The Jacobian Conjecture, a reduction of the degree via a Combinatorial Physics approach

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arXiv:1411.6558[math.AG], *Annales Henri Poincaré (2016)*

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Plan

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Jacobian Conjecture

strinkingly simple and natural conjecture

(a metro/tram ticket (size) conjecture)

"high school algebra"

strinkingly simple and natural conjecture

(a metro/tram ticket (size) conjecture)

"high school algebra"

O. Keller, *Monats. Math. Phys.* (1939)
(for $n = 2$ and polynomials with integral coefficients)

<u>Jacobian Conjecture (JC_n):</u>
Let $n \geq 1$. If a polynomial function $F : \mathbb{C}^n \to \mathbb{C}^n$ has Jacobian determinant which is a non-vanishing constant, then the function F has a polynomial inverse.

Example.
$n = 2,\ F(z_1, z_2) = (z_1 + z_2^3, z_2)$ and $F^{-1}(z_1, z_2) = (z_1 - z_2^3, z_2)$.

$$\text{Jacobian} = \det \begin{pmatrix} \frac{dF_1}{dz_1} & \frac{dF_1}{dz_2} \\ \frac{dF_2}{dz_1} & \frac{dF_2}{dz_2} \end{pmatrix} = det \begin{pmatrix} 1 & 3z_2^2 \\ 0 & 1 \end{pmatrix} = 1$$

Relations to the Dixmier conjecture

J. Dixmier, *Bull. Soc. Math. France* (1968)

Dixmier conjecture (DC_n):
Any endomorphism of the $n-$th Weyl algebra (the algebra of polynomial differential operators in n variables) is invertible.

$DC_n \Rightarrow JC_n$

A. Belov-Kanel and M. Kontsevitch, *Moscow Math. J.* (2007)

The Jacobian conjecture in dimension $2n$ implies the Dixmier conjecture in rank n.

Back to the Jacobian conjecture - faulty proofs

- before 1982 (see H. Bass *et. al.*, *Bull. Amer. Math. Soc.*)
  - W. Engel (*Math. Ann.* ('55)) claimed to prove the case $n = 2$. A. Vitushkin (1975) published 2 essential errors
  - B. Segre published 3 incomplete proofs ('56,'57,'60) ; Canals and Lluis ('70) noted an error. Abhyankar and Moh pointed out a faul in Segre's proof and also in Canals and Lluis's correction
  - Gröbner proposed a proof in '61. Zariski pointed out that the argument is faulty
  - Oda in '80 proposed a proof - false
  - *etc.*
- after 1982 : other faulty proofs ...

A. van den Essen, "Polynomial automorphisms and the Jacobian conjecture", Birkhäuser (2000)

Internet blog: "How not to prove the Jacobian conjecture" :)

Back to the Jacobian conjecture - some notations

$$J_F(z) = \left(\frac{\mathrm{d}}{\mathrm{d}z_i} F_j(z)\right)_{1\leq i,j\leq n}$$

$\mathcal{P}_n$ - the set of polynomial systems F (all its coordinate functions F_j ($j = 1, \ldots, n$) are polynomials)

$$\mathcal{J}_n^{\mathrm{lin}} := \{F \in \mathcal{P}_n \,|\, \det J_F(z) = c \in \mathbb{C}^\times\},$$
$$\mathcal{J}_n := \{F \in \mathcal{P}_n \,|\, F \text{ is invertible}\}.$$

Jacobian Conjecture (JC_n):

$$\mathcal{J}_n^{\mathrm{lin}} = \mathcal{J}_n \qquad \forall n\,.$$

Some more notations

$\deg(F) := \max_j \deg(F_j(z))$,

$$
\begin{aligned}
\mathcal{P}_{n,d} &:= \{F \in \mathcal{P}_n \mid \deg(F) \leq d\} \\
\mathcal{J}^{\mathrm{lin}}_{n,d} &:= \{F \in \mathcal{P}_n \mid \det J_F(z) = c \in \mathbb{C}^\times,\ \deg(F) \leq d\} \\
\mathcal{J}_{n,d} &:= \{F \in \mathcal{P}_n \mid F \text{ is invertible},\ \deg(F) \leq d\}
\end{aligned}
$$

Some (substantial?) progress

1. theorem for the quadratic case

Theorem

(S. Wang, J. Alg. (1980))

$$\mathcal{J}^{\text{lin}}_{n,2} = \mathcal{J}_{n,2} \qquad \forall n\,.$$

2. reduction theorem to the cubic case

Theorem

(H. Bass et. al., Bull. Am. Math. Soc. (1982))

$$\mathcal{J}^{\text{lin}}_{n,3} = \mathcal{J}_{n,3} \quad \forall n \qquad \Longrightarrow \qquad \mathcal{J}^{\text{lin}}_{n} = \mathcal{J}_{n} \quad \forall n\,.$$

$F_i(z) = z_i +$ *homogeneous pol. of degree* 3.

