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Single-exponential bounds for diagonals of D-finite power series

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Abstract D-finite power series appear ubiquitously in combinatorics, number theory, and mathematical physics. They satisfy systems of linear partial differential equations whose solution spaces are finite-dimensional, which makes them enjoy a lot of nice properties. After attempts by others in the 1980s, Lipshitz was the first to prove that the class they form in the multivariate case is closed under the operation of diagonal. In particular, an earlier work by Gessel had addressed the D-finiteness of the diagonals of multivariate rational power series. In this paper, we give another proof of Gessel's result that fixes a gap in his original proof, while extending it to the full class of D-finite power series. We also provide a single exponential bound on the degree and order of the defining differential equation satisfied by the diagonal of a D-finite power series in terms of the degree and order of the input differential system.

Keywords D-finite power series, diagonal theorem, order bound, degree bound

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1 Introduction

Diagonals of multivariate formal power series appear frequently in different areas: diagonals of rational power series play an important role in enumerative combinatorics, especially the lattice paths enumeration (see the books [22, 23, 25, 28] and the survey [24]); Christol's number-theoretic conjecture, which predicts that globally bounded D-finite power series are diagonals of rational power series [12], remains largely open (see the nice survey [14] by himself); intensive studies on diagonals also appear in computer algebra with connection to mathematical physics [1, 4, 7, 8].

In these contexts, formal power series are commonly given implicitly as solutions to either algebraic or (linear) differential equations, and the corresponding diagonals also satisfy such equations. This is

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in particular the case for *D-finite power series*. Recall that these series are defined (Definition 2.2) as multivariate formal power series in variables $x_1, \ldots, x_n$ whose infinite set of higher-order partial derivatives generates a finite-dimensional vector space over the field of rational functions in the variables. D-finite power series were first introduced and studied by Stanley in 1980 in the univariate case [27] and later systematically investigated by Lipshitz in the multivariate case [20, 21]. In the early 1980's, Gessel, Stanley, Zeilberger, and many combinatorists conjectured that the diagonal of a rational power series in several variables is D-finite. Zeilberger [30] in 1980 and Gessel [18] in 1981 independently claimed to have proved this conjecture. Later, in 1988, Lipshitz [20] pointed out that both proofs were not complete and he used a different, elementary idea to prove that D-finite power series are closed under taking diagonals, so that, in particular, diagonals of rational power series are D-finite. In parallel, Christol had used the finiteness of some De Rham cohomology to prove the result: first under some regularity assumption of a Jacobian variety [10]; then in full generality [11, 13]. In 1990, Zeilberger [31] then completed his own proof with the theory of holonomic D-modules. Later, Wu and Chen [29] provided a similar result for the case of bivariate rational functions as a follow-up of Gessel's work.

The problem we address in this paper is to bound the degrees and orders of linear differential equations satisfied by the diagonal of a given series in terms of degrees and orders of the given differential systems that the series satisfies. We view this as a crucial preliminary step to the computational complexity analysis of algorithms for computing diagonals, and to the longer-term development of fast algorithms in a complexity-driven way.

Diagonals of multivariate series come in several flavors (see Definition 2.4): first, *primary diagonals* collapse just two variables; next, *complete diagonals* collapse all variables to a single one.

Starting with primary diagonals, we get a polynomial increase of the order and degree bounds (Corollary 3.18). A naive iteration of primary diagonals (Section 4.1.3) would thus lead to double-exponential bounds for complete diagonals (Section 4.1.3). Our first and main contribution is therefore to derive a *single-exponential* bound (Theorems 4.2 and 4.12). Note however that in the bivariate case ($n = 2$), no iteration is necessary so that the double-exponential bound is in fact just polynomial, and the bounds of Corollary 3.18 are better than those of Theorems 4.2 and 4.12.

After Lipshitz's work [20], the general belief was that the gaps in Gessel's proof do not seem easy to fill. As a secondary contribution, we however fully fix and generalize Gessel's proof [18] by elaborating on his original proof strategy (Theorems 3.1 and 3.2). Because Gessel's approach does not need any change of variables, as opposed to Lipshitz's, it leads more directly to explicit filtrations, from which we benefit in our bound estimates of the Lipshitz way.

It is worth comparing the bounds we obtained in this paper with the situation in positive characteristic. In that context, a result by Furstenberg [17] and Deligne [16] states that the diagonal of any algebraic function is algebraic. A quantitative version of this theorem by Adamczewski and Bell [2] provides bounds on the algebraic degree of a diagonal and on the maximal degree (height) of a polynomial equation is satisfies, which, even in the case of the diagonal of a *rational* function, is doubly exponential of the form $O(p^{n^n})$, where p is the characteristic and n is the number of variables. As our bounds are singly exponential and might be useful also in characteristic p, this is another instance of the phenomenon [26] that representing an algebraic function by differential equations is more compact than by a polynomial equation. The bound in [2] has very recently been significantly improved in [3, Theorem 5.2].

In the case of characteristic zero, the first bound on the order of an annihilator of the diagonal of a rational power series was given by Christol [10], under a regularity assumption. In [9], single-exponential bounds were announced for both order and degree, still in the rational case. Other single-exponential bounds have been announced for differential operators cancelling Hadamard products of rational series (and therefore diagonals of rational series) in the extended version [6] of a work [5] related with a theoretical study on automata: this indicates the existence of an annihilating operator satisfying single-exponential bounds on its order, its degree, as well as the height of its coefficients. Our contribution can therefore be viewed as a generalization of these results on order and degree to general D-finite power series.

The remainder of this paper is organized as follows. We recall some basic terminology about rings of differential operators and introduce D-finite power series and their diagonals in Section 2. In Section 3, we first prove the Diagonal Theorem (Theorem 3.1) on D-finite power series in the way suggested in Gessel's work and we then derive an explicit polynomial bound for annihilators of diagonals in the bivariate case. Then, a single-exponential bound is given for the general multivariate situation in Section 4 by analyzing Lipshitz's proof.

2 Differential operators, D-finiteness, and diagonals

Throughout this article, we assume that K is a field of characteristic 0. Let $K[\boldsymbol{x}]$ be the ring of polynomials in $\boldsymbol{x} = x_1, \dots, x_n$ over K and $K(\boldsymbol{x})$ be the field of rational functions in $\boldsymbol{x}$ over K. Let $K[[\boldsymbol{x}]]$ be the ring of formal power series in $\boldsymbol{x}$ over K, which is a domain. Denote $\mathcal{S} := K(\boldsymbol{x}) \otimes_{K[\boldsymbol{x}]} K[[\boldsymbol{x}]]$. Let $D_{x_1}, \dots, D_{x_n}$ denote the usual partial derivations $\partial/\partial x_1, \dots, \partial/\partial x_n$ on $\mathcal{S}$. This is the basic notation that we will use in Sections 1 and 3, but it will need to be generalized in Section 4.

The *Weyl algebra* $\mathbb{W}_n$ is the non-commutative polynomial ring in the variables $\boldsymbol{x} = x_1, \dots, x_n$ and $\boldsymbol{D}_x = D_{x_1}, \dots, D_{x_n}$, in which the following multiplication rules hold: $x_i x_j = x_j x_i, D_{x_i} D_{x_j} = D_{x_j} D_{x_i}$ for all $i, j \in \{1, \dots, n\}$ and $D_{x_i} a = a D_{x_i} + \partial a/\partial x_i$ for all $i \in \{1, \dots, n\}$ and $a \in K[\boldsymbol{x}]$. We will also write $K[\boldsymbol{x}]\langle \boldsymbol{D}_x \rangle$ for $\mathbb{W}_n$, as, here and throughout, we use angled brackets $R\langle \dots \rangle$ to denote a twisted extension of a ring R, when generators between brackets always commute with one another. The Weyl algebra can be interpreted as the ring of linear partial differential operators with polynomial coefficients. We will also use the ring $K(\boldsymbol{x})\langle \boldsymbol{D}_x \rangle$ of linear partial differential operators with *rational* function coefficients. The elements of this ring act on $\mathcal{S}$ by interpreting D_{x_i} as $\partial/\partial x_i$, which turns $\mathcal{S}$ into a left $K(\boldsymbol{x})\langle \boldsymbol{D}_x \rangle$-module. For a given $f \in \mathcal{S}$, the *annihilating ideal* of f in $K(\boldsymbol{x})\langle \boldsymbol{D}_x \rangle$ is defined as the set $\{L \in K(\boldsymbol{x})\langle \boldsymbol{D}_x \rangle \mid L(f) = 0\}$. Note that this is indeed a left $K(\boldsymbol{x})\langle \boldsymbol{D}_x \rangle$-module, therefore in particular a vector space over $K(\boldsymbol{x})$.

Notation 2.1. Given a polynomial $a \in K[\boldsymbol{x}]$ and an operator P (in $K[\boldsymbol{x}]\langle \boldsymbol{D}_x \rangle$ or in $K(\boldsymbol{x})\langle \boldsymbol{D}_x \rangle$), we will distinguish the expression Pa, with no parentheses, from the expression $P(a)$, with parentheses: the former will always denote the product in the operator algebra; the latter will always denote application of the operator to a, viewed as a series in $\mathcal{S}$.

In contrast, for a series f in $\mathcal{S}$, we will never have to denote a product, and both Pf and $P(f)$ will denote application.

Definition 2.2 (D-finiteness)**.** An element $f \in K[[\boldsymbol{x}]]$ is *D-finite* over $K(\boldsymbol{x})$ if the $K(\boldsymbol{x})$-vector space generated in $\mathcal{S}$ by the derivatives $D_{x_1}^{\alpha_1} \cdots D_{x_n}^{\alpha_n}(f)$ when $\alpha_1, \dots, \alpha_n$ range over $\mathbb{N}$ is finite-dimensional.

Note that $L(f)$ is also D-finite for any operator $L \in K(\boldsymbol{x})\langle \boldsymbol{D}_x \rangle$.

Definition 2.3 (Order and degree)**.** Assume that $f \in \mathcal{S}$ is D-finite over $K(\boldsymbol{x})$. Then for each $i \in \{1, \dots, n\}$, there exists a non-zero operator L_i in the subalgebra $K[\boldsymbol{x}]\langle D_{x_i} \rangle$ of $\mathbb{W}_n$ such that $L_i(f) = 0$. Write

$$L_i = \ell_{i,0} + \ell_{i,1} D_{x_i} + \cdots + \ell_{i,r_i} D_{x_i}^{r_i} \tag{2.1}$$

with $\ell_{i,0}, \dots \ell_{i,r_i} \in K[\boldsymbol{x}]$ and $\ell_{i,r_i} \neq 0$. We call r_i the *order* of the operator L_i, denoted by $\operatorname{ord}(L_i)$. The *degree* of L_i is defined as the maximum total degree of its polynomial coefficients: $\deg(L_i) := \max_{j=0}^{r_i} \operatorname{tdeg}(\ell_{i,j})$, where tdeg means the total degree with respect to $x_1, \dots, x_n$. Let $r_f := \max_{i=1}^{n} r_i$ and $d_f := \max_{i=1}^{n} d_i$ where $d_i = \deg(L_i)$.

