5 EXAMPLES

The running examples that follow are split into smaller paragraphs to be referred to from the main body of the text: Ex. 5.3, 5.5, and 5.6 continue Ex. 5.1 in independent ways; Ex. 5.7 continues Ex. 5.4.

Example 5.1. For planar maps, the bivariate generating function $F(t,u)$ satisfies the DDE (2). In this case, $a=1$, $f(u)=1$ and

$$Q = u^2x^2 + ux + uy, \quad \tilde{Q} = u^2x^2 + ux + \frac{u\,(x-z)}{u-1}, \qquad (12)$$
$$P = t\,u^2(u-1)x^2 + (t\,u^2 - u + 1)x - tuz + u - 1.$$

Note that setting $u=1$ in $P(F(t,u), F(t,1), t, 1) = 0$ yields a tautologic identity. The algorithms presented in the paper will compute, by different means, the polynomial $R(z,t) = 27t^2z^2 + (1-18t)\,z + 16t - 1$ (or a multiple of it) as an annihilating polynomial for $F(t,1)$.

Example 5.2. Taking (6) with $f(u)=1$, $a=0$ and $Q := 97t^2 - 73u^2 - 56x^2 - 62y^2 + 87x$ gives the polynomial equation (7), with

$$\mathrm{disc}_x\, P = -\,16352t^2u^6 + (21728t^4 - 10535t^2 + 50t + 1)u^4$$
$$+\, 248t(97t^3 - 56t\,z^2 + 87tz - z + 1)u^2,$$

which has a double root at $u=0$.

Example 5.3. For P as in (12), Algorithm DD computes

$$D_0 = D_1 = t\,(4tz + t - 4)\,u^4 - 2t\,(2tz-3)\,u^3 + (1-2t)\,u^2 - 2u + 1,$$

then it computes the polynomial of bi-degree $(8,4)$ in (t,z)

$$D_2 = -\,6912t^8z^4 + 256t^6(72t-1)z^3 - 512t^5(8t^2+31t-1)z^2$$
$$+\, 256(32t^2+16t-1)t^4z - 256(16t-1)t^4,$$

and it finally returns the polynomial of bi-degree $(4,3)$ in (t,z)

$$R = 27t^4z^3 - t^2(45t-1)z^2 + t(16t^2+17t-1)z - t(16t-1).$$

If one further completely factors the bivariate polynomial R,

$$R = t(tz-1)(27t^2z^2 - 18tz + 16t + z - 1),$$

one concludes that $F(t,1)$ is a root of $27t^2z^2 + (1-18t)\,z + 16t - 1$, as announced in the introduction.

Example 5.4. Doing $u=0$ in the trivial functional equation

$$F(t,u) = 1 + t(uF(t,u)^2 + F(t,u) - F(t,0)) \qquad (13)$$

solves it immediately, but this example is to show a specific phenomenon. Here, $\tilde{Q} = u\,x^2 + x - z$ and $P = 1 - x + t(ux^2 + x - z)$. Therefore, $\partial_x P(x,z,0,u) = -1$, hence assumption (**H1**) is violated. Now, Algorithm DD computes $\mathrm{disc}_x\, P = 4t\,(tz-1)\,u + (t-1)^2$, then its discriminant in u, which is the constant polynomial 1. The output of Algorithm DD is $R=1$, which is obviously wrong. In fact, the unique solution $F(t,u)$ of (13) in $\mathbb{Q}[u][[t]]$ satisfies $F(t,0)=1$, and is a root of $tux^2 + (t-1)x + 1 - t$.

Example 5.5. For P as in (12), Zeilberger's method computes the first 20 terms in the expansion

$$F(t,u) = 1 + (u+1)ut + (2u^3 + 3u^2 + 2u + 2)ut^2 + \cdots,$$

and from there it guesses a polynomial $S = 27t^3u^4\,(u-1)^2\,x^4 + 54t^2u^2\,(u-1)\,\left(u^2t - u + 1\right)x^3 + \cdots$ of degree $(4,3,6)$ in (x,t,u). Alternatively, the Gessel-Zeilberger variant computes the first 8 terms of $F(t,u)$, then deduces that $F(t,1) = 1 + 2t + 9t^2 + 54t^3 + 378t^4 + \cdots$, from where it guesses the polynomial $R = 27t^2z^2 + (1-18t)\,z + 16t - 1$ that cancels (conjecturally) $F(t,1)$, and then takes its resultant with P from (12); this gives $t \cdot S$. Then, Step (2) is a nontrivial one: one way to prove the existence of a root G in $\mathbb{K}[u][[t]]$ of S is to use rational parametrizations, but this proof requires human cleverness, in addition to nontrivial algorithms, see e.g. [8, p. 365]. Finally, Step (3) can be performed by algebraic elimination: the intersection of $\langle S(x,t,u), S(z,t,1), T-P\rangle$ and $\mathbb{Q}[T,t,u]$ is generated by $T\left(1728t^3u^2 + 729T^2t^2 - 432t^2u^2 + 36u^2t - u^2\right)$. Since by construction $P(G(t,u), G(t,1), t, u) = O(t)$ is a root of this polynomial, it is 0. By uniqueness of the solution of (2), one concludes that $F(t,u) = G(t,u)$, and hence $F(t,1)$ is a root of $R = S(z,t,1)$.