A remark on H. Bass *et. al.* proof

The proof of H. Bass *et. al.* involves manipulations under which the dimension n is increased, thus this proof does *not* imply the corresponding statement without the "$\forall n$" quantifier, i.e. that

$$\mathcal{J}_{n,3}^{\mathrm{lin}} = \mathcal{J}_{n,3} \ \Rightarrow \ \mathcal{J}_{n}^{\mathrm{lin}} = \mathcal{J}_{n}.$$

Our result - a further reduction of the degree; notations

For $n' \leq n$ and $F \in \mathcal{P}_{n,d}$, we write

$$z = (z_1, z_2)$$

and

$$F = (F_1, F_2)$$

to distinguish components in the two subspaces

$$\mathbb{C}^{n'} \times \mathbb{C}^{n-n'} \equiv \mathbb{C}^n.$$

We set

$$R(z_2; z_1) = F_2(z_1, z_2),$$

emphasizing that, in R, we consider z_2 as the variables in a polynomial system, and z_1 as parameters.
The invertibility of R, denoted by $R(\cdot; z_1) \in \mathcal{J}_{n-n',d}$, for a fixed z_1, means that there exists a pol. R^{-1} with variables $y_2 \in \mathbb{C}^{n-n'}$, and depending on z_1, s. t.

$$\forall z_2 \in \mathbb{C}^{n-n'}, \quad R^{-1}(R(z_2; z_1); z_1) = z_2.$$

A few more notations

We define the subspaces of $\mathcal{P}_{n,d}$:

$$\mathcal{J}_{n,d;n'} := \{F \in \mathcal{P}_{n,d} \,|\, R(\cdot; z_1) \in \mathcal{J}_{n-n',d} \ \forall z_1 \in \mathbb{C}^{n'} \text{ and } F^{-1} \text{ restricted to } \mathbb{C}^{n'} \times \{0\} \text{ is in } \mathcal{P}_{n'}\}$$

$$\mathcal{J}^{\mathrm{lin}}_{n,d;n'} := \{F \in \mathcal{P}_{n,d} \,|\, R(\cdot; z_1) \in \mathcal{J}_{n-n',d} \ \forall z_1 \in \mathbb{C}^{n'} \text{ and } (\det J_F)(z_1, R^{-1}(0, z_1)) = c \in \mathbb{C}^{\times}, \ \forall z_1 \in \mathbb{C}^{n'}\}$$

generalizations of $\mathcal{J}_{n,d}$ and resp. $\mathcal{J}^{\mathrm{lin}}_{n,d}$

linear subspace of dimension $n - n'$ (the last $n - n'$ variables) on which z vanishes

QFT-inspired choices - they should become clear in the sequel

Our result - a further reduction of the degree

A. de Goursac *et. al.*, *Annales Henri Poincaré* (2016)

Theorem

For $n \in \mathbb{N}$ *and* $d \geq 3$*, there exists an injective map* $\Phi : \mathcal{P}_{n,d} \to \mathcal{P}_{n(n+1),d-1}$ *satisfying*

$$\Phi(\mathcal{J}^{\text{lin}}_{n,d}) \equiv \mathcal{J}^{\text{lin}}_{n(n+1),d-1;n} \cap \text{Im}(\Phi)\,; \quad \Phi(\mathcal{J}_{n,d}) \equiv \mathcal{J}_{n(n+1),d-1;n} \cap \text{Im}(\Phi)\,,$$

where $\text{Im}(\Phi) = \Phi(\mathcal{P}_{n,d})$.