Definition 2.4. Let $f = \sum_{i_1,\dots,i_n \geqslant 0} a_{i_1,\dots,i_n} x_1^{i_1} \cdots x_n^{i_n} \in K[[\boldsymbol{x}]]$. We call the power series

$$\Delta_{1,2}(f) := \sum_{i_1,i_3,\dots,i_n \geqslant 0} a_{i_1,i_1,i_3,\dots,i_n} x_1^{i_1} x_3^{i_3} \cdots x_n^{i_n} \in K[[x_1, x_3, \dots, x_n]]$$

a *primary diagonal* of f. Other primary diagonals $\Delta_{i,j}$ are defined similarly, so that $\Delta_{i,j}(f)$ and $\Delta_{j,i}(f)$ are the same series except for the variable names. A *diagonal* is defined as any composition of the $\Delta_{i,j}$.

The *complete diagonal* of f, denoted by $\Delta(f)$, is defined as

$$\Delta(f) := \Delta_{n,n-1}\Delta_{n-1,n-2}\cdots\Delta_{2,1}(f) = \sum_{i\geqslant 0} a_{i,\dots,i}x_n^i \in K[[x_n]]. \tag{2.2}$$

By *the* diagonal of f, we mean its complete diagonal when no ambiguity arises.

For future reference, we recall here the following well-known consequence of Cramer's rules that will be used in the subsequent sections.

Lemma 2.5. *Let $A = (a_{i,j}) \in K[\boldsymbol{x}]^{n\times m}$ be a matrix with entries of total degree at most d. Assume the inequality $n < m$, so that the matrix has a non-trivial right nullspace. Then, there exists a non-zero vector $v = (v_1, \dots, v_m) \in K[\boldsymbol{x}]^m$ that solves $Av = 0$ and has total degree at most nd.*

Proof. Let ρ denote the rank of A. Because $\rho \leqslant n$, we can fix ρ linearly independent rows of A and form a $\rho \times m$ submatrix B of A of rank ρ. In turn, consider ρ linearly independent columns of B, thus forming a $\rho \times \rho$ submatrix C, and an additional column c of B. The system $Cw = -c$ admits a non-zero solution w with $\operatorname{tdeg}(w) \leqslant \rho d$ that can be expressed by Cramer's rules. Padding w with zeros results in a non-zero v satisfying $Av = 0$ and $\operatorname{tdeg}(v) \leqslant \rho d \leqslant nd$ as wanted. □

3 Diagonal theorem in the multivariate case

In this section, we give a proof of the following "Diagonal theorem" in the spirit of Gessel [18].

Theorem 3.1 (Diagonal Theorem)**.** *Let $f \in K[[\boldsymbol{x}]]$ be D-finite over $K(\boldsymbol{x})$. Then $\Delta(f) \in K[[x_n]]$ is D-finite over $K(x_n)$.*

The proof of Theorem 3.1 is just an iteration of the following result for primary diagonals.

Theorem 3.2. *Let $f \in K[[\boldsymbol{x}]]$ be D-finite over $K(\boldsymbol{x})$. Then $\Delta_{1,2}(f)$ is D-finite over $K(x_1, x_3, \dots, x_n)$.*

The rest of the present section is devoted to the proof of Theorem 3.2.

The following objects will serve as generators in relevant algebras:

$$\begin{aligned} D_{x_1,x_2} &:= D_{x_1}D_{x_2}, \\ \theta_{x_i} &:= x_i D_{x_i} \quad \text{for each } i \in \{1, \dots, n\}, \\ T_{x_1,x_2} &:= \theta_{x_1} - \theta_{x_2}. \end{aligned}$$

We use bold notation to abbreviate monomials: for example, $\boldsymbol{x}^{\boldsymbol{\alpha}}$ denotes $x_1^{\alpha_1}\dots x_n^{\alpha_n}$ and $\boldsymbol{D}_x^{\boldsymbol{\beta}}$ denotes $D_{x_1}^{\beta_1}\dots D_{x_n}^{\beta_n}$. By [15, Proposition 2.1], the set $\{\boldsymbol{x}^{\boldsymbol{\alpha}}\boldsymbol{D}_x^{\boldsymbol{\beta}} \mid \boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{N}^n\}$ is a basis of $\mathbb{W}_n = K[\boldsymbol{x}]\langle \boldsymbol{D}_x\rangle$ as a vector space over K. Similarly, Lemmas 3.3, 3.4, and 3.5 are lemmas providing canonical bases for several subalgebras of $\mathbb{W}_n$: $K[x_1x_2, x_3, \dots, x_n]\langle T_{x_1,x_2}, D_{x_1,x_2}\rangle$, $K[x_1, x_3, \dots, x_n]\langle D_{x_1}\rangle$, and $K[x_1x_2, x_3, \dots, x_n]\langle T_{x_1,x_2}, D_{x_h}\rangle$ for $h \in \{3, \dots, n\}$.

Lemma 3.3. *The set*

$$\{(x_1x_2)^i x_3^{k_3}\cdots x_n^{k_n} T_{x_1,x_2}^j D_{x_1,x_2}^\ell \mid i, j, \ell, k_3, \dots, k_n \in \mathbb{N}\}$$

is a basis of $K[x_1x_2, x_3, \dots, x_n]\langle T_{x_1,x_2}, D_{x_1,x_2}\rangle$ as a vector space over K.

Proof. It suffices to show that the monomials $(x_1x_2)^i T_{x_1,x_2}^j D_{x_1,x_2}^\ell$ are linearly independent over K. Suppose that

$$L = \sum_{(i,j,\ell)\in\Lambda} c_{i,j,\ell}\ (x_1x_2)^i T_{x_1,x_2}^j D_{x_1,x_2}^\ell = 0$$

for some non-empty finite set Λ and $c_{i,j,\ell} \in K \setminus \{0\}$. Let $\succ$ be the lexicographical order on the algebra $\mathbb{W}_2 = K[x_1, x_2]\langle D_{x_1}, D_{x_2}\rangle$ with $D_{x_1} \succ D_{x_2} \succ x_2 \succ x_1$. For any non-zero element Q of $\mathbb{W}_2$, we write $\operatorname{lm}(Q)$ to denote its leading monomial, that is, the highest monomial with respect to $\succ$ occurring in Q with a non-zero coefficient. It can be proved by induction that there exist Q_1 and Q_2 in $\mathbb{W}_2$ such that

$$T_{x_1,x_2}^j = \theta_{x_1}^j + Q_1 = x_1^j D_{x_1}^j + Q_2 \quad \text{and} \quad \operatorname{lm}(Q_1) < \operatorname{lm}(\theta_{x_1}^j),\ \operatorname{lm}(Q_2) < x_1^j D_{x_1}^j.$$

Then we have that there exists Q_3 in $\mathbb{W}_2$ such that

$$(x_1x_2)^i T_{x_1,x_2}^j D_{x_1,x_2}^\ell = x_1^{i+j} x_2^i D_{x_1}^{j+\ell} D_{x_2}^\ell + Q_3 \quad \text{and} \quad \operatorname{lm}(Q_3) < x_1^{i+j} x_2^i D_{x_1}^{j+\ell} D_{x_2}^\ell.$$

Note that the map $(i,j,\ell) \mapsto (i+j, i, j+\ell, \ell)$ is injective. Since the set $\{\boldsymbol{x}^{\boldsymbol{i}} \boldsymbol{D}_x^{\boldsymbol{j}} \mid \boldsymbol{i}, \boldsymbol{j} \in \mathbb{N}^n\}$ is a basis of $\mathbb{W}_n = K[\boldsymbol{x}]\langle \boldsymbol{D}_x \rangle$ as a vector space over K, this forces all $c_{i,j,\ell} = 0$, which contradicts our assumption. □

Lemma 3.4. *The set*

$$\{x_1^k x_3^{k_3} \cdots x_n^{k_n} D_{x_1}^\ell \mid k, k_3, \ldots, k_n, \ell \in \mathbb{N}\}$$

is a basis of $K[x_1, x_3, \ldots, x_n]\langle D_{x_1} \rangle$ *as a vector space over* K.

Lemma 3.5. *For each* $h \in \{3, \ldots, n\}$*, the set*

$$\{(x_1x_2)^i x_3^{k_3} \cdots x_n^{k_n} T_{x_1,x_2}^j D_{x_h}^\ell \mid i, j, \ell, k_3, \ldots, k_n \in \mathbb{N}\}$$

is a basis of $K[x_1x_2, x_3, \ldots, x_n]\langle T_{x_1,x_2}, D_{x_h} \rangle$ *as a vector space over* K.

We omit the proofs of Lemma 3.4 and 3.5 that are very similar to the proof of Lemma 3.3. Next we present some commutation rules between the diagonal operator $\Delta_{1,2}$ and the operators $x_1x_2, D_{x_1,x_2}, \theta_{x_i}$ and T_{x_1,x_2}.

Proposition 3.6. *For any power series* $f(\boldsymbol{x}) \in K[[\boldsymbol{x}]]$*, we have*

1. $\Delta_{1,2}(x_1x_2f) = x_1\Delta_{1,2}(f)$;
2. $\Delta_{1,2}(D_{x_1,x_2}(f)) = D_{x_1}\theta_{x_1}(\Delta_{12}(f))$;
3. $\Delta_{1,2}(\theta_{x_1}(f)) = \theta_{x_1}(\Delta_{1,2}(f))$;
4. $\Delta_{1,2}(\theta_{x_2}(f)) = \theta_{x_1}(\Delta_{1,2}(f))$;
5. $\Delta_{1,2}(T_{x_1,x_2}(f)) = 0$;
6. $D_{x_1,x_2}T_{x_1,x_2} = T_{x_1,x_2}D_{x_1,x_2}$;
7. $T_{x_1,x_2}\ x_1x_2 = x_1x_2\ T_{x_1,x_2}$.

