

Explicit computations of the radius of convergence for first order differential equations

*From a Baldassarri's question,
with the decisive Pulita's contribution,*

but mistakes, I did it alone.

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Berkovich points

and Dwork's generic points

Constant fields

- Let K be a complete extension of \mathbb{Q}_p , $|p| = p^{-1}$
 \bar{K} the residue field, $|K^*|$ the absolute values group,
 K^{alg} the algebraic closure of K , \hat{K}^{alg} its completion.
- Let Ω be a “big enough” extension of K , namely
 $\bar{\Omega}/\bar{K}$ is transcendental, $|\Omega^*| = \mathbb{R}^{>0}$,
 Ω spherically complete and algebraically closed.

For $t \in \Omega$ let us define :

- the *multiplicative semi-norm* $|\cdot|_t$ over $K[x]$ by $|P|_t \stackrel{\text{def}}{=} |P(t)|$,
- the *radius* of t by

$$\rho(t) \stackrel{\text{def}}{=} \inf_{f \in K[x] \neq 0} \frac{|f|_t}{\left| \frac{d}{dx} f \right|_t} = d(t, \hat{K}^{\text{alg}}) = d(t, K^{\text{alg}}).$$

Berkovich points

Let $\mathcal{M}(K)$ be the set of multiplicative semi-norm on $K[x]$.

- Elements of $\mathcal{M}(K)$ are called *Berkovich points* of \mathbb{A}_K^1 .
- Then $\mathcal{M}(K) \cup \{\infty\}$ is the set of Berkovich points of \mathbb{P}_K^1 .

Theorem

There is an isomorphism :

$$\begin{array}{ccc} \Omega/\text{Gal}^{\text{cont}}(\Omega/K) & \xrightarrow{\sim} & \mathcal{M}(K) \\ t & \longmapsto & |\cdot|_t \end{array}$$

Remark on notations

We will often write abusively t instead of $|\cdot|_t$ confusing t and its image modulo $\text{Gal}^{\text{cont}}(\Omega/K)$.

Dwork's generic points

For $a \in K^{\text{alg}}$ and $\rho > 0$,

$$\sup_{x \in D(a, \rho^-)} |P(x)| = |P|_{t_{a, \rho}}$$

if and only if $t_{a, \rho} \in \Omega$ satisfies

$$|t_{a, \rho} - a| = \rho = \rho(t_{a, \rho}) = d(t_{a, \rho}, K^{\text{alg}}).$$

“Tree” structure on $\mathcal{M}(K) \cup \{\infty\}$

$\mathcal{M}(K)$ is a “quasi-polyhedron” :

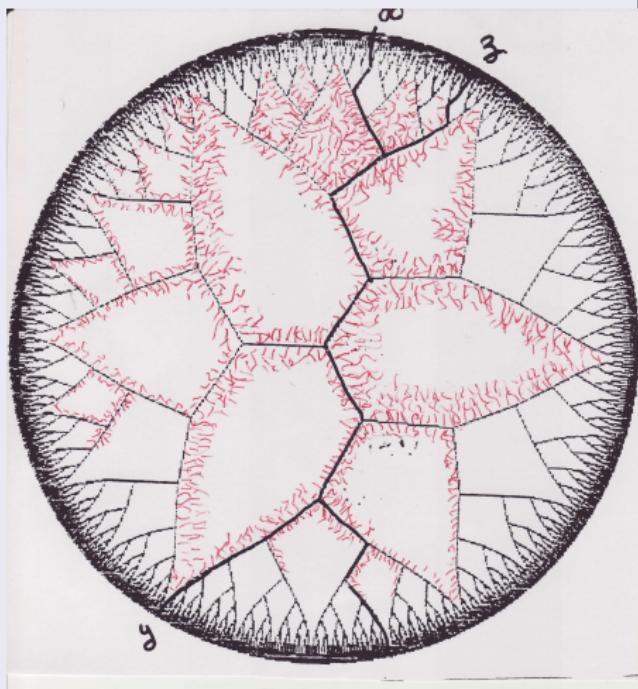
- “branches” made of paths

$$]a, \infty[\stackrel{\text{def}}{=} \left\{ |\cdot|_{t_{a, \rho}} \right\}_{\rho \in]0, \infty[} \quad (a \in K^{\text{alg}})$$

- “leaves” = $\widehat{K}^{\text{alg}} / \text{Gal}^{\text{cont}}(\widehat{K}^{\text{alg}} / K)$

... and “points of fourth kind”.

Example : $\mathcal{M}(\mathbb{Q}_2) \cup \{\infty\}$



Function defined by a sub-tree

Let $\mathfrak{T} \subset \mathcal{M}(K)$ be a sub-tree and $\mathfrak{p} : \overrightarrow{\text{Edges}}(\mathfrak{T}) \rightarrow \mathbb{R}$, $\mathfrak{p}(\overleftarrow{\mathfrak{b}}) = -\mathfrak{p}(\overrightarrow{\mathfrak{b}})$.