A first consequence

Combining Bass *et. al.* theorem and the theorem above, the full Jacobian Conjecture reduces to the question whether

$$\mathcal{J}^{\mathrm{lin}}_{n(n+1),2;n} \cap \mathrm{Im}(\Phi) = \mathcal{J}_{n(n+1),2;n} \cap \mathrm{Im}(\Phi).$$

- this question seems as difficult as the original Jacobian conjecture ...
- it involves only a quadratic degree, and this might simplify the resolution, in the light of Wang Theorem

A stronger version of JC

Conjecture

For all $n \geq n' \geq 0$, and all $d \geq 1$,

$$\mathcal{J}^{\mathrm{lin}}_{n,d;n'} = \mathcal{J}_{n,d;n'}\,.$$

JC follows from the above conjecture

(0−dim.) Quantum Field Theory in a nutshell

A theory defined by means of a (functional) integral representation of the partition function, in which the fields are linearly coupled to sources;

from this, all the correlation functions of the respective physical system can be obtained by (functional) differentiation

A. Abdsselam, *Sém. Loth. Comb.* (2002),

A. Tanasă, *Sém. Loth. Comb.* (2012)

0−dim. Quantum Field Theory in a nutshell

Usually in QFT, the fields φ_i are functions of space(-time) ($\mathbb{R}^D$)
$D = 0$
($D \neq n$, the dimension of the linear system $F(z)$)
the scalar field φ_i is not a function of space-time
(there is no space-time)!
φ_i is a (real or complex) variable

partition function (generating function)

$$Z = \int_{\mathbb{R}} d\varphi \, e^{-\frac{1}{2}\varphi^2 + \frac{\lambda}{4!}\varphi^4}.$$

λ - the coupling constant
the quadratic part + interaction non-quadratic (here quartic) part

In 0−dim. QFT, the functional integral become usual (real or complex) integrals!

Combinatorial QFT

QFT: sources

One (still) needs to evaluate integrals of type

$$\frac{\lambda^n}{n} \int d\varphi \, e^{-\varphi^2/2} \left(\frac{\varphi^4}{4!} \right)^n .$$

one can (still) use standard QFT techniques:
$Z_0(J) := \int d\varphi \, e^{-\varphi^2/2 + J\phi}$
J - the source
computations of $(2k)-$point correlation functions:

$$\int d\phi \, e^{-\phi^2/2} \varphi^{2k} = \frac{\partial^{2k}}{\partial J^{2k}} \int d\varphi \, e^{-\varphi^2/2 + J\varphi} |_{J=0} = \frac{\partial^{2k}}{\partial J^{2k}} e^{J^2/2} |_{J=0}.$$

QFT - perturbation theory and Feynman graphs

perturbation theory - formal series in λ

$\rightarrow$ (abstract) Feynman graphs and Feynman integrals

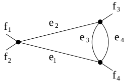
(use of Wick Theorem)

A. Zvonkine, *Math. and Computer Modelling* (1997)

the quadratic part $\rightarrow$ edges

the interaction part $\rightarrow$ vertices

example:



(related to the physical information of a theory - interactions of elementary particle (in colliders a. s. o.))

0−dimensional QFT - interesting "laboratories" for testing theoretical physics tools

V. Rivasseau and Z. Wang, *J. Math. Phys.* (2010)

The intermediate field method in QFT - the idea

idea: introduing a new field, σ, to rewrite the interaction

the degree of the interaction has been reduced!

example: φ^6 model

$$Z(\lambda) = \int \frac{d\varphi}{\sqrt{2\pi}} e^{-\frac{1}{2}\varphi^2} e^{-\lambda\varphi^6} = \int \frac{d\varphi}{\sqrt{2\pi}} e^{-\frac{1}{2}\varphi^2} \int \frac{d\sigma}{\sqrt{2\pi}} e^{-\frac{1}{2}\sigma^2} e^{\imath\sqrt{2\lambda}\varphi^3\sigma}.$$

JC as a QFT model - the Abdesselam-Rivasseau model

A. Abdesselam, *Annales H. Poincaré* (2013)

$F \in \mathcal{P}_{n,d}$.

$$F_i(z) = z_i - \sum_{k=2}^{d} \sum_{j_1,\ldots,j_k=1}^{n} w^{(k)}_{i,j_1\ldots j_k} z_{j_1} \ldots z_{j_k} =: z_i - \sum_{k=2}^{d} W_i^{(k)}(z)\,,$$

for $i \leq n$ and $w^{(k)}_{i,j_1\ldots j_k}$ some coefficients (the coupling constants)