Proof. Given $f = \sum_{i_1,\ldots,i_n \geqslant 0} a_{i_1,\ldots,i_n} x_1^{i_1} \cdots x_n^{i_n} \in K[[\boldsymbol{x}]]$, we have

$$\begin{aligned}\Delta_{1,2}(x_1x_2f) &= \Delta_{1,2}\Bigg(\sum_{i_1,\ldots,i_n \geqslant 0} a_{i_1,\ldots,i_n} x_1^{i_1+1} x_2^{i_2+1} x_3^{i_3} \cdots x_n^{i_n}\Bigg) \\ &= \sum_{i_1,i_3,\ldots,i_n \geqslant 0} a_{i_1,i_1\ldots,i_n} x_1^{i_1+1} x_3^{i_3} \cdots x_n^{i_n} = x_1\Delta_{1,2}(f),\end{aligned}$$

which proves Point 1. Points 2, 3, and 4 are proved in [29, Lemma 4.3]. Point 5 immediately follows by linearity from Points 3 and 4. Taking the difference of the two identities

$$\begin{aligned}D_{x_1,x_2}(x_1D_{x_1}) &= D_{x_1}(x_1D_{x_1})D_{x_2} = (x_1D_{x_1}+1)D_{x_1,x_2}, \\ D_{x_1,x_2}(x_2D_{x_2}) &= D_{x_2}(x_2D_{x_2})D_{x_1} = (x_2D_{x_2}+1)D_{x_1,x_2},\end{aligned}$$

we obtain Point 6. Similarly, taking the difference of the two identities

$$\begin{aligned}(x_1D_{x_1})\ x_1x_2 &= x_1(x_1D_{x_1}+1)x_2 = x_1x_2\ (x_1D_{x_1}+1), \\ (x_2D_{x_2})\ x_1x_2 &= x_2(x_2D_{x_2}+1)x_1 = x_1x_2\ (x_2D_{x_2}+1),\end{aligned}$$

proves Point 7. □

Lemma 3.7. *Let* $f(\boldsymbol{x}) \in K[[\boldsymbol{x}]]$*. Then, there exists* $s \in \mathbb{N}$ *such that* $T_{x_1,x_2}^s(f) = 0$ *if and only if there exists* g *in* $n-1$ *variables such that* $f(\boldsymbol{x}) = g(x_1x_2, x_3, \ldots, x_n)$.

Proof. If $f(\boldsymbol{x}) = g(x_1x_2, x_3, \ldots, x_n)$, write

$$g(x_1, x_3, \ldots, x_n) = \sum_{i_1,i_3,\ldots,i_n \geqslant 0} b_{i_1,i_3,\ldots,i_n} x_1^{i_1} x_3^{i_3} \cdots x_n^{i_n}.$$

Take $s = 1$, then

$$\begin{aligned} T_{x_1,x_2}(f) &= T_{x_1,x_2}(g(x_1x_2, x_3, \ldots, x_n)) \\ &= \sum_{i_1,i_3,\ldots,i_n \geqslant 0} (i_1 - i_1)\, b_{i_1,i_3,\ldots,i_n}\, x_1^{i_1} x_2^{i_1} x_3^{i_3} \cdots x_n^{i_n} = 0. \end{aligned}$$

For the converse statement, assume there exists $s \in \mathbb{N}$ such that $T^s_{x_1,x_2}(f) = 0$. If $s = 0$, then $f = 0$. Take $g = 0$. If $s > 0$, write $f = \sum_{i_1,\ldots,i_n \geqslant 0} a_{i_1,\ldots,i_n} x_1^{i_1} \cdots x_n^{i_n}$. Then

$$T^s_{x_1,x_2}(f) = \sum_{i_1,\ldots,i_n \geqslant 0} (i_1 - i_2)^s a_{i_1,\ldots,i_n} x_1^{i_1} \cdots x_n^{i_n} = 0.$$

Hence $(i_1 - i_2)^s a_{i_1,\ldots,i_n} = 0$ for all integers $i_1, \ldots, i_n \geqslant 0$, so that $a_{i_1,\ldots,i_n} = 0$ for all $i_1 \neq i_2$. Take

$$g(x_1, x_3, \ldots, x_n) = \sum_{i_1,i_3,\ldots,i_n \geqslant 0} a_{i_1,i_1,i_3,\ldots,i_n} x_1^{i_1} x_3^{i_3} \cdots x_n^{i_n}.$$

Then $f(\boldsymbol{x}) = g(x_1x_2, x_3, \ldots, x_n)$. □

Lemma 3.8. *Let $f(\boldsymbol{x}) \in K[[\boldsymbol{x}]]$ be D-finite over $K(\boldsymbol{x})$. Write $\boldsymbol{y}$ for $y_1, \ldots, y_m$ and consider power series*

$$g_1(\boldsymbol{y}), \ldots, g_n(\boldsymbol{y}) \in K[[\boldsymbol{y}]]$$

that are algebraic over $K(\boldsymbol{y})$. Assume that the substitution $f(g_1(\boldsymbol{y}), \ldots, g_n(\boldsymbol{y}))$ is well-defined in $K[[\boldsymbol{y}]]$. Then the series $f(g_1(\boldsymbol{y}), \ldots, g_n(\boldsymbol{y}))$ is D-finite over $K(\boldsymbol{y})$. In particular, let $f(\boldsymbol{x})$ be D-finite over $K(\boldsymbol{x})$ and suppose that the evaluation of $f(\boldsymbol{x})$ at $x_2 = 1$ is well-defined as a series in $K[[x_1, x_3, \ldots, x_n]]$, then $f(x_1, 1, x_3, \ldots, x_n)$ is D-finite over $K(x_1, x_3, \ldots, x_n)$.

Proof. See [21, Proposition 2.3]. □

From the definition of $K(\boldsymbol{x})\langle \boldsymbol{D}_x \rangle$,

$$D_{x_i} a = a D_{x_i} + D_{x_i}(a) \quad \text{for all } a \in K(\boldsymbol{x}).$$

More generally, we have the formulae: for all $a \in K(\boldsymbol{x})$ and $i \in \{1, \ldots, n\}$,

$$D^k_{x_i} a = \sum_{\ell=0}^{k} \binom{k}{\ell} D^\ell_{x_i}(a) D^{k-\ell}_{x_i} \tag{3.1}$$

and

$$a D^k_{x_i} = \sum_{\ell=0}^{k} (-1)^\ell \binom{k}{\ell} D^{k-\ell}_{x_i} D^\ell_{x_i}(a). \tag{3.2}$$

The relations (3.1) and (3.2) can be proved by a straightforward induction. In the sequel, we merely use the facts that, for all $a \in K[\boldsymbol{x}]$ and $i \in \{1, \ldots, n\}$,

$$D^k_{x_i} a = f D^k_{x_i} + P \qquad \text{and} \qquad a D^k_{x_i} = D^k_{x_i} a - P, \tag{3.3}$$

where $P \in K[\boldsymbol{x}]\langle D_{x_i} \rangle$ with $\operatorname{ord}(P) < k$ and $\deg(P) \leqslant \deg(a)$. Denote by $K[\boldsymbol{x}]_{\leqslant d}$ the set of polynomials in $K[\boldsymbol{x}]$ with total degree less than or equal to d.

A number of similar arguments in the rest of the article will differ only by a choice of variables. This is why we make some notation depend on a set S in the following definition. The reader is invited to pay attention to this implicit dependency in what follows. We will indeed use $S = \{1, 2\}$ and $S = \{1, 2, h\}$ in the present Section 3, and we will additionally use $S = \{1, \ldots, n\}$ in Section 4.

Notation 3.9. Fix a subset $S \subseteq \{1,2,\dots,n\}$. Let $L_i \in \mathbb{W}_n$, for $1 \leqslant i \leqslant n$, be operators defined by (2.1) as in Definition 2.3. In particular, recall $L_i = \ell_{i,r_i} D_{x_i}^{r_i} + \dots$ for some non-zero polynomial $\ell_{i,r_i} \in K[\boldsymbol{x}]$. We give the following definitions and notation:

1. Given $\boldsymbol{\beta} \in \mathbb{N}^S$, write $\boldsymbol{D}_S^{\boldsymbol{\beta}}$ for the product $\prod_{j\in S} D_{x_j}^{\beta_j}$,
2. $C := \operatorname{lcm}_{j\in S}(\ell_{j,r_j}) \in K[\boldsymbol{x}]$,
3. for each $j \in S$, $\tilde{L}_j := (C/\ell_{j,r_j}) L_j \in \mathbb{W}_n$,
4. $d_C := \sum_{j\in S} d_j$,
5. $B := \prod_{j\in S}\{0,1,\dots r_i - 1\} \subseteq \mathbb{N}^{\#S}$,
6. $\mathcal{F}_{d,r} := \bigoplus_{|\boldsymbol{\beta}|\leqslant r} K[\boldsymbol{x}]_{\leqslant d}\, \boldsymbol{D}_S^{\boldsymbol{\beta}}$,
7. $\mathcal{H}_{d,r} := \bigoplus_{|\boldsymbol{\beta}|\leqslant r \text{ or } \boldsymbol{\beta}\in B} K[\boldsymbol{x}]_{\leqslant d}\, \boldsymbol{D}_S^{\boldsymbol{\beta}}$,
8. $J := \sum_{j\in S} \mathbb{W}_n \tilde{L}_j$,

where the dependency in S is kept implicit in the notation.

Immediately we have

Lemma 3.10. *For any non-empty set $S \subseteq \{1,2,\dots,n\}$, consider the quantities in Definition 3.9. Then*

1. $\operatorname{tdeg}(C) \leqslant d_C$,
2. *for each $i \in S$, $\tilde{L}_i(f) = 0$ and $\deg \tilde{L}_i \leqslant d_C$,*
3. *for each $i \in S$, $CD_{x_i}^{r_i} = (CD_{x_i}^{r_i} - \tilde{L}_i) + \tilde{L}_i \in \mathcal{F}_{d_C, r_i - 1} + J$.*

Continuing in analogy with [20, Lemma 3], we have the following lemmas:

Lemma 3.11. *For all $\boldsymbol{\alpha} \in \mathbb{N}^S$, $C\boldsymbol{D}_S^{\boldsymbol{\alpha}}$ is an element of $\mathcal{F}_{d_C,|\boldsymbol{\alpha}|-1} + J$.*

Proof. If $\boldsymbol{\alpha} \in B$, nothing needs to be proven. So suppose, for instance, $n \in S$ and $\alpha_n \geqslant r_n$. Then multiply $\boldsymbol{D}_S^{\boldsymbol{\alpha} - r_n \mathrm{e}_n}$ with $CD_{x_n}^{r_n}$, where $e_n := (0,0,\dots,1)$, which yields

$$\begin{aligned}\boldsymbol{D}_S^{\boldsymbol{\alpha} - r_n \mathrm{e}_n} C D_{x_n}^{r_n} &\in \bigoplus_{j=0}^{r_n-1} \boldsymbol{D}_S^{\boldsymbol{\alpha} - r_n \mathrm{e}_n} K[\boldsymbol{x}]_{\leqslant d_C} D_{x_n}^{j} + J \\ &\subseteq \sum_{j=0}^{r_n-1} \left(K[\boldsymbol{x}]_{\leqslant d_C} \boldsymbol{D}_S^{\boldsymbol{\alpha} - r_n \mathrm{e}_n} + \mathcal{F}_{d_C, |\boldsymbol{\alpha}| - r_n - 1} \right) D_{x_n}^{j} + J \\ &\subseteq \mathcal{F}_{d_C,|\boldsymbol{\alpha}|-1} + \mathcal{F}_{d_C,|\boldsymbol{\alpha}|-2} + J = \mathcal{F}_{d_C,|\boldsymbol{\alpha}|-1} + J.\end{aligned}$$

