neq p :=$ char $R/\mathfrak{m} \in \mathfrak{m}^2$ then it was not clear how looks a regular local $\mathbf{Z}$-algebra, essentially of finite type. But now an information is given by the following:

Theorem 3

(P, [4, Theorem 17]) Every regular local ring is a filtered inductive limit of regular local rings, essentially smooth over a regular local $\mathbf{Z}$*-algebra of type* $A/(p-b)$*, where* $(A, \mathfrak{a})$ *is a regular local ring, essentially smooth over* $\mathbf{Z}$*, and* $b \in \mathfrak{a}^2$*.*

Corollary 4

All regular local rings are filtered inductive limits of excellent regular local rings.

This corollary was useful to Kestutis Cesnavicius to reduce the purity conjecture to the case of regular local rings which are complete.

Corollary 5

(P) Let B be a regular local $\mathbf{Z}$*-algebra essentially of finite type. Then B is essentially smooth over a regular local* $\mathbf{Z}$*-algebra of type* $A/(p-b)$*, where* $(A, \mathfrak{a})$ *is a regular local ring, essentially smooth over* $\mathbf{Z}$*, and* $b \in \mathfrak{a}^2$*.*

Now we see how the above question could be applied to the BQ Conjecture. We need the following two Lemmas

Lemma 6

Let R be a regular local ring, a filtered inductive limit of regular local rings $(R_i)_{i\in I}$. If the BQ Conjecture holds for all R_i, $i \in I$ then it holds for R too.

Next lemma follows Lindel's idea.

Lemma 7

Let $R \to R'$ be an essentially smooth morphism of regular local rings. If the BQ Conjecture holds for R then it holds for R' too.

Proof.

It results that R' is an etale neighborhood of a localization A of a polynomial R-agebra. If the BQ Conjecture holds for R it holds also for A by a result of Roitman. To pass from A to R' one should apply a result of Lindel. $\square$

Theorem 8

(P) If the above question has a positive answer then the BQ Conjecture holds for all regular local rings.

Proof.

With a so called Quillen's Patching Theorem we reduce to show the BQ Conjecture only for the regular local rings. Let $(R, \mathfrak{m}, k)$ be a regular local ring. Using Corollary 2 we may assume that $0 \neq p :=\text{char } k \in \mathfrak{m}^2$.
By Theorem 3 R is a filtered inductive limit of regular local rings of type B as above. Then the BQ Conjecture holds for A from Lemma 7 and for $A/(p-b)$ if the above question has a positive answer. Applying again Lemma 7 we get the BQ Conjecture for B and so also for R with Lemma 6. $\square$

A theorem related to Swan's Question is the following weaker form of **Zariski's Uniformization Theorem**.

Theorem 9

A valuation ring $V \supset \mathbf{Q}$ is the filtered inductive union of its regular local subrings essentially of finite type over $\mathbf{Q}$.

This theorem was useful for example for Desingularization. In particular, this says that V is the filtered inductive union of its smooth sub-$\mathbf{Q}$-algebras. For us this said that among non Noetherian local rings there exist some good ones which are filtered inductive limits of regular local rings. We have two candidates: the valuation rings and the ultrapowers of regular local rings with respect to a non principal ultrafilter on $\mathbf{N}$.

Meanwhile, together with Kestutis Cesnavicius we showed

Theorem 10

(Cesnavicius-P [1]) Let $V \subset V'$ be an extension of valuation rings containing $\mathbf{Q}$. Then V' is a filtered inductive limit of smooth V-algebras if and only if the following conditions hold:

1. *each prime ideal of V extends to a prime ideal of V',*
2. *for any prime ideals $q \in \operatorname{Spec} V$ and $q' \in \operatorname{Spec} V'$ such that $\operatorname{height}(qV'/q') = 1$ the extension $V_q/(q' \cap V)V_q \subset V'_{qV'}/q'V'_{qV'}$ of one dimensional valuation rings has the same value group*

In 1984 [2] we state this theorem in the case dim $V < \infty$ and by mistake without the condition ii). The above theorem gives in particular a weaker form of Zariski's unformization theorem, that is the smooth V-algebras are not necessarily subrings of V'. We also wanted to have such a weak form of Zariski's uniformization theorem in characteristic $p > 0$ but in 1987 [3] we had an example, inspired by the work of Ostrowski, of an immediate algebraic extension of valuation rings, that is with the same residue field and the same value groups, which was not dense as it is in characteristic zero. This happens since in characteristic $p > 0$, the Defektless Theorem of Ostrowski does not hold. So in spite of the work of Abhyankar and others we are not convinced that the Desingularization in characteristic $p > 0$ could hold in general.

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