JC as a QFT model - the Abdesselam-Rivasseau model

A. Abdesselam, *Annales Henri Poincaré* (2013)

the partition function

$$Z(J,K) = \int_{\mathbb{C}^n} \mathrm{d}\varphi \mathrm{d}\varphi^\dagger e^{-\varphi^\dagger\varphi + \varphi^\dagger \sum_{k=2}^{d} W^{(k)}(\varphi) + J^\dagger\varphi + \varphi^\dagger K},$$

where J, K are vectors in $\mathbb{C}^n$ (the sources)
measure: $\mathrm{d}\varphi \mathrm{d}\varphi^\dagger := \prod_{i=1}^{n} \frac{\mathrm{dRe}\varphi_i \, \mathrm{dIm}\varphi_i}{\pi}$
$\varphi^\dagger K := \sum_{i=1}^{n} \varphi_i^\dagger K$, a. s. o.

setting the coupling constants to zero (free theory), the partition function is calculated by Gaussian integration:

$$\int_{\mathbb{C}^n} \mathrm{d}\varphi \mathrm{d}\varphi^\dagger e^{-\varphi^\dagger\varphi + J^\dagger\varphi + \varphi^\dagger K} = e^{J^\dagger K}.$$

very particular combinatorics of this QFT model

JC as a QFT model - the Abdesselam-Rivasseau model

- The partition function Z coincides with the inverse of the Jacobian
- The inverse G of F corresponds to the (standard) $1-$point correlation function:

$$G_i(u) = \frac{\int_{\mathbb{C}^n} \mathrm{d}\varphi \mathrm{d}\varphi^\dagger \varphi_i e^{-\varphi^\dagger \varphi + \varphi^\dagger \sum_{k=2}^{d} W^{(k)}(\varphi) + \varphi^\dagger u}}{\int_{\mathbb{C}^n} \mathrm{d}\varphi \mathrm{d}\varphi^\dagger e^{-\varphi^\dagger \varphi + \varphi^\dagger \sum_{k=2}^{d} W^{(k)}(\varphi) + \varphi^\dagger u}} \qquad (1)$$

The sets of polynomial functions involved in JC can be rephrased in this framework:

$$\mathcal{J}_{n,d}^{\mathrm{lin}} = \{F \in \mathcal{P}_{n,d} \,|\, Z(0,u) = 1,\ \forall u \in \mathbb{C}^n\},$$
$$\mathcal{J}_{n,d} = \{F \in \mathcal{P}_{n,d} \,|\, G_i(u) \text{ given by (1) is in } \mathcal{P}_n\}.$$

The intermediate field method for the JC QFT model

- This will reduce the degree d of F.
- We will add n^2 "intermediate fields" σ_{ij} to the model.

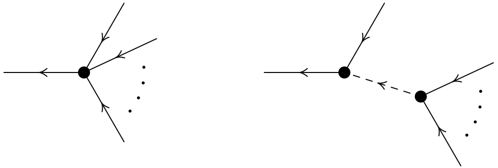
$i, j = 1, \ldots, n$

- intermediate field identity

Using the general formula of Gaussian integration, one has:

$$e^{(\varphi_i^\dagger \varphi_j)\left(\sum_{j_2,\ldots,j_d=1}^{n} w^{(d)}_{i,j,j_2\ldots j_d} \varphi_{j_2}\ldots\varphi_{j_d}\right)}$$

$$= \int_{\mathbb{C}^{n^2}} d\sigma_{i,j} d\sigma^\dagger_{i,j} e^{-\sigma^\dagger_{i,j}\sigma_{i,j} + \sigma^\dagger_{i,j}\left(\sum_{j_2,\ldots,j_d=1}^{n} w^{(d)}_{i,j,j_2\ldots j_d} \varphi_{j_2}\ldots\varphi_{j_d}\right) + (\varphi_i^\dagger \varphi_j)\sigma_{i,j}}$$



The intermediate field method for the JC QFT model

use the intermediate field identity for each pair (i,j), in the partition function $Z(J,K)$ of the JC QFT model with n dimensions and degree d, in order to to re-express the monomials of degree d in the fields φ

$$\Longrightarrow Z(J,K) = \int_{\mathbb{C}^n} \mathrm{d}\varphi \mathrm{d}\varphi^\dagger \int_{\mathbb{C}^{n^2}} \mathrm{d}\sigma \mathrm{d}\sigma^\dagger e^{-\varphi^\dagger\varphi + \varphi^\dagger \sum_{k=2}^{d-1} W^{(k)}(\varphi) + J^\dagger\varphi + \varphi^\dagger K}$$

$$e^{\sum_{i,j=1}^{n}\left(-\sigma_{i,j}^\dagger \sigma_{i,j} + \sigma_{i,j}^\dagger \sum_{j_2,\ldots,j_d=1}^{n} w_{i,j,j_2\ldots j_d}^{(d)} \varphi_{j_2}\ldots\varphi_{j_d} + \varphi_i^\dagger \varphi_j \sigma_{i,j}\right)}.$$