Hence

$$\begin{aligned}C\boldsymbol{D}_S^{\boldsymbol{\alpha}} \in (\boldsymbol{D}_S^{\boldsymbol{\alpha} - r_n \mathrm{e}_n} C + \mathcal{F}_{d_C,|\boldsymbol{\alpha}| - r_n - 1}) D_{x_n}^{r_n} &\subseteq \boldsymbol{D}_S^{\boldsymbol{\alpha} - r_n \mathrm{e}_n} C D_{x_n}^{r_n} + \mathcal{F}_{d_C,|\boldsymbol{\alpha}|-1} \\ &\subseteq \mathcal{F}_{d_C,|\boldsymbol{\alpha}|-1} + J.\end{aligned}$$

□

Lemma 3.12. *For any $t \in \mathbb{N}, r \in \mathbb{Z}$, $C\mathcal{H}_{t,r} \subseteq \mathcal{H}_{d_C + t, r-1} + J$.*

Proof. We have the chain of equalities and inclusions:

$$\begin{aligned}C\mathcal{H}_{t,r} = C \bigoplus_{|\boldsymbol{\beta}|\leqslant r \text{ or } \boldsymbol{\beta}\in B} K[\boldsymbol{x}]_{\leqslant t}\, \boldsymbol{D}_S^{\boldsymbol{\beta}} &= \bigoplus_{|\boldsymbol{\beta}|\leqslant r \text{ or } \boldsymbol{\beta}\in B} K[\boldsymbol{x}]_{\leqslant t}\, C\boldsymbol{D}_S^{\boldsymbol{\beta}} \\ &\subseteq \sum_{\boldsymbol{\beta}\in B} K[\boldsymbol{x}]_{\leqslant d_C + t}\, \boldsymbol{D}_S^{\boldsymbol{\beta}} + \sum_{|\boldsymbol{\beta}|\leqslant r \text{ and } \boldsymbol{\beta}\notin B} K[\boldsymbol{x}]_{\leqslant t}\, \mathcal{F}_{d_C,|\boldsymbol{\beta}|-1} + J \\ &\subseteq \mathcal{H}_{d_C+t,r-1} + J,\end{aligned}$$

where the first inclusion is by Lemma 3.11. □

Lemma 3.13. *For any $u, t \in \mathbb{N}, v \in \mathbb{Z}$, if $u \geqslant v$, then $C^u \mathcal{H}_{t,v} \subseteq \mathcal{H}_{t+ud_C,0} + J$. In particular, for all $\boldsymbol{\alpha} \in \mathbb{N}^S$, $C^{|\boldsymbol{\alpha}|}\boldsymbol{D}_S^{\boldsymbol{\alpha}} \in \mathcal{H}_{|\boldsymbol{\alpha}|d_C,0} + J$.*

Proof. Note that for all $r' \leqslant 0$, $\mathcal{H}_{t,r'} = \mathcal{H}_{t,0}$. The result is obtained by making u repetitions of Lemma 3.12. □

Lemma 3.13 is specialized as follows.

Lemma 3.14. *Set $u := v + 1 - \min_{i\in S} r_i$. Then $C^u \mathcal{H}_{t,v} \subseteq \mathcal{H}_{t+ud_C,0} + J$.*

Proof. Observe that for any $\boldsymbol{\beta} \in \mathbb{N}^S$, if $|\boldsymbol{\beta}| < \min r_i$, then $\boldsymbol{\beta} \in B$. Hence for any $r' < \min r_i$, $\mathcal{H}_{t,r'} = \mathcal{H}_{t,0}$. Again, the result is obtained by repeating the use of Lemma 3.12 u times. □

Observation 3.15. For positive integers D and R, define $N = 3D^2R$, then

$$\binom{N+3}{3} - R\binom{DN+2}{2} > 0.$$

Proof. The result follows from the equality

$$\binom{N+3}{3} - R\binom{DN+2}{2} = 9R^2D^3\left(D - \frac{1}{2}\right) + R\left(\frac{11}{2}D^2 - 1\right) + 1.$$

□

The following result provides structured annihilating operators of f whose existence will be used in the proof of Theorem 3.2. It also provides degree bounds for all the announced annihilating operators, of which only those concerning P will be used, in the specific situation of Corollary 3.18 ($n = 2$).

Theorem 3.16. *Let $f \in K[[\boldsymbol{x}]]$ be a D-finite power series over $K(\boldsymbol{x})$. Then, there exists a non-zero annihilating operator P of f that satisfies*

- *$P \in K[x_3, \ldots, x_n][x_1x_2]\langle T_{x_1,x_2}, D_{x_1,x_2}\rangle$,*
- *P is of degree $O(d_f^2 r_f^2)$ in x_1x_2, of total degree $O(d_f^9 r_f^8)$ in $x_3, \ldots, x_n$, and of total degree $O(d_f^2 r_f^2)$ in T_{x_1,x_2}, D_{x_1,x_2},*

and for each $h \in \{3, \ldots, n\}$, there exists a non-zero annihilating operator Q_h of f that satisfies

- *$Q_h \in K[x_3, \ldots, x_n][x_1x_2]\langle T_{x_1,x_2}, D_{x_h}\rangle$,*
- *Q_h is of degree $O(d_f^2 r_f^3)$ in x_1x_2, of total degree $O(d_f^9 r_f^{12})$ in $x_3, \ldots, x_n$, and of total degree $O(d_f^2 r_f^3)$ in T_{x_1,x_2}, D_{x_h}.*

Proof. First we prove the existence of the operator P. We apply the counting argument used in [18, 20]. Use Definition 3.9 with $S = \{1, 2\}$. For any positive integer N, set

$$V_N = \operatorname{span}_{K(x_3,\ldots,x_n)} \left\{ C^{2N}(x_1x_2)^i T_{x_1,x_2}^j D_{x_1,x_2}^\ell \mid i + j + \ell \leqslant N \right\}$$

and

$$W_N = \operatorname{span}_{K(x_3,\ldots,x_n)} \mathcal{H}_{2N(d_1+d_2+1),0}.$$

By degree considerations, for any integers i, j, ℓ satisfying $i + j + \ell \leqslant N$ we have

$$(x_1x_2)^i T_{x_1,x_2}^j D_{x_1,x_2}^\ell \in \mathcal{F}_{j+2i,\, j+2\ell} \subseteq \mathcal{H}_{j+2i,\, j+2\ell} \subseteq \mathcal{H}_{2N,\, 2N}.$$

Note that $\operatorname{tdeg}(C) \leqslant d_C = d_1 + d_2$. Hence by Lemma 3.13,

$$C^{2N}(x_1x_2)^i T_{x_1,x_2}^j D_{x_1,x_2}^\ell \in \mathcal{H}_{2N(d_1+d_2+1),\,0} + J. \tag{3.4}$$

Consequently, we have the inclusion $V_N \subseteq W_N + K(x_3, \ldots, x_n)\, J$ between $K(x_3, \ldots, x_n)$-vector spaces. Note the asymptotic estimates

$$\dim_{K(x_3,\ldots,x_n)} V_N = \binom{N+3}{3} = \Theta(N^3),$$

where the first equality is by Lemma 3.3, and

$$\dim_{K(x_3,\ldots,x_n)} W_N = r_1 r_2 \binom{2N(d_1+d_2+1)+2}{2} = \Theta(N^2).$$

Choosing sufficient large N results in $\dim(V_N) > \dim(W_N)$. So, some non-zero element of V_N is in $K(x_3,\ldots,x_n)J$ and without loss of generality we can choose it in $\mathbb{W}_n \cap V_N$. Observe that this operator has C^{2N} as a left factor. So, dividing by C^{2N} yields a non-zero annihilating operator of f in $K[x_1x_2, x_3, \ldots, x_n]\langle D_{x_1}, D_{x_2}\rangle$.

To control the degree and order of such an annihilating operator, we now make a more specific choice that will lead to the announced operator P. To this end, we make (3.4) explicit in the form

$$C^{2N}(x_1x_2)^i T^j_{x_1,x_2} D^\ell_{x_1,x_2} \in \sum_{\substack{i_1<r_1,\ i_2<r_2,\\ k_1+k_2\leqslant 2N(d_1+d_2+1)}} q_{i,j,\ell,i_1,i_2,k_1,k_2} x_1^{k_1} x_2^{k_2} D^{i_1}_{x_1} D^{i_2}_{x_2} + J,$$

for polynomials $q_{i,j,\ell,i_1,i_2,k_1,k_2}$ of $K[x_3,\ldots,x_n]$ of total degree bounded by $2N(d_1+d_2+1)$, and we set up an ansatz of the form

$$\begin{aligned} C^{2N}P &= \sum_{i+j+\ell\leqslant N} p_{i,j,\ell} C^{2N}(x_1x_2)^i T^j_{x_1,x_2} D^\ell_{x_1,x_2} \\ &\in \sum_{\substack{i_1<r_1,\ i_2<r_2,\\ k_1+k_2\leqslant 2N(d_1+d_2+1)}} q_{i_1,i_2,k_1,k_2} x_1^{k_1} x_2^{k_2} D^{i_1}_{x_1} D^{i_2}_{x_2} + J, \end{aligned} \tag{3.5}$$

where the $p_{i,j,\ell}$ are undetermined polynomials from $K[x_3,\ldots,x_n]$ and the resulting coefficients q_{i_1,i_2,k_1,k_2} are polynomials of $K[x_3,\ldots,x_n]$ given as linear combinations of the $p_{i,j,\ell}$ by

$$q_{i_1,i_2,k_1,k_2} = \sum_{i+j+\ell\leqslant N} p_{i,j,\ell}\, q_{i,j,\ell,i_1,i_2,k_1,k_2}.$$