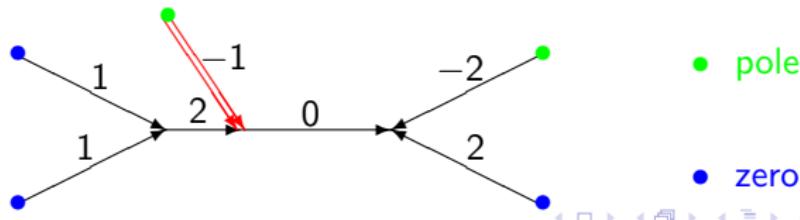
Then $\phi_{\mathfrak{T}, \mathfrak{p}} : \mathcal{M}(K) \rightarrow [0, \infty]$ is defined (up to a constant) by

- For paths \mathfrak{b} meeting \mathfrak{T} in at most one point, $\phi_{\mathfrak{T}, \mathfrak{p}}$ is constant on \mathfrak{b} .
- For $\overrightarrow{\mathfrak{b}} = \overrightarrow{[t_{a,r}, t_{a,R}]} \in \overrightarrow{\text{Edges}}(\mathfrak{T})$, $\phi_{\mathfrak{T}, \mathfrak{p}}(t_{a,\rho}) = C^{\text{ste}} \rho^{\mathfrak{p}(\overrightarrow{\mathfrak{b}})}$ $\rho \in [r, R]$.

Basic example : the function $t \mapsto |f|_t$ for $f \in K(x)$

$\phi_{\mathfrak{T}_f, \mathfrak{p}_f}(t) = C^{\text{st}} |f|_t$, when $\mathfrak{T}_f = \text{tree}(\text{zeros and poles of } f)$

and $\mathfrak{p}_f([\overrightarrow{t_{a,r}, t_{a,R}}]) = \#\text{zeros} - \#\text{poles of } f \text{ in } D(a, r^+) \quad (r < R)$.



Function Radius (of convergence)

Definition and properties

Definition

Various radii associated to the differential equation $y' = f y$

For $f \in K(x)$ let us define recursively : $f_0 \stackrel{\text{def}}{=} 1$, $f_{s+1} \stackrel{\text{def}}{=} \frac{d}{dx}(f_s) + f_s f$.

Let $\mathcal{P}_f \stackrel{\text{def}}{=} \{\text{poles of } f \text{ in } \mathbb{P}^1\}$ and $\rho_f(t) \stackrel{\text{def}}{=} d(t, \mathcal{P}_f) \geq \rho(t)$. Then

For $t \in \Omega - \mathcal{P}_f$ $\text{Rad}(f, t) \stackrel{\text{def}}{=} \liminf_{s \rightarrow \infty} \left| \frac{1}{s!} f_s \right|_t^{-1/s}$,

$\text{Rad}_{\text{DW}}(f, t) \stackrel{\text{def}}{=} \min \{\rho(t), \text{Rad}(f, t)\}$,

$\text{Rad}_{\text{BV}}(f, t) \stackrel{\text{def}}{=} \min \{\rho_f(t), \text{Rad}(f, t)\}$. (1)

These radii clearly rely only on $|\cdot|_t$. Our aim is to study the function

$\text{Rad}_{\text{BV}}(f, \cdot) : \mathcal{M}(K) - \mathcal{P}_f / \text{Gal}^{\text{cont}}(\Omega/K) \rightarrow \mathbb{R}^{>0}$.

Except when $|f|_t$ is “big” and hence when $\text{Rad}(f, t)$ is “small”,
the definition (1) do not allow explicit computations.

Taylor solution

For $f \in K(x)$, $t \in \Omega - \mathcal{P}_f$, let : $\tilde{f}(x, t) \stackrel{\text{def}}{=} \sum_{s=0}^{\infty} f_s(t) \frac{1}{s!} (x - t)^s$

One checks easily $\tilde{f}(t, t) = 1$ and $\frac{d}{dx} \tilde{f}(x, t) = f(x) \tilde{f}(x, t)$ hence

Basic formula

$$\tilde{f}(t + y, t) = \exp \left(\sum_{s=1}^{\infty} f^{(s-1)}(t) \frac{1}{s!} y^s \right) \in \Omega[[y]] \quad (2)$$

$\text{Rad}(f, t)$ as the radius of convergence of a power series

Let $\text{Roc } \tilde{f}$ denote the radius of convergence of $\tilde{f} \in \Omega[[y]]$ and let us set

$$\text{Roc}_{\text{BV}} \tilde{f} = \min \left\{ \text{Roc } \tilde{f}, |\text{zeros of } \tilde{f}| \right\} = \min \left\{ \text{Roc } \tilde{f}, \text{Roc } \frac{1}{\tilde{f}} \right\}.$$

Then $\text{Rad}(f, t) = \text{Roc } \tilde{f}(t + y, t)$, $\text{Rad}_{\text{BV}}(f, t) = \text{Roc}_{\text{BV}} \tilde{f}(t + y, t)$.

Theoretical results

Small radius theorem

If $|f|_t > \frac{1}{\rho(t)}$ then $\text{Rad}_{\text{BV}}(f, t) = \text{Rad}(f, t) = \frac{p^{-\frac{1}{p-1}}}{|f|_t} < p^{-\frac{1}{p-1}} \rho(t)$.