Setting this proper

1. We define the new fields $\phi \in \mathbb{C}^{n+n^2}$ by
$\phi = (\varphi_1, \ldots, \varphi_n, \sigma_{1,1}, \ldots, \sigma_{1,n}, \cdots, \sigma_{n,1}, \ldots, \sigma_{n,n})$.
2. We define the coupling constants $\tilde{w}$ as:
   - for $k = d-1$, we set $\tilde{w}^{(d-1)}_{i,j,j_2\ldots j_d} := w^{(d-1)}_{i,j,j_2\ldots j_d}$ and $\tilde{w}^{(d-1)}_{i\cdot n+j,j_2\ldots j_d} = w^{(d)}_{i,j,j_2\ldots j_d}$ with $i, j, j_2, \ldots j_n \le n$
   - for $k \in \{3, \ldots, d-2\}$, we set $\tilde{w}^{(k)}_{i,j,j_2\ldots j_k} := w^{(k)}_{i,j,j_2\ldots j_k}$ with $i, j, j_2, \ldots j_n \le n$
   - for $k = 2$, we set $\tilde{w}^{(2)}_{i,j,j_2} := w^{(2)}_{i,j,j_2}$ and $\tilde{w}^{(2)}_{i,j,i\cdot n+j} = 1$ with $i, j, j_2 \le n$.

   The remaining coefficients of $\tilde{w}$ are set to 0.
3. The sources are defined to be $\tilde{J} := (J, 0)$ and $\tilde{K} := (K, 0)$, (the number of extra vanishing coordinates is n^2).

The resulting QFT model

1. The partition function:

$$Z(J,K)=\int_{\mathbb{C}^{n+n^2}}\mathrm{d}\phi\mathrm{d}\phi^\dagger e^{-\phi^\dagger\phi+\phi^\dagger\sum_{k=2}^{d-1}\tilde{W}^{(k)}(\phi)+\tilde{J}^\dagger\phi+\phi^\dagger\tilde{K}}$$

2. The $1-$point correlation functions:

$$G_i(u)=\frac{\int_{\mathbb{C}^{n+n^2}}\mathrm{d}\phi\mathrm{d}\phi^\dagger\phi_i e^{-\phi^\dagger\phi+\phi^\dagger\sum_{k=2}^{d-1}\tilde{W}^{(k)}(\phi)+\phi^\dagger\tilde{u}}}{\int_{\mathbb{C}^{n+n^2}}\mathrm{d}\phi\mathrm{d}\phi^\dagger e^{-\phi^\dagger\phi+\phi^\dagger\sum_{k=2}^{d-1}\tilde{W}^{(k)}(\phi)+\phi^\dagger\tilde{u}}},$$

for $i\in\{1,\dots,n\}$.

So, what have we showed?

The partition function (resp. the $1-$point correlation function) of the JC QFT model with dimension n and degree d is equal to the partition function (resp. the n first coordinates of the $1-$point correlation function) of the model with dimension $n(n+1)$ and degree $d-1$, up to a redefinition of

1. the fields
2. the coupling constant $w \mapsto \tilde{w}$
3. the sources.

Since the partition function corresponds to the inverse of the Jacobian (resp. the $1-$point correlation function corresponds to the formal inverse), *this gives a QFT proof of the theorem.*

However ...

This is not a proof of the Jacobian Conjecture (unfortunately)!

Alternative proof

In

A. de Goursac *et. al. Annales H. Poincaré* (2016)

algebraic (no-QFT-like) proof of our reduction result

For further reading

various purely combinatorial approaches to JC were given:

- proposition of Joyal's combinatorial species as a tool

  D. Zeilberger (1987)

- reformulation of the JC using trees

  D. Wright (1999)

- reformulation of the JC using rooted trees

  D. Singer, *Electron. J. Comb.* (2011)

- *etc.*

Conclusion and perspectives

- We have proved, using QFT-inspired techniques, a reduction theorem to the quadratic case for JC, up to the addition of a new parameter n' (related to the introduction of additional intermediate fields σ)

- immediate perspective: adaptation of Wang's proof to our modified quadratic case
- reformulation of Wang's proof in a QFT language

- revisit the Dixmier Conjecture from the perspective of non-commutative QFT

Thank you for your attention!

Vă mulțumesc pentru atenție!