After applying to f to obtain

$$C^{2N}P(f) = \sum_{\substack{i_1<r_1,\ i_2<r_2,\\ k_1+k_2\leqslant 2N(d_1+d_2+1)}} q_{i_1,i_2,k_1,k_2} x_1^{k_1} x_2^{k_2} D^{i_1}_{x_1} D^{i_2}_{x_2}(f),$$

we can enforce $P(f) = 0$ by forcing each q_{i_1,i_2,k_1,k_2} to be zero. This gives a linear system over $K(x_3,\ldots,x_n)$ with $\binom{N+3}{3}$ variables and a number S of equations that is

$$S := \dim_{K(x_3,\ldots,x_n)} W_N = r_1 r_2 \binom{2N(d_1+d_2+1)+2}{2}.$$

Set $R := r_1r_2$ and $D := 2(d_1+d_2+1) \geqslant 2$. By Observation 3.15, we can choose $N := 3D^2R$ so as to get a system with more variables than equations and thus a system with a non-trivial solution. Because the corresponding polynomial matrix is of size $S \times \binom{N+3}{3}$ with entries of total degree $2N(d_1+d_2+1)$, by Lemma 2.5 we have, for a suitable non-zero solution $(p_{i,j,\ell})$,

$$\operatorname{tdeg}(p_{i,j,\ell}) \leqslant 2N(d_1+d_2+1) r_1 r_2 \binom{2N(d_1+d_2+1)+2}{2} = O(d_f^9 r_f^8),$$

where the total degree is with respect to $x_3,\ldots,x_n$. This non-trivial solution leads to a non-zero annihilator $P \in K[x_1x_2, x_3, \ldots, x_n]\langle T_{x_1,x_2}, D_{x_1,x_2}\rangle$ of f. From the ansatz form (3.5), P has its degree in x_1x_2 bounded by $N = O(d_f^2 r_f^2)$ and its total degree in T_{x_1,x_2}, D_{x_1,x_2} not exceeding $N = O(d_f^2 r_f^2)$. This leads to the desired degree and order bounds for P.

For each $h \in \{3,\ldots,n\}$, the proof of the existence of the operator Q_h is similar. Using Definition 3.9 with $S = \{1,2,h\}$, we set

$$V_N = \operatorname{span}_{K(x_3,\ldots,x_n)} \left\{ C^N (x_1x_2)^i T^j_{x_1,x_2} D^\ell_{x_h} \ \middle|\ i+j+\ell \leqslant N \right\},$$

and

$$W_N = \mathrm{span}_{K(x_3,\ldots,x_n)} \mathcal{H}_{N(d_1+d_2+d_h+2),\,0}.$$

This time we derive $V_N \subseteq \mathcal{H}_{2N,\,N}$ (not $\mathcal{H}_{2N,\,2N}$) and we have the additional term d_h in $\mathrm{tdeg}(C) \leqslant d_C = d_1 + d_2 + d_h$, so that the analogue of (3.4) is

$$C^N (x_1x_2)^i T_{x_1,x_2}^j D_{x_h}^\ell \in \mathcal{H}_{N(d_1+d_2+d_h+2),\,0} + J.$$

Set $R := r_1r_2r_h$ and $D := d_1 + d_2 + d_h + 2 \geqslant 2$. We can still choose

$$N := 3D^2R = 3\,(d_1 + d_2 + d_h + 2)^2 r_1r_2r_h.$$

Then by Observation 3.15

$$\dim_{K(x_3,\ldots,x_n)} V_N - \dim_{K(x_3,\ldots,x_n)} W_N = \binom{N+3}{3} - R\binom{DN+2}{2} > 0.$$

Continuing as we did for P, we obtain that there exists a non-zero operator

$$Q_h \in K[x_1x_2, x_3, \ldots, x_n]\langle T_{x_1,x_2}, D_{x_h}\rangle$$

such that $Q_h(f) = 0$. By a similar argument, we have that Q_h is of degree at most $N = O(d_f^2 r_f^3)$ in x_1x_2, of total degree $O(d_f^9 r_f^{12})$ in $x_3, \ldots, x_n$, and of total degree at most $N = O(d_f^2 r_f^3)$ in T_{x_1,x_2}, D_{x_h}. □

After the preparation above, let us prove the diagonal theorem.

Proof of Theorem 3.2. Let $u_1, \ldots, u_n$ be new variables. Write $K\langle\langle u_1, \ldots, u_n\rangle\rangle$ for the associative K-algebra over the free non-commutative monoid generated by $\{u_1, \ldots, u_n\}$. Assume that $f \in K[[\boldsymbol{x}]]$ is D-finite over $K(\boldsymbol{x})$. By Theorem 3.16, there exists a non-zero operator P in $K[x_1x_2, x_3, \ldots, x_n]\langle T_{x_1,x_2}, D_{x_1,x_2}\rangle$ and, for each $h \in \{3, \ldots, n\}$, a non-zero operator Q_h in $K[x_1x_2, x_3, \ldots, x_n]\langle T_{x_1,x_2}, D_{x_h}\rangle$ such that $P(f) = 0$ and for each $h \in \{3, \ldots, n\}$, $Q_h(f) = 0$.

We first show that there is a non-zero operator $\bar{P} \in K(x_1, x_3, \ldots, x_n)\langle D_{x_1}\rangle$ such that $\bar{P}(\Delta_{1,2}(f)) = 0$. Recall that T_{x_1,x_2} commutes with x_1x_2 and D_{x_1,x_2}. Consider the maximal integer s such that

$$P = T_{x_1,x_2}^s \tilde{P} \quad \text{with} \quad \tilde{P} = \sum_{i=0}^m T_{x_1,x_2}^i A_i(x_1x_2, x_3, \ldots, x_n, D_{x_1,x_2}) \tag{3.6}$$

for some $A_i \in K\langle\langle u_1, \ldots, u_n\rangle\rangle$, where $A_i(\sigma_1, \ldots, \sigma_n)$ denotes the evaluation at $u_1 = \sigma_1, \ldots, u_n = \sigma_n$ of A_i for elements $\sigma_1, \ldots, \sigma_n \in \mathbb{W}_n$. The maximality of s implies $A_0 \neq 0$. By Lemma 3.7, we have

$$\tilde{P}(f) = \sum_{i=0}^m T_{x_1,x_2}^i A_i(f) = g(x_1x_2, x_3, \ldots, x_n) \tag{3.7}$$

for some power series g in $n-1$ variables. Since $\Delta_{1,2}T_{x_1,x_2} = 0$ and by Proposition 3.6, taking the diagonal of the two sides of (3.7) yields

$$\Delta_{1,2}\tilde{P}(f) = A_0(x_1, x_3, \ldots, x_n, D_{x_1}\theta_{x_1})(\Delta_{1,2}(f)) = g(x_1, x_3, \ldots, x_n).$$

The operator $H := A_0(x_1, x_3, \ldots, x_n, D_{x_1}\theta_{x_1})$ is non-zero, since

$$x_1, x_3, \ldots, x_n, D_{x_1}\theta_{x_1}$$

are linearly independent over K by Lemma 3.4. Because f is D-finite over $K(\boldsymbol{x})$, the series $\tilde{P}(f)$ is also D-finite over $K(\boldsymbol{x})$. Hence $g(x_1, x_3, \ldots, x_n) = \tilde{P}(f)|_{x_2=1}$ is D-finite over $K(x_1, x_3, \ldots, x_n)$ by Lemma 3.8. Therefore there exists a non-zero operator $G \in K(x_1, x_3, \ldots, x_n)\langle D_{x_1}\rangle$ such that $G(g) = 0$. Then the operator $\bar{P} := GH$ is non-zero and $\bar{P}(\Delta_{1,2}(f)) = 0$.

The existence of a non-zero operator $\bar{Q}_h \in K(x_1, x_3, \ldots, x_n)\langle D_{x_h}\rangle$ such that $\bar{Q}_h(\Delta_{1,2}(f)) = 0$ for each $h \in \{3, \ldots, n\}$ is proved similarly. The only difference is the variation in the formula

$$\begin{aligned}\Delta_{1,2}A_0(x_1x_2, x_3, \ldots, x_n, D_{x_h})(f) &= A_0(x_1, x_3, \ldots, x_n, D_{x_h})\Delta_{1,2}(f)\\ &= g(x_1, x_3, \ldots, x_n).\end{aligned}$$

Hence we conclude that $\Delta_{1,2}(f)$ is D-finite over $K(x_1, x_3, \ldots, x_n)$. □

The following result is very much inspired by [19], which we merely generalize to the bivariate situation. The reader will pay attention that it combines bounds about a function f provided by a system of equations, each in a single derivative like in Definition 2.3, with bounds on a (potentially) partial differential operator L, to derive bounds on equations in a single derivative for $L(f)$.

Lemma 3.17. *Fix $n = 2$ and a bivariate D-finite function f. Given a system of linear differential equations with known order and degree bounds r_f and d_f exhibiting the D-finiteness of f, as well as an operator L of order r_L and degree d_L, there exists a system of linear differential equations exhibiting the D-finiteness of $g = L(f)$, whose order r_g and degree d_g are bounded by*

$$d_g \leqslant (d_L + 2d_f(r_f^2 + r_L))r_f^2 \quad \text{and} \quad r_g \leqslant r_f^2. \tag{3.8}$$

Proof. Use Definition 3.9 when $S = \{1, 2\}$. We look for non-zero operators $A \in K[x_1x_2]\langle D_{x_1}\rangle$ annihilating g, that is, such that $(AL)(f) = 0$. Write r_A and d_A for the order and degree of a potential A. For $l \in K[x_1, x_2]$, if $\deg(l) \leqslant d_L$, $0 \leqslant k \leqslant r_A$, and $0 \leqslant i + j \leqslant r_L$, then, by Lemma 3.13 we have

$$C^{r_A+r_L} D_{x_1}^k l(x_1, x_2)(D_{x_1}^i D_{x_2}^j) \in \mathcal{H}_{d_L+d_C(r_A+r_L),0} + J,$$

hence for a potential $A = \sum_{k=0}^{r_A} a_k(x_1, x_2)D_{x_1}^k$ we need to have

$$C^{r_A+r_L}(AL)(f) = \sum_{0\leqslant i<r_1,\, 0\leqslant j<r_2} \sum_{k=0}^{r_A} a_k q_{i,j,k} D_{x_1}^i D_{x_2}^j(f)$$

for explicit polynomials $q_{i,j,k} \in K[x_1, x_2]$ of degree at most $d_L + d_C(r_A + r_L)$. Now, for this to be zero, the $r_A + 1$ polynomial coefficients of A need to cancel the $r_1r_2 = O(r_f^2)$ equations obtained by equating the coefficients of the $K[x_1, x_2]$-linearly independent elements $D_{x_1}^i D_{x_2}^j(f)$ that appear in the sum. Setting $r_A = r_1r_2$ ensures a non-zero solution exist, and Lemma 2.5 guarantees there exists a solution with degree d_A at most $(d_L + d_C(r_A + r_L))r_A$. Looking for $A \in K[x_1x_2]\langle D_{x_2}\rangle$ leads to the same bounds, which leads to (3.8).