When $|f|_t \leq \frac{1}{\rho(t)}$, there exist an integer m and a “*Frobenius antecedent*” $\psi^m(f)$ with “small radius” and such that

$$\text{Rad}(f, t) = \text{Rad}(\psi^m(f), t^{p^m})^{1/p^m}.$$

Constructing $\psi^m(f)$ is not an effective process.
Moreover, m relies deeply on the unknown $\text{Rad}(f, t)$!

Regular and singular branches of a Berkovich points

For $a \in K^{\text{alg}}$, $t \in \mathcal{M}(K)$ let $B(t, a) \stackrel{\text{def}}{=} \{u \in \mathcal{M}(K) \cup \{\infty\},]t, u[\cup]t, a[\neq \emptyset\}$,

$$\text{sgbr}_f(t) \stackrel{\text{def}}{=} U_{c \in \mathcal{P}_f} B(t, c) \quad \text{rgbr}_f(t) \stackrel{\text{def}}{=} \mathcal{M}(K) \cup \{\infty\} - \text{sgbr}_f(t).$$

In particular, $t \in \text{rgbr}_f(t)$.

Properties of the radius function $\text{Rad}_{\text{BV}}(f, .)$

- It is *continuous* for both topologies on $\mathcal{M}(K)$,
- If $\text{Rad}_{\text{BV}}(f, t) \geq \rho(t)$ then $\text{Rad}_{\text{BV}}(f, .)$ is constant on $\text{rgbr}_f(t)$,
- When $\mathcal{P}_f \cap \{x ; r < |x - a| < R\} = \emptyset$ it is *logarithmically concave* on the path $[t_{a,r}, t_{a,R}]$,
- It is defined by a sub-tree (Baldassarri conjecture) s.t. :
 - *slope integrality* : $\mathfrak{p}(\vec{\mathfrak{b}}) \in \mathbb{Z}$,
 - *harmonicity* : the sum of the log-slopes on edges starting from a vertex is 0 [counting edges with multiplicity].

Equations without singular point

$$f \in K[x]$$

Exponentials of polynomial have no zero in their disk of convergence. So it is useless to distinguish between Roc and Roc_{BV} for them. Consequently, when $f \in K[x]$, one has $\text{Rad}_{\text{BV}}(f, t) = \text{Rad}(f, t)$ and we will omit $_{\text{BV}}$.

Radius of convergence of a product

General setting

For $\tilde{f}, \tilde{g} \in \Omega[[y]]$, $\text{Roc } \tilde{f} \tilde{g} \geq \min \{ \text{Roc } \tilde{f}, \text{Roc } \tilde{g} \}$

But inequality can be strict : $(1-y) \sum_{s \geq 0} y^s = 1 \dots$ (3)

The improvement when introducing Roc_{BV}

If $\text{Roc}_{\text{BV}} \tilde{f} > \text{Roc}_{\text{BV}} \tilde{g}$ then $\text{Roc}_{\text{BV}} \tilde{f} \tilde{g} = \text{Roc}_{\text{BV}} \tilde{g}$. (4)

But, beyond example (3), there is the basic Dwork's example :

$$\text{Roc } \exp(\pi_0 y - \pi_0 y^p) = p^{\frac{p-1}{p^2}} > 1 = \text{Roc } \exp(\pi_0 y) = \text{Roc } \exp(-\pi_0 y^p).$$

We produce now situations where (4) is an equality even for equal radii.

The sequence π_m

Definition

Let $P(x) = x^p + px$,

[More generally $P(x) = x^p + px \pmod{px^2\mathbb{Z}[x]}$]

and let define $\{\pi_m\}_{m \geq 0}$ recursively by $P(\pi_0) = 0$, $P(\pi_{m+1}) = \pi_m$.

For the sake of simplicity, we will suppose $\pi_m \in K$.

Some properties of the π_m useful for computations

$$|\pi_m| = p^{-\frac{1}{(p-1)p^m}} \quad \text{and, for } m \geq 1 \quad \text{and} \quad \zeta^{p^m} = \frac{1}{p},$$

$$\left| \frac{\pi_m}{\pi_{m-1}} \right| = p^{\frac{1}{p^m}}, \quad \left| \frac{\pi_m}{\pi_{m-1}} + \zeta \right| = p^{\frac{1}{p^{m+1}}}, \quad \left| \frac{\pi_m^p}{\pi_{m-1}} - 1 + p\zeta \right| = p^{-1 + \frac{1}{p^{m+1}}}$$

$$\left[|\zeta| = p^{\frac{1}{p^m}} \quad \text{and, for} \quad \zeta'^{p^m} = \zeta^{p^m} = \frac{1}{p}, \quad |\zeta - \zeta'| \leq p^{-\frac{1}{(p-1)p^m}} \right].$$

Robba exponentials (following Pulita)