Example 5.6. For P as in (12), the hybrid method loops and eventually (or immediately) uses some $\sigma \geq 8$: one could check that any smaller σ will get rejected at Step (2). With $\sigma = 8$, Step (0) computes the first 8 terms in the expansion $F(t,1) = 1 + 2t + 9t^2 + 54t^3 + 378t^4 + \cdots$, which are enough to guess in Step (1) the polynomial $R = 27t^2z^2 + (1-18t)\,z + 16t - 1$. In Step (2), as $\delta = 4$, one first computes the $\sigma \cdot (2\delta^3) + 1 = 1025$ terms of $F(t,1)$, using the polynomial system (9) and the Newton iteration (10) with initial point $\mathbf{F} = [1,1,1]^{\mathrm{T}}$, where $J = \mathrm{Jac}_{x,z,u}(P, \partial_x P, \partial_u P)$ is the matrix

$$\begin{bmatrix} 2t\,u^2\,(u-1)\,x + t\,u^2 - u + 1 & -ut & 3t\,u^2x^2 - 2xt\,(x-1)\,u - tz - x + 1 \\ 2t\,u^2\,(u-1) & 0 & 6t\,u^2x - 4tux + 2ut - 1 \\ 2ut\,(3u-2)\,x + 2ut - 1 & -t & 2xt\,(3ux - x + 1) \end{bmatrix}$$

which yields the simultaneous expansion of

$$\begin{bmatrix} F(t,U(t)) \\ F(t,1) \\ U(t) \end{bmatrix} = \begin{bmatrix} 1 + 2t + 12t^2 + 90t^3 + 756t^4 + 6804t^5 + \cdots \\ 1 + 2t + 9t^2 + 54t^3 + 378t^4 + 2916t^5 + \cdots \\ 1 + t + 4t^2 + 25t^3 + 190t^4 + 1606t^5 + \cdots \end{bmatrix}.$$

Checking $R(F(t,1),t) \bmod t^{1025} = 0$ *proves* that $R(F(t,1),t) = 0$. The needed precision 1025 is so large because of the very pessimistic a priori bound $2\delta^3 = 128$ on $\deg R = 4$. Any improvement in this bound would result in (much) less computations.

Example 5.7. Consider the functional equation (13). The corresponding deformed functional equation is

$$G(h,u,\epsilon) = 1 + \epsilon h \Delta_0 G(h,u,\epsilon) + h^2(uG(h,u,\epsilon)^2 + u\Delta_0 G(h,u,\epsilon)).$$

Hence, the deformed polynomial equation is given by

$$P_\epsilon := (1-x)u + \epsilon h(x-z) + uh^2(ux^2 + x - z).$$

The determinant of the Jacobian matrix $\mathrm{Jac}_{x,u,z}(P_\epsilon, \partial_x P_\epsilon, \partial_u P_\epsilon)$ is

$$\mathcal{D} = (8u^3x^2 + 4u^2x - 2u^2z)h^6 + (12\epsilon u^2x^2 + 4\epsilon ux)h^5$$
$$-\,(4u^2x - 2u^2)h^4 - (8\epsilon ux + \epsilon)h^3 + \epsilon h.$$

From there, one determines $\mathcal{I}_\epsilon$ and computes $\mathcal{J}_\epsilon \cap \mathbb{K}[z,h,\epsilon]$. A generator of the principal ideal $\mathcal{J}_\epsilon \cap \mathbb{K}[z,h,\epsilon]$ is

$$R_\epsilon := 16\epsilon^2h^6z^3 - (h^6 + 18\epsilon h^5 + 36\epsilon h^3 - 3h^4 + 3h^2 - 1)z$$
$$-\,(\epsilon h^7 - 20\epsilon h^5 - 8\epsilon h^3)z^2 + h^6 + 27\epsilon h^3 - 3h^4 + 3h^2 - 1.$$

Consequently, $R_\epsilon(G(h,a,\epsilon),h,\epsilon) = 0$, and specializing ϵ to 0 gives $R_0(G(h,a,0),h,0) = 0$ where $R_0 = -\,(h-1)^3(h+1)^3(z-1)$. Setting $h = t^2$, we recover that $z-1$ annihilates $F(t,0) = 1$.