□

Corollary 3.18. Let $f \in K[[x_1, x_2]]$ be D-finite over $K(x_1, x_2)$. Then $\Delta_{1,2}(f)$ is D-finite over $K(x_1)$. In addition, there exists a non-zero operator $\bar{P}$ that satisfies $\bar{P}(\Delta_{1,2}(f)) = 0$ and

$$\deg(\bar{P}) = O(d_f^3 r_f^4) \quad \text{and} \quad \operatorname{ord}(\bar{P}) = O(d_f^2 r_f^2).$$

Proof. The first statement is just Theorem 3.2 in the case $n = 2$. For the degree bounds, we continue in the context of the proof of Theorem 3.2. Specifically, we have found:

- an operator $\tilde{P} = \tilde{P}(x_1x_2, T_{x_1,x_2}, D_{x_1,x_2})$ that is a factor of an operator P that we obtained by Theorem 3.16 and therefore satisfies that its degree in x_1x_2 and its degree in D_{x_1,x_2} are both $O(d_f^2 r_f^2)$,
- a univariate power series g such that $\tilde{P}(f) = g(x_1x_2)$,
- a non-zero operator $H = H(x_1, D_{x_1}\theta_{x_1})$ such that $H(x_1x_2, D_{x_1,x_2})$ is the coefficient of T_{x_1,x_2}^0 in $\tilde{P}$ and $H(\Delta_{1,2}(f)) = g(x_1)$.

By construction, both $\tilde{P}$ and H admit the same bounds on order and degree as P, in particular, both $\operatorname{ord}(H)$ and $\deg(H)$ are in $O(d_f^2 r_f^2)$. Now, Lemma 3.17 applies to the D-finite function f and the operator H to prove the existence of a non-zero annihilator $G \in K[x_1]\langle D_{x_1}\rangle$ of g satisfying

$$\deg(G) \leqslant (\deg(H) + 2d_f(r_f^2 + \operatorname{ord}(H)))r_f^2 = O(d_f^3 r_f^4) \quad \text{and} \quad \operatorname{ord}(G) \leqslant r_f^2$$

as a consequence of (3.8). Setting $\bar{L} = GH$ and observing that H has lower bounds than g gives the announced result. □

Remark 3.19. It is unsatisfactory that we could not find and apply a one-stage variant of Gessel's approach, especially in view of the bivariate case in which it outperforms Lipshitz's approach that is developed in the next section. After this work, it would still be of interest to derive such a direct variant.

4 Lipshitz's method for bounds of diagonal

In this section, we analyze the method of Lipshitz [20] and we make specific choices in it so as to construct annihilating operators of a diagonal and to derive upper bounds on their order and degree.

Let us provide definitions that generalize those of Section 1. Given integers n and m satisfying $0 \leqslant m \leqslant n-1$, we use the notation $\boldsymbol{s}$ for $s_1, \ldots, s_m$ and $\hat{\boldsymbol{x}}$ for $x_{m+1}, \ldots, x_n$. In particular, the list $\boldsymbol{s}$ is empty if $m = 0$, which was the setting in Section 1. The variable x_{m+1} is denoted by t if $m \geqslant 1$: in this new situation, our goal is to take a diagonal with respect to $\boldsymbol{s}, t = s_1, \ldots, s_m, x_{m+1}$, keeping $\hat{\boldsymbol{x}} = x_{m+2}, \ldots, x_n$ as parameters. For primary diagonals there is a single s_i ($m = 1$), and we simply denote s_1 by s. In other words, we have:

$$\boldsymbol{s}, \hat{\boldsymbol{x}} = \begin{cases} x_1, \ldots, x_n & \text{if} \quad m = 0, \\ s, t\,(= x_2), x_3, \ldots, x_n & \text{if} \quad m = 1, \\ s_1, \ldots, s_m, t\,(= x_{m+1}), x_{m+2}, \ldots, x_n & \text{if} \quad m \geqslant 2. \end{cases}$$

The definitions of τ that will be needed, (4.3) in the present section and (4.32) in Section 4.2, motivate that we accommodate series with negative exponents by defining

$$M := \bigcup_{k \in \mathbb{N}} \bigoplus_{|\boldsymbol{\alpha}|+|\boldsymbol{\beta}| \geqslant -k} K \boldsymbol{s}^{\boldsymbol{\alpha}} \hat{\boldsymbol{x}}^{\boldsymbol{\beta}} \subseteq K^{\mathbb{Z}^m \times \mathbb{N}^{n-m}},$$

where $\boldsymbol{\alpha} := (\alpha_1, \ldots, \alpha_m) \in \mathbb{Z}^m$ and $\boldsymbol{\beta} := (\beta_{m+1}, \ldots, \beta_n) \in \mathbb{N}^{n-m}$. This set M is a module over $K[\boldsymbol{s}, \hat{\boldsymbol{x}}]\langle \boldsymbol{D}_s, \boldsymbol{D}_{\hat{\boldsymbol{x}}} \rangle$, but it is not a $K(\boldsymbol{s}, \hat{\boldsymbol{x}})$-vector space. If $m = 0$, then $\boldsymbol{x} = \hat{\boldsymbol{x}}$ and M is just the ring $K[[\boldsymbol{x}]]$ of formal power series.

Definition 4.1 (D-finiteness)**.** An element $F \in M$ is *D-finite* over $K(\boldsymbol{s}, \hat{\boldsymbol{x}})$ if the $K(\boldsymbol{s}, \hat{\boldsymbol{x}})$-vector space generated by the derivatives of F in $\mathcal{T} := K(\boldsymbol{s}, \hat{\boldsymbol{x}}) \otimes_{K[\boldsymbol{s}, \hat{\boldsymbol{x}}]} M$ is finite-dimensional, after identifying each element $m \in M$ with $1 \otimes m \in \mathcal{T}$.

The reader will pay attention to the redefinition of a number of quantities in Sections 4.1.1 and 4.1.2, including M, S, B, C, d_C, R, N, $\mathcal{G}_N$, V_N, W_N, ϕ.

4.1 Bounds for primary diagonal

We analyze the behavior of the primary diagonal operator $\Delta_{2,1}$ and derive the following theorem, which gives bounds on order and degree for linear differential operators that annihilate $\Delta_{2,1}(f)$. The rest of the section consists of the proof of this theorem, with the bounds (4.1) proven by Lemma 4.9 and the bounds (4.2) proven by Lemma 4.11.

Theorem 4.2. *Let $f \in K[[\boldsymbol{x}]]$ be D-finite over $K(\boldsymbol{x})$ and let d_i, f_i, d_f, r_f be as in Definition 2.3. Then, there exists a non-zero annihilating operator P_α of $\Delta_{2,1}(f)$ in $K[t, x_3, \ldots, x_n]\langle D_t \rangle$ that satisfies*

$$\begin{aligned} \deg(P_\alpha) &\leqslant 8(d_1 + d_2 + 1)^2 (r_1 r_2)^2 (8(d_1 + d_2 + 1)^2 r_1 r_2 + 1) = O(d_f^4 r_f^6), \\ \operatorname{ord}(P_\alpha) &\leqslant 4(d_1 + d_2 + 1)\, r_1 r_2 = O(d_f r_f^2), \end{aligned} \tag{4.1}$$

and for each $h \in \{3, \ldots, n\}$, there exists a non-zero annihilating operator P_{h,α_h} of $\Delta_{2,1}(f)$ in $K[t, x_3, \ldots, x_n]\langle D_{x_h} \rangle$ that satisfies

$$\begin{aligned} \deg(P_{h,\alpha_h}) &\leqslant 8(d_1 + d_2 + d_h + 1)^2 (r_1 r_2\, r_h)^2 (8(d_1 + d_2 + d_h + 1)^2 r_1 r_2\, r_h + 1) \\ &= O(d_f^4 r_f^9), \\ \operatorname{ord}(P_{h,\alpha_h}) &\leqslant 4(d_1 + d_2 + d_h + 1)\, r_1 r_2\, r_h = O(d_f r_f^3). \end{aligned} \tag{4.2}$$

We specialize our setting by choosing $m = 1$, that is, we make $\boldsymbol{s}, \hat{\boldsymbol{x}} = s, t, x_3, \ldots, x_n$. We aim to refine Lipshitz's proof [20, Lemma 3] of existence of annihilating operators in $K[\hat{\boldsymbol{x}}]\langle D_s, D_{x_i} \rangle$ for $i = 2, \ldots, n$.

Recall the notation $\mathcal{S} = K(\boldsymbol{x}) \otimes_{K[\boldsymbol{x}]} K[[\boldsymbol{x}]]$ from the introduction. We define two maps σ and τ from $\mathcal{S}$ to M by

$$\tau(h(\boldsymbol{x})) = h\left(s, \frac{t}{s}, x_3, \ldots, x_n\right) \quad \text{and} \quad \sigma(h(\boldsymbol{x})) = \frac{\tau(h(\boldsymbol{x}))}{s}. \tag{4.3}$$

Hence, τ is a ring morphism and we have

$$\sigma(gh) = \tau(g)\sigma(h) \quad \text{for any } g, h \text{ in } \mathcal{S}. \tag{4.4}$$

Lemma 4.3. *Let P be any non-zero operator*

$$P = P(\hat{\boldsymbol{x}}; D_t, D_s) = \sum_{j=\alpha}^{\beta} P_j(\hat{\boldsymbol{x}}; D_t) D_s^j \in K[\hat{\boldsymbol{x}}]\langle D_t, D_s\rangle \tag{4.5}$$

for which $P_\alpha \neq 0$, and let $g \in \sum_{i\in\mathbb{Z}} g_i(\hat{\boldsymbol{x}})s^i$ be any element of M. Then, the coefficient of $s^{-1-\alpha}$ in $P(g)$ is $P_\alpha(g_{-1})$.

Similarly, for any $h \in \{3, \ldots, n\}$, if P_h is a non-zero operator

$$P_h = P_h(\hat{\boldsymbol{x}}; D_{x_h}, D_s) = \sum_{j=\alpha_h}^{\beta_h} P_{h,j}(\hat{\boldsymbol{x}}; D_{x_h}) D_s^j \in K[\hat{\boldsymbol{x}}]\langle D_{x_h}, D_s\rangle, \tag{4.6}$$

for which $P_{h,\alpha_h} \neq 0$, then the coefficient of $s^{-1-\alpha}$ in $P_h(g)$ is $P_{\alpha_h}(g_{-1})$.