Definition of Robba polynomials and Robba exponentials

For any integer $n = d p^m \geq 1$ ($p \nmid d$ and $m \geq 0$), let us set

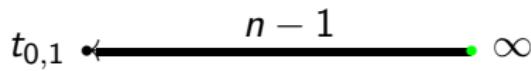
$$e_n(x) \stackrel{\text{def}}{=} \pi_m d x^{d-1} + \pi_{m-1} d x^{dp-1} + \cdots + \pi_0 d x^{dp^m-1},$$

$$\tilde{e}_n(y) \stackrel{\text{def}}{=} \tilde{e}_n(y, 0) = \exp\left(\pi_m y^d + \frac{1}{p} \pi_{m-1} y^{dp} + \cdots + \frac{1}{p^m} \pi_0 y^{dp^m}\right).$$

Theorem

$$\tilde{e}_n(y) \in \mathbb{Z}_p[\pi_m][[y]], \quad \text{Rad}(e_n, t) = \begin{cases} 1 & \text{if } |t| \leq 1, \\ |t|^{1-n} & \text{if } |t| > 1. \end{cases}$$

Hence $\text{Rad}(e_n, t)$ is defined by the following sub-tree



Product of Robba exponentials

Natural decomposition for exponentials of polynomial

$$\forall P(y) = \sum_{s=1}^S c_s y^s \in \Omega[y] \quad \exists \lambda_s \in \Omega \quad \text{s.t.} \quad \exp P(y) = \prod_{s=1}^S \tilde{e}_s(\lambda_s y) .$$

Then $\text{Roc } \exp P(y) = \min_s \text{Roc } \tilde{e}_s(\lambda_s y) = \min_s |\lambda_s|^{-1} .$

But in general neither the λ_s neither any of their powers are in $K[c_i]$.

Product of Robba exponentials

Witt decomposition for exponentials of polynomial

$$\tilde{f}(y) \stackrel{\text{def}}{=} \exp\left(\sum_{s=1}^S c_s y^s\right) = \prod_{p \nmid d, d=1}^D \prod_{i=0}^{m(d)} \tilde{e}_{p^{m(d)-i}}(a_{i,d} y^{dp^i}) ,$$

- for $\frac{p^i}{\pi_{m(d)-i}} c_{dp^i} = a_{0,d}^{p^i} + p a_{1,d}^{p^{i-1}} + \cdots + p^i a_{i,d}$,

- i.e. $a_{i,d} \in \mathbb{Q} \left[\frac{1}{\pi_{m(d)}} c_d, \dots, \frac{1}{\pi_{m(d)-i}} c_{dp^i} \right] \quad [0 \leq i \leq m(d)]$.

Then $\text{Roc } \tilde{f} = \min_{d,i} \text{Roc } \tilde{e}_{p^{m(d)-i}}(a_{i,d} y^{dp^i}) = \min_{d,i} |a_{i,d}|^{-\frac{1}{dp^i}}$.

The $m(d)$ are far from unique : $c_{dp^i} = 0$ for $i > m(d)$ is enough.

Hence there are several decompositions ... giving the same radius !

Example : polynomial of degree p^2

$$\sum_{s=1}^{p^2} c_s y^s = c_1 y + c_p y^p + c_{p^2} y^{p^2} + \sum_{d=2}^{p-1} c_d y^d + c_{dp} y^{dp} + \sum_{\substack{p^2-1 \\ p \nmid d, d=p+1}} c_d y^d$$

$$\exp(P(y)) = \tilde{e}_{p^2}(a_{0,1}y) \tilde{e}_p(a_{1,1}y^p) \tilde{e}_1(a_{2,1}y^{p^2})$$

$$\prod_{d=2}^{p-1} \tilde{e}_p(a_{0,d}, y^d) \tilde{e}_1(a_{1,d} y^{dp}) \prod_{\substack{p^2-1 \\ p \nmid d, d=p+1}} \tilde{e}_d(a_{0,d} y^d)$$

$$(d=1) \quad a_{0,1} = \frac{1}{\pi_2} c_1, \quad a_{1,1} = \frac{1}{\pi_1} c_p - \frac{1}{p\pi_2^p} c_1^p,$$

$$a_{2,1} = \frac{1}{\pi_0} c_{p^2} - \frac{1}{p} \left(\frac{1}{\pi_1} c_p - \frac{1}{p\pi_2^p} c_1^p \right)^p - \frac{1}{p^2 \pi_2^{p^2}} c_1^{p^2}.$$

$$(2 \leq d < p) \quad a_{0,d} = \frac{1}{\pi_1} c_d, \quad a_{1,d} = \frac{1}{\pi_0} c_{dp} - \frac{1}{p\pi_1^p} c_d^p,$$

$$(d > p) \quad a_{0,d} = \frac{1}{\pi_0} c_d.$$

Equations without singular point : the algorithm

Given $f \in K[x]$.

- ① Compute the *Taylor solution* $\tilde{f}(t + y, t) \in K[t][[y]]$ (see (2))
- ② Compute the *polynomials* $a_{i,d} \in K[t]$ s.t.

$$\tilde{f}(t + y, t) = \prod_{p \nmid d, d=1}^D \prod_{i=0}^{m(d)} \tilde{e}_{p^{m(d)-i}} \left(a_{i,d}(t) y^{dp^i} \right),$$

- ③ Construct the sub-trees associated to the functions $t \mapsto |a_{i,d}|_t$,
- ④ Comparing them, construct the sub-tree associated to the function

$$\text{Rad}(f, t) = \min_{i,d} |a_{i,d}|_t^{-\frac{1}{dp^i}}.$$

Example : $f(x) = p^2 x^{p^2-1}$

① $\tilde{f}(t+y, t) = \exp\left((t+y)^{p^2} - t^{p^2}\right) = \exp\left(\sum_{s=1}^{p^2} \binom{p^2}{s} t^{p^2-s} y^s\right)$

②

$$|a_{0,d}|_t^{-\frac{1}{d}} = p^{\frac{2}{d} - \frac{1}{d(p-1)}} \quad |t|^{1-\frac{p^2}{d}}, \quad (p < d)$$

$$|a_{0,d}|_t^{-\frac{1}{d}} = p^{\frac{2}{d} - \frac{1}{dp(p-1)}} \quad |t|^{1-\frac{p^2}{d}}, \quad (2 < d < p)$$

$$|a_{0,1}|_t^{-1} = p^{2 - \frac{1}{p^2(p-1)}} \quad |t|^{1-p^2},$$

$$|a_{1,d}|_t^{-\frac{1}{dp}} = p^{\frac{1}{pd} - \frac{1}{dp(p-1)}} \quad |t|^{1-\frac{p}{d}} \quad |T - \alpha_{1,d}|^{-\frac{1}{dp}}, \quad (2 < d < p)$$

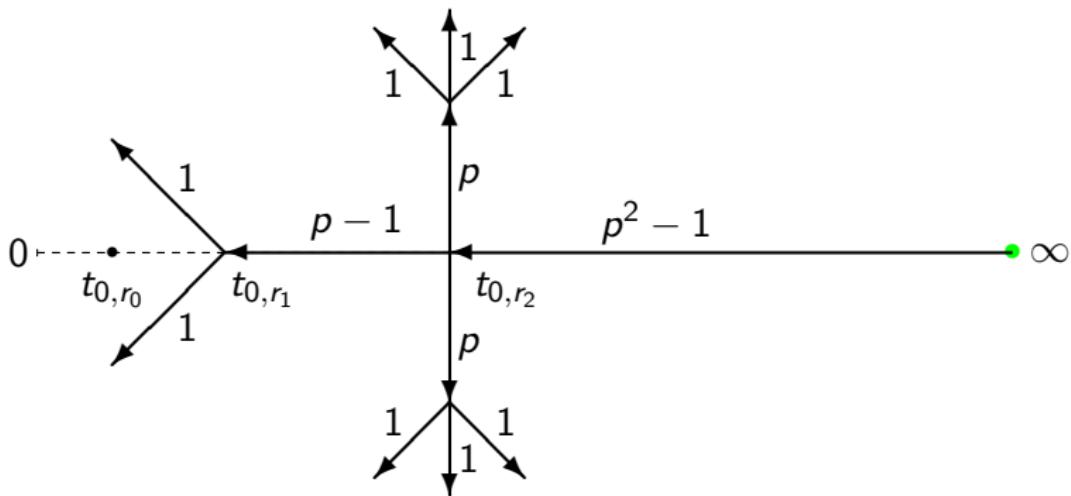
$$|a_{1,1}|_t^{-\frac{1}{p}} = p^{\frac{1}{p} - \frac{1}{p^2(p-1)}} \quad |t|^{1-p} \quad |T - \alpha_2|^{-\frac{1}{p}},$$

$$|a_{2,1}|_t^{-\frac{1}{p^2}} = p^{\frac{1}{p} - \frac{1}{p^2} - \frac{1}{p^2(p-1)}} \quad |T(T - \alpha_2)^p + \alpha_3 p^{p-1} (1 - \alpha_4 T^{p+1})|^{-\frac{1}{p^2}}.$$

with $|\alpha_* - 1| \leq p^{-1 + \frac{1}{p^2}}$ and setting $T = (p^2 t^{p^2})^{p-1}$

Sub-tree of the function $\text{Rad}(p^2x^{p^2-1}, t)$ ($p = 3$)

4



$$r_0 = p^{-\frac{1}{p^2(p-1)}} = \text{Roc} \exp(x^{p^2})$$

$$r_1 = p^{\frac{1}{p^2}}$$

$$r_2 = p^{\frac{2}{p^2}}$$

General Case

$$f \in K(x)$$

The trouble

Because it contains infinitely many factors, the power series

$$\tilde{f}(t+y, t) = \exp \left(\sum_{s=1}^{\infty} f^{(s-1)}(t) \frac{1}{s!} y^s \right) \quad (2)$$

cannot be written directly as a product of Robba exponentials .

Basic idea to (uniformly) “cut off the tail”

When $\infty \notin \mathcal{P}_f$, one has $\left| \frac{1}{s!} f^{(s)} \right|_t \leq \rho_f(t)^{-s} |f|_t$. Hence

$$\text{Roc } \exp \left(f^{(s-1)}(t) \frac{1}{s!} y^s \right) \geq R \quad \text{as soon as} \quad s \left(\frac{R}{\rho_f(t)} \right)^s \leq \left(\frac{p^{-\frac{1}{p-1}}}{\rho_f(t) |f|_t} \right).$$

Hence if $\frac{\text{Rad}(f, t)}{\rho_f(t)} \leq \theta < 1$ somewhere, we can “cut off the tail”

in the formula (2) namely drop terms for $s > S$ for an explicit index S .