Proof. Note that

$$D_s^j(g) = D_s^j\left(\sum_{i\leqslant -2} g_i(\hat{\boldsymbol{x}})s^i\right) + (-1)^j j!\, g_{-1}(\hat{\boldsymbol{x}})s^{-1-j} + D_s^j\left(\sum_{i\geqslant 0} g_i(\hat{\boldsymbol{x}})s^i\right),$$

where the first term has all exponents less than $-1-j$ and the last has all exponents at least 0: only the middle term contributes to the coefficient of s^{-1-j}. So, for $j \geqslant \alpha$, some contribution to the coefficient of $s^{-1-\alpha}$ is only possible if $j = \alpha$, proving the result for the case $P = P(\hat{\boldsymbol{x}}; D_t, D_s)$. The proof for the other cases is the same. □

Consider any non-necessarily D-finite series

$$f = \sum_{i_1,\ldots,i_n \geqslant 0} a_{i_1,\ldots,i_n} x_1^{i_1} \cdots x_n^{i_n} \in K[[\boldsymbol{x}]] \tag{4.7}$$

and the corresponding element $\sigma(f)$ of $M \subseteq \mathcal{T}$. By Definition 2.4 (diagonals) and because we write t for x_2, the primary diagonal $\Delta_{2,1}(f)$ is

$$\Delta_{2,1}(f) = \sum_{i_1,i_3,\ldots,i_n \geqslant 0} a_{i_1,i_1,i_3,\ldots,i_n} t^{i_1} x_3^{i_3} \cdots x_n^{i_n} \in K[[\hat{\boldsymbol{x}}]].$$

By the definition (4.3) of τ and σ, this diagonal is the coefficient of degree s^{-1} in $\sigma(f)$. The following lemma immediately follows, as a consequence of Lemma 4.3.

Lemma 4.4. *Let f be as in (4.7). If $P(f) = 0$ for P and $P_\alpha \neq 0$ as in (4.5), then P_α annihilates $\Delta_{2,1}(f)$. For any $h \in \{3, \ldots, n\}$, if $P_h(f) = 0$ for P_h and $P_{h,\alpha_h} \neq 0$ as in (4.6), then P_{h,α_h} annihilates $\Delta_{2,1}(f)$.*

In the next two subsections, when f is D-finite we will construct operators P and P_h to be used in the previous lemma.

4.1.1 *Controlling and combining the $D_s^i D_t^j(\sigma(f))$*

We construct an operator $P \in K[\hat{\boldsymbol{x}}]\langle D_t, D_s\rangle$ such that $P(\sigma(f)) = 0$. To this end, we introduce two vector spaces depending on $N \in \mathbb{N}$,

$$V_N = A_N(s, \hat{\boldsymbol{x}})\ \mathrm{span}_{K(\hat{\boldsymbol{x}})}\{D_s^i D_t^j \mid i + j \leqslant N\} \tag{4.8}$$

and

$$W_N = \operatorname{span}_{K(\hat{\boldsymbol{x}})}\{s^\alpha \sigma(\boldsymbol{D}_x^{\boldsymbol{\beta}} f) \mid \alpha \leqslant DN, \boldsymbol{\beta} \in B\}, \tag{4.9}$$

where B is a finite set and $A_N(s, \hat{\boldsymbol{x}})$ is a polynomial, both to be determined (see Lemma 4.8). We will prove that the map defined by $\phi(P) := P(\sigma(f))$ is $K(\hat{\boldsymbol{x}})$-linear from V_N to W_N, that it is non-injective for large enough N (see Lemma 4.9). As a by-product, we will get an annihilator P_α of $\Delta_{2,1}(f)$ with controled degree and order (see again Lemma 4.9).

Denote $D_i := D_{x_i}$ for $i = 1, \ldots, n$.

Lemma 4.5. *We have for all $g \in \mathcal{S}$:*

$$\begin{aligned} D_s(\sigma(g)) &= \sigma\left(\left(-x_1^{-1} + D_1 - x_1^{-1}x_2D_2\right)(g)\right), \\ D_t(\sigma(g)) &= \sigma\left((x_1^{-1}D_2)(g)\right), \\ D_{x_h}(\sigma(g)) &= \sigma(D_h(g)), \quad h = 3, \ldots, n. \end{aligned}$$

Proof. For the first two identities, write the following two equations by the chain rule, then use the formulas $\tau(x_1) = s$, $\tau(x_2) = t/s$, and (4.4):

$$\begin{aligned} D_s(\sigma(g)) &= -\frac{1}{s}\sigma(g) + \sigma(D_1(g)) - \frac{t}{s^2}\sigma(D_2(g)), \\ D_t(\sigma(g)) &= \frac{1}{s}\sigma(D_2(g)). \end{aligned}$$

The third identity is obvious. □

Define for any $N \in \mathbb{N}$:

$$\mathcal{G}_N := \bigoplus_{a+b \leqslant N} x_1^{-N} K[x_1, x_2]_{\leqslant N} D_1^a D_2^b. \tag{4.10}$$

Lemma 4.6. *For all $g \in \mathcal{S}$ and all non-negative integers i and j, $D_t^j D_s^i(\sigma(g))$ is an element of $\sigma(\mathcal{G}_{i+j}(g))$.*

Proof. It follows immediately from Lemma 4.5 that, for all $i, j \in \mathbb{N}$,

$$\begin{aligned} D_t^j D_s^i(\sigma(g)) &= D_t^j \sigma\left((-x_1^{-1} + D_1 - x_1^{-1}x_2D_2)^i(g)\right) \\ &= \sigma\left((x_1^{-1}D_2)^j(-x_1^{-1} + D_1 - x_1^{-1}x_2D_2)^i(g)\right). \end{aligned} \tag{4.11}$$

Consider an element $x^{-i}pD_1^aD_2^b$ of $\mathcal{G}_i$, or equivalently, integers a and b and a polynomial $p \in K[x_1, x_2]$ satisfying $a + b \leqslant i$ and $\operatorname{tdeg}(p) \leqslant i$. We observe that

$$\left(-\frac{1}{x_1} + D_1 - \frac{x_2}{x_1}D_2\right)\left(\frac{p}{x_1^i}D_1^aD_2^b\right) = \frac{1}{x_1^{i+1}}\Big(-pD_1^aD_2^b - ipD_1^aD_2^b + x_1D_1(p)D_1^aD_2^b + x_1pD_1^{a+1}D_2^b - x_2D_2(p)D_1^aD_2^b - x_2pD_1^aD_2^{b+1}\Big)$$

is in $\mathcal{G}_{i+1}$. Therefore, $\left(-\frac{1}{x_1} + D_1 - \frac{x_2}{x_1}D_2\right)\mathcal{G}_i \subseteq \mathcal{G}_{i+1}$, by linearity. We derive similarly

$$\left(\frac{1}{x_1}D_2\right)\left(\frac{1}{x_1^i}p(x_1, x_2)D_1^aD_2^b\right) = \frac{1}{x_1^{i+1}}\left(D_2(p)D_1^aD_2^b + pD_1^aD_2^{b+1}\right) \in \mathcal{G}_{i+1},$$

and $\left(\frac{1}{x_1}D_2\right)\mathcal{G}_i \subseteq \mathcal{G}_{i+1}$. Since $1 \in \mathcal{G}_0$, we get by induction that for all $i, j \in \mathbb{N}$,

$$\left(\frac{1}{x_1}D_2\right)^j\left(-\frac{1}{x_1} + D_1 - \frac{x_2}{x_1}D_2\right)^i \in \mathcal{G}_{i+j}.$$

□

Lemma 4.7. *For any integers p and q, we have:*

$$\tau\left(\frac{1}{x_1^q}K[x_1,x_2]_{\leqslant p}\right) \subseteq \frac{K[s,t]_{\leqslant 2p}}{s^{p+q}} \quad \text{and} \quad \tau\left(K[\boldsymbol{x}]_{\leqslant p}\right) \subseteq \frac{K[s,\hat{\boldsymbol{x}}]_{\leqslant 2p}}{s^p}.$$

Proof. Both formulas follow by linearity from the action of τ on monomials:

$$\tau(x_1^i x_2^j) = \frac{s^{i+(p-j)}t^j}{s^p} \in \frac{K[s,t]_{\leqslant p+i}}{s^p} \quad \text{if} \quad i+j \leqslant p;$$
$$\tau(\boldsymbol{x}^{\boldsymbol{i}}) = \frac{s^{i_1+(p-i_2)}x_2^{i_2}\cdots x_n^{i_n}}{s^p} \in \frac{K[s,\hat{\boldsymbol{x}}]_{\leqslant p+i_1}}{s^p} \quad \text{if} \quad |\boldsymbol{i}| \leqslant p.$$

□

Lemma 4.8. *Consider $B := \{0,1,\ldots,r_1-1\}\times\{0,1,\ldots,r_2-1\}$, the polynomial C, and $d_C = d_1+d_2 \leqslant 2d_f$ as set by Definition 3.9 for $S := \{1,2\}$. Fix $N \in \mathbb{N}$ and set $D := 2+2d_C \geqslant 2$ and $A_N(s,\hat{\boldsymbol{x}}) := s^{(d_C+2)N}\tau(C^N) \in K[s,\hat{\boldsymbol{x}}]$. Then, if $i+j \leqslant N$, then*

$$D_s^i D_t^j(\sigma(f)) \in \sum_{\substack{\alpha\leqslant DN \\ \boldsymbol{\beta}\in B}} \frac{K[\hat{\boldsymbol{x}}]_{\leqslant DN}}{A_N(s,\hat{\boldsymbol{x}})} s^\alpha \sigma(\boldsymbol{D}_x^{\boldsymbol{\beta}} f). \tag{4.12}$$

Proof. If $i+j \leqslant N$, then Lemma 4.6, Equation (4.4) and Lemma 4.7 imply

$$\begin{aligned} D_s^i D_t^j(\sigma(f)) \in \sigma(\mathcal{G}_{i+j}(f)) &\subseteq \sigma(\mathcal{G}_N(f)) \\ &= \sum_{a+b\leqslant N} \tau(x_1^{-N}K[x_1,x_2]_{\leqslant N})\,\sigma(D_1^a D_2^b(f)) \\ &\subseteq \sum_{a+b\leqslant N} \frac{K[s,t]_{\leqslant 2N}}{s^{2N}}\,\sigma(D_1^a D_2^b(f)). \end{aligned} \tag{4.13}$$

Next, by Definition 3.9 for $S := \{1,2\}$ and by Lemma 3.13 with $u := N \geqslant v := a+b$ and $t := 0$, we have

$$D_1^a D_2^b \in \mathcal{H}_{0,a+b} \subseteq \frac{1}{C^N}\,\mathcal{H}_{Nd_C,0} + \frac{1}{C^N}\,J.$$

Applying to f, then applying σ, yields, appealing again to (4.4), next again to Lemma 4.7:

$$\sigma(D_1^a D_2^b(f)) \in \frac{1}{\tau(C^N)}\,\sigma(\mathcal{H}_{Nd_C,0}(f)) \subseteq \sum_{\boldsymbol{\beta}\in B} \frac{K[s,\hat{\boldsymbol{x}}]_{\leqslant 2d_C N}}{s^{d_C N}\tau(C^N)}\sigma(\boldsymbol{D}_x^{\boldsymbol{\beta}} f). \tag{4.14}$$

Combining (4.13) and (4.14) and using $t = x_2$, we obtain (4.12) where D and A_N are set as in the lemma statement. □

Lemma 4.9. *There exists a non-zero annihilator $P_\alpha(\hat{\boldsymbol{x}}; D_t)$ of $\Delta_{2,1}(f)$ satisfying* (4.1).