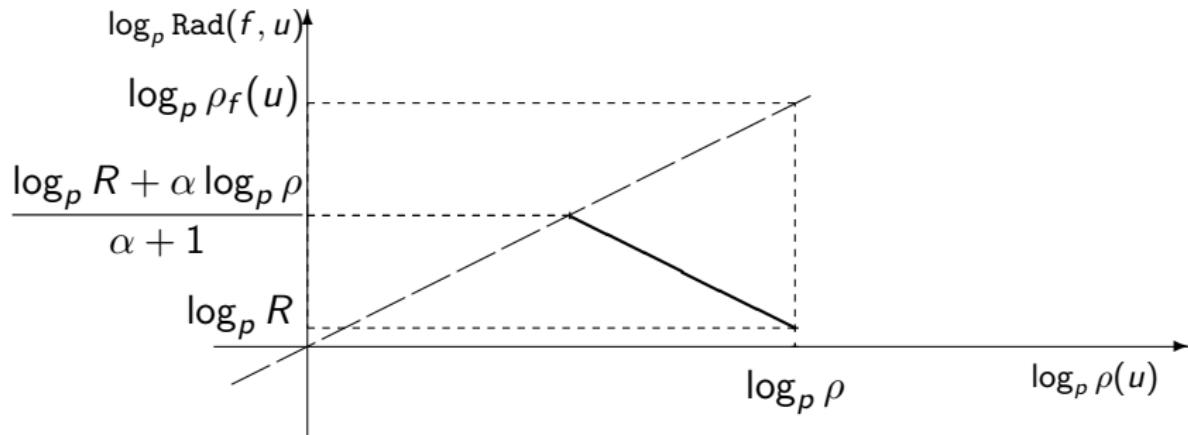
The case of regular branches of a point

For $u \in B(t, a) \subset \text{rgbr}_f(t)$, and $R \stackrel{\text{def}}{=} \text{Rad}(f, t) < \rho \stackrel{\text{def}}{=} \rho(t)$, one has

$$\text{Rad}_{\text{BV}}(f, u) = \text{Rad}(f, u) \leq \rho^{\frac{\alpha}{\alpha+1}} R^{\frac{1}{\alpha+1}} < \rho$$

$\alpha = \text{sing}_f(t) \stackrel{\text{def}}{=} \sum_{c \in \mathcal{P}_f} \text{log-slope of } \text{Rad}(f, u) \text{ in } t^- \text{ along } \overrightarrow{[c, t]}.$

[Some extra complications for “disks” centered at infinity are omitted.]



Indeed the log-slope of $\text{Rad}(f, t)$ along $\overleftarrow{[a, t_{a, \rho}]}$ is at most α .

Method of steepest descent : case of a single point

Let $t \in \Omega - \mathcal{P}_f$, let us set $\rho \stackrel{\text{def}}{=} \rho(t)$, $R \stackrel{\text{def}}{=} \text{Rad}(f, t)$, $\alpha \stackrel{\text{def}}{=} \text{sing}_f(t)$.

Then, if $R < \rho$, when computing $\text{Rad}(f, u)$ for $u \in \text{rgbr}_f(t)$,

terms $f^{(s-1)}(u) \frac{1}{s!} y^s$ can be dropped as soon as

$$s^{\alpha+1} \left(\frac{R}{\rho} \right)^s \leq \left(\frac{p^{-\frac{1}{p-1}}}{\rho |f|_t} \right)^{\alpha+1}.$$

Moreover, the factor $s^{\alpha+1}$ can be omitted when f has no residues.

Actually, for $u \in \text{rgbr}_f(t)$, one gets

$$\text{Roc exp} \left(f^{(s-1)}(u) \frac{1}{s!} y^s \right) \geq \rho \left(p^{-\frac{1}{p-1}} \frac{1}{s\rho} |f|_t^{-1} \right)^{\frac{1}{s}}.$$

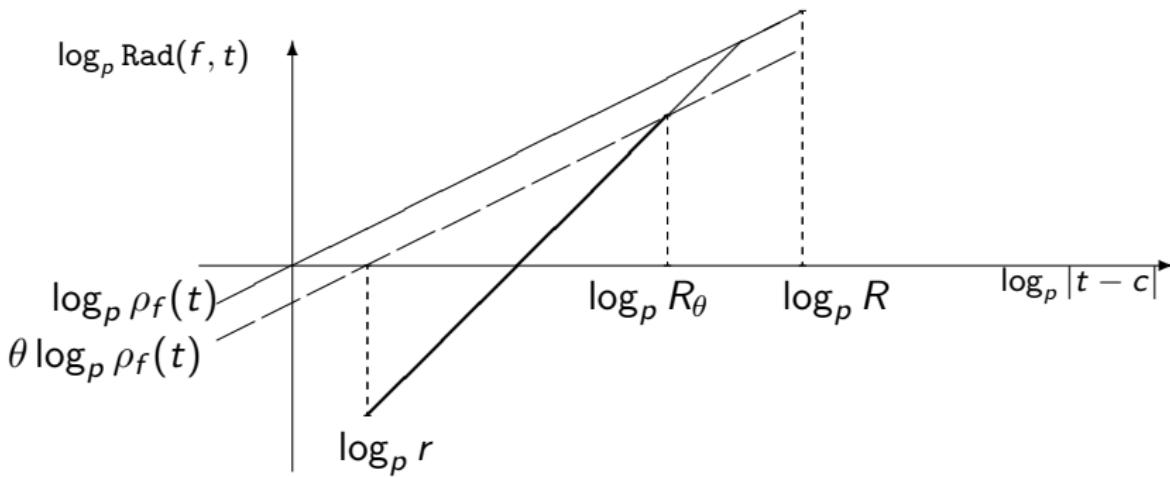
[if $R = \rho$, one already knows that $\text{Rad}(f, u) = \rho$ for $u \in \text{rgbr}_f(t)$].

Let $\mathcal{T}_f \stackrel{\text{def}}{=} \text{sub-tree of } \mathcal{P}_f = \{t \in \mathcal{M}_K, \rho(t) = \rho_f(t)\}$.

Method of steepest descent for edges of \mathcal{T}_f

Let $[t_{c,r}, t_{c,R}]$ be an edge of \mathcal{T}_f . Let suppose that $\text{Rad}(f, t_{c,r}) < r$ and the log-slope $\alpha > 1$ of $\text{Rad}(f, t)$ in $t_{c,r}$ along $[t_{c,r}, t_{c,R}]$ are known. Then choosing a $\theta < 1$ one can compute the function $\text{Rad}(f, t)$ for

$$r \leq |t - c| \leq R_\theta = r^{\frac{\alpha}{\alpha-\theta}} \text{Rad}(f, t_{c,r})^{-\frac{1}{\alpha-\theta}}.$$



Sketch of an algorithm for the general case $f \in K(x)$

- ① Write partial *fraction decomposition* ($a_c \in K^{\text{alg}}$ and $f_c \in K^{\text{alg}}[X]$)

$$f(x) = \sum_{c \in \mathcal{P}_f} \frac{a_c}{x - c} + f_\infty(x) + \sum_{c \in \mathcal{P}_f - \{\infty\}} \frac{-1}{(x - c)^2} f_c\left(\frac{1}{x - c}\right)$$

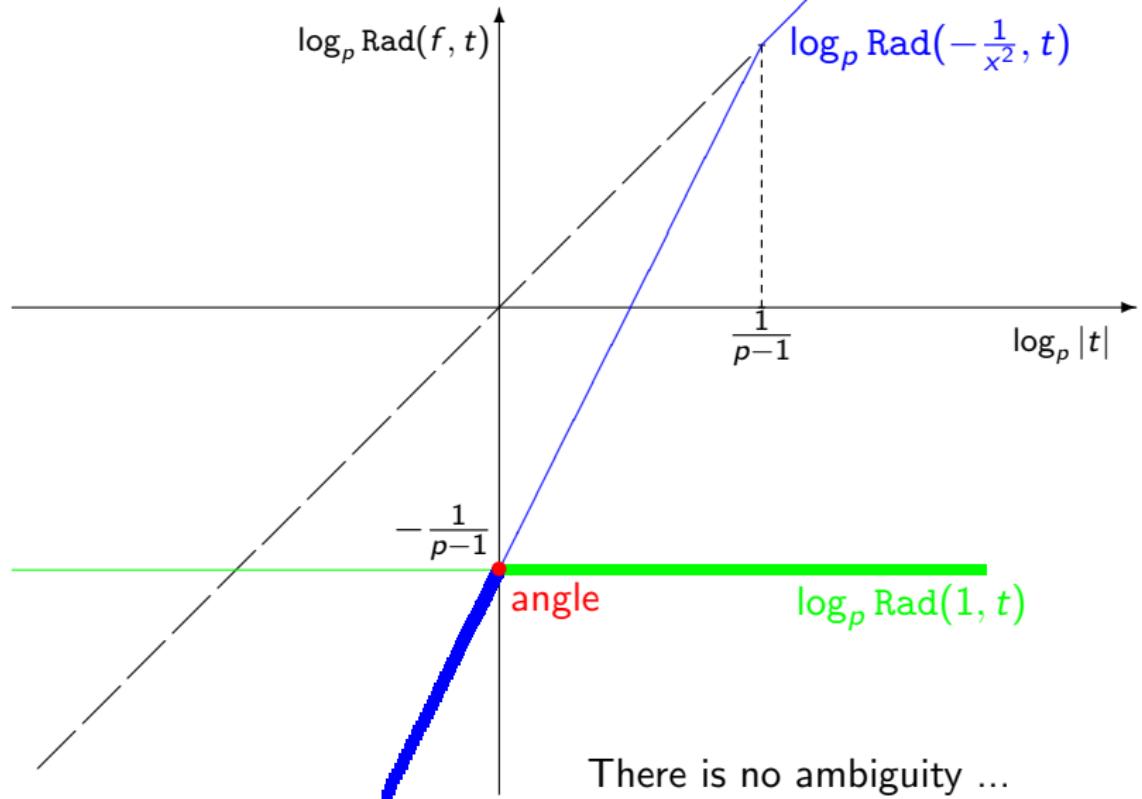
- ② For $t \in \mathcal{T}_f$, compute the $\text{Rad}(f_c, t)$ and find

$$\min \left\{ \text{Roc}(1 + y)^{a_c} |t - c|; \text{Rad}_{\text{BV}}(f_c, \frac{1 + ct}{t}); \text{Rad}_{\text{BV}}(f_\infty, t) \right\} \quad (5)$$