Proof. Recall the definitions (4.8) and (4.9) of V_N and W_N, where A_N and B are now fixed. Lemma 4.8 has proved that the $K(\hat{\boldsymbol{x}})$-linear map defined by $\phi(P) := P(\sigma(f))$ is from V_N to W_N. Note that

$$\dim_{K(\hat{\boldsymbol{x}})} V_N = \binom{N+2}{2}, \quad \dim_{K(\hat{\boldsymbol{x}})} W_N \leqslant R(DN+1), \tag{4.15}$$

where $R := r_1 r_2 = O(r_f^2)$. Fix

$$N = 2DR = 4(d_1+d_2+1)\,r_1 r_2 = O(d_f r_f^2), \tag{4.16}$$

so that

$$\dim_{K(\hat{\boldsymbol{x}})} V_N - \dim_{K(\hat{\boldsymbol{x}})} W_N = (3D-1)R+1 > 0 \tag{4.17}$$

and ϕ is non-injective. For all i, j with $i + j \leqslant N$, by Lemma 4.8 there exist polynomials $q^{(i,j)}_{\alpha,\boldsymbol{\beta}} \in K[\hat{\boldsymbol{x}}]$ satisfying $\operatorname{tdeg}(q^{(i,j)}_{\alpha,\boldsymbol{\beta}}) \leqslant DN$ and

$$A_N(s,\hat{\boldsymbol{x}}) D_s^i D_t^j(\sigma(f)) = \sum_{\substack{\alpha \leqslant DN \\ \boldsymbol{\beta} \in B}} q^{(i,j)}_{\alpha,\boldsymbol{\beta}}\, s^{\alpha} \sigma(\boldsymbol{D}_x^{\boldsymbol{\beta}} f) \in W_N.$$

A witness of non-injectivity will be provided by polynomials $p_{i,j} \in K[\hat{\boldsymbol{x}}]$ such that

$$\sum_{i+j \leqslant N} p_{i,j}(\hat{\boldsymbol{x}}) A_N(s,\hat{\boldsymbol{x}})\, D_s^i D_t^j(\sigma(f)) = 0,$$

that is, by coefficient extraction, such that for all $\alpha \leqslant DN$ and $\boldsymbol{\beta} \in B$,

$$\sum_{i+j \leqslant N} p_{i,j}\, q^{(i,j)}_{\alpha,\boldsymbol{\beta}} = 0.$$

Hence we have a linear system

$$\begin{pmatrix} \cdots \\ \cdots q^{(i,j)}_{\alpha,\boldsymbol{\beta}} \cdots \\ \cdots \end{pmatrix} \begin{pmatrix} \vdots \\ p_{i,j} \\ \vdots \end{pmatrix} = 0,$$

where the polynomials $q^{(i,j)}_{\alpha,\boldsymbol{\beta}}$ have total degree at most DN. This system has $\dim_{K(\hat{\boldsymbol{x}})} W_N$ rows and $\dim_{K(\hat{\boldsymbol{x}})} V_N$ columns, where those dimensions are given by (4.15), and by the inequality (4.17) it has more columns than rows. So, Lemma 2.5 applies and leads to a non-zero solution $(p_{i,j})$ satisfying

$$\operatorname{tdeg}(p_{i,j}) \leqslant DN \times R(ND+1) = O(D^4R^3) = O(d_f^4 r_f^6),$$

where we used (4.16). The operator $P := \sum_{i+j\leqslant N} p_{i,j}\, D_s^i D_t^j$ satisfies $P(\sigma(f)) = 0$ and can be written

$$P = \sum_{i=\alpha}^{\beta} P_i(\hat{\boldsymbol{x}}; D_t) D_s^i$$

with $P_\alpha(\hat{\boldsymbol{x}}; D_t) \neq 0$. Then P_α annihilates $\Delta_{2,1}(f)$ and satisfies the announced bounds (4.1). □

4.1.2 *Controlling and combining the* $D_s^i D_{x_h}^j(\sigma(f))$

For each $h \in \{3, \ldots, n\}$, we proceed by an argument similar to the argument of Section 4.1.1 to construct an operator $P_h \in K[\hat{\boldsymbol{x}}]\langle D_{x_h}, D_s\rangle$ such that $P_h(\sigma(f)) = 0$. The proof is a bit simpler, because the action of D_{x_h} on $\sigma(f)$ is simpler that the action of D_t on it. This time, we consider $B = \{0, 1, \ldots, r_1\} \times \{0, 1, \ldots, r_2\} \times \{0, 1, \ldots, r_h\}$, the polynomial C, and $d_C = d_1 + d_2 + d_h \leqslant 3d_f$ as set by Definition 3.9 for $S := \{1, 2, h\}$. In analogy with (4.1) and (4.2), for each $N \in \mathbb{N}$, we introduce

$$V_N = A_N(s,\hat{\boldsymbol{x}})\ \operatorname{span}_{K(\hat{\boldsymbol{x}})}\{D_s^i D_{x_h}^j \mid i + j \leqslant N\}, \tag{4.18}$$

where $A_N = s^{(d_C+2)N}\tau(C^N) \in K[s, \hat{\boldsymbol{x}}]$, and

$$W_N = \operatorname{span}_{K(\hat{\boldsymbol{x}})}\{s^{\alpha}\sigma(\boldsymbol{D}_x^{\boldsymbol{\beta}} f) \mid \alpha \leqslant DN, \boldsymbol{\beta} \in B\}, \tag{4.19}$$

where $D = 2+2d_C = O(d_f)$. We will again prove that the map defined by $\phi(P) := P(\sigma(f))$ is $K(\hat{\boldsymbol{x}})$-linear from V_N to W_N.

In analogy with (4.10), define for any $N \in \mathbb{N}$:

$$\mathcal{G}_N := \bigoplus_{a+b+c\leqslant N} x_1^{-N} K[x_1, x_2]_{\leqslant N} D_1^a D_2^b D_{x_h}^c. \tag{4.20}$$

Lemma 4.10. *Let B, C, and d_C be as defined at the beginning of Section 4.1.2, that is, as set by Definition 3.9 for $S := \{1, 2, h\}$. Then, if $i + j \leqslant N$, then*

$$D_s^i D_{x_h}^j(\sigma(f)) \in \sum_{\substack{\alpha \leqslant DN \\ \boldsymbol{\beta} \in B}} \frac{K[\hat{\boldsymbol{x}}]_{\leqslant DN}}{A_N(s, \hat{\boldsymbol{x}})} s^\alpha \sigma(\boldsymbol{D}_x^{\boldsymbol{\beta}} f). \tag{4.21}$$

Proof. If $i + j \leqslant N$, then Lemma 4.6, the definition (4.20), Equation (4.4) and Lemma 4.7 imply

$$\begin{aligned} D_s^i D_{x_h}^j(\sigma(f)) &\in D_{x_h}^j \sigma(\mathcal{G}_i(f)) \subseteq \sigma(\mathcal{G}_{i+j}(f)) \subseteq \sigma(\mathcal{G}_N(f)) \\ &= \sum_{a+b+c \leqslant N} \tau(x_1^{-N} K[x_1, x_2]_{\leqslant N})\, \sigma(D_1^a D_2^b D_{x_h}^c(f)) \\ &\subseteq \sum_{a+b+c \leqslant N} \frac{K[s, t]_{\leqslant 2N}}{s^{2N}}\, \sigma(D_1^a D_2^b D_{x_h}^c(f)). \end{aligned} \tag{4.22}$$

Next, by Lemma 3.13 with $u := N \geqslant v := a + b + c$ and $t := 0$, we have

$$D_1^a D_2^b D_{x_h}^c \in \mathcal{H}_{0,a+b+c} \subseteq \frac{1}{C^N}\, \mathcal{H}_{Nd_C,0} + \frac{1}{C^N}\, J.$$

Applying to f, then applying σ, yields, appealing again to (4.4), next again to Lemma 4.7:

$$\sigma(D_1^a D_2^b D_{x_h}^c(f)) \in \frac{1}{\tau(C^N)}\, \sigma(\mathcal{H}_{Nd_C,0}(f)) \subseteq \sum_{\boldsymbol{\beta} \in B} \frac{K[s, \hat{\boldsymbol{x}}]_{\leqslant 2d_C N}}{s^{d_C N} \tau(C^N)} \sigma(\boldsymbol{D}_x^{\boldsymbol{\beta}} f). \tag{4.23}$$

Combining (4.22) and (4.23) and using $t = x_2$, we obtain (4.21) where D and A_N are set as in the lemma statement. □

Lemma 4.11. *There exists a non-zero annihilator $P_\alpha(\hat{\boldsymbol{x}}; D_{x_h})$ of $\Delta_{2,1}(f)$ satisfying* (4.2).

Proof. Recall the definitions (4.18) and (4.19) of V_N and W_N. Lemma 4.10 has proved that the $K(\hat{\boldsymbol{x}})$-linear map defined by $\phi(P) := P(\sigma(f))$ is from V_N to W_N. Note that

$$\dim_{K(\hat{\boldsymbol{x}})} V_N = \binom{N+2}{2}, \quad \dim_{K(\hat{\boldsymbol{x}})} W_N \leqslant R(DN + 1),$$

where $R := r_1 r_2 f_h = O(r_f^3)$, and fix

$$N = 2DR = 4(d_1 + d_2 + d_h + 1)\, r_1 r_2 r_h = O(d_f r_f^3). \tag{4.24}$$

The thast three formulas in terms of R and D are the same as in Lemma 4.11, with only the values of R and D changed, so the inequality

$$\dim_{K(\hat{\boldsymbol{x}})} V_N - \dim_{K(\hat{\boldsymbol{x}})} W_N = (3D - 1)R + 1 > 0 \tag{4.25}$$

holds again, and ϕ is non-injective. The proof by linear algebra continues as in the proof of Lemma 4.9, recombining expressions $A_N\, D_s^i D_{x_h}^j(\sigma(f))$ instead of expressions $A