$\left[\text{Roc}(1 + y)^{a_c}$ is explicitly given when knowing $d(a_c, \mathbb{Z}_p)$ $\right].$

- ③ Compute the function $\text{Rad}(f, t)$ on \mathcal{T}_f using (recursively) the method of steepest descent for “*solving ambiguity*” namely edges where the radius is not small and the minimum (5) is reached by several terms.
- ④ “*Smooth angles*”, by using method of steepest descent for isolated points of \mathcal{T}_f where harmonicity condition is not fulfilled.

Example $f(x) = 1 - \frac{1}{x^2}$: $\text{Rad}(f, t)$ for $t \in \mathcal{T}_f =]0, \infty[$



Example $f(x) = 1 - \frac{1}{x^2}$: smoothing the angle

Let $t = t_{0,1}$ then : $\text{rgbr}_f(t) = \{u \in \mathcal{M}_K ; |u| = 1\}$, $\rho = \rho(t) = 1$
 $\alpha = \text{sing}_f(t) = 2$, $R = \text{Rad}(f, t) = p^{-\frac{1}{p-1}}$ and $|f(t)| = 1$.

We can drop terms such that $(p^{-\frac{1}{p-1}})^s \leq (p^{-\frac{1}{p-1}})^{\alpha+1}$

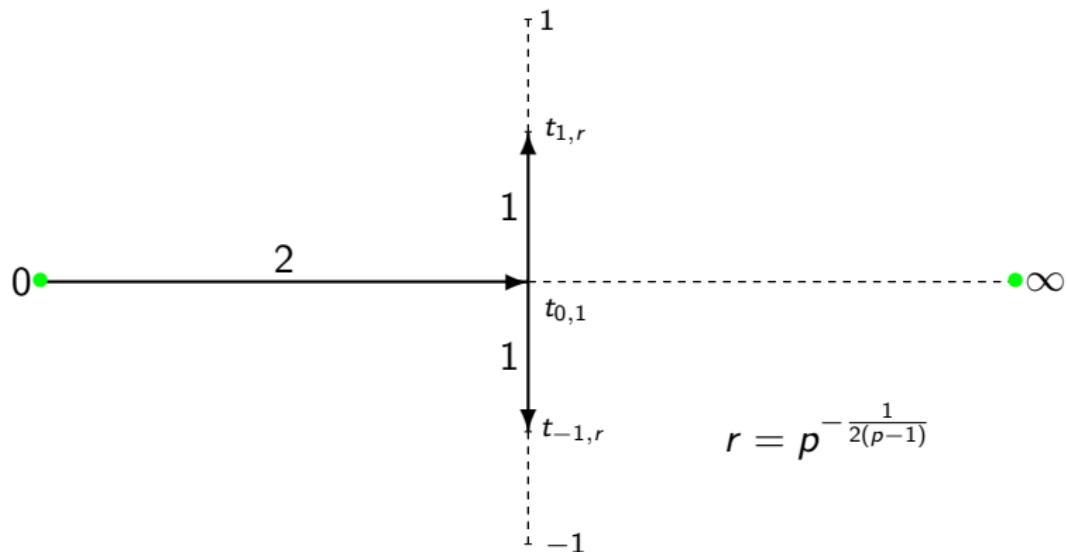
i.e. $s \geq 3$.

When $u \in \text{rgbr}_f(t)$ and $\text{Rad}(f, u) < \rho(u)$, we get :

$$\begin{aligned}\text{Rad}(f, u) &= \text{Roc} \exp \sum_{s=1}^2 f^{(s-1)}(u) \frac{1}{s!} y^s = \text{Roc} \exp \left(\left(\frac{u^2 - 1}{u^2} \right) y + \frac{1}{u^3} y^2 \right) \\ &= \min \left(p^{-\frac{1}{p-1}} |u^2 - 1|^{-1}, p^{-\frac{1}{2(p-1)}} \right) = p^{-\frac{1}{p-1}} |u^2 - 1|^{-1}\end{aligned}$$

Sub-tree of the function

$$\text{Rad}_{\text{BV}} \left(1 - \frac{1}{x^2}, t \right)$$



Final remarks

- ➊ When f has three or more distinct poles, ambiguities can occur. It is not clear that the method of steepest descent enables to conclude in any case : if there is an hidden angle, it can be found by choosing a convenient θ but how to be certain that there is no hidden angle ?
- ➋ To better understand the underlying problems, further examples would be needed. But they imply often very tedious computations. Hence, to go further, it would be essential to enjoy the help of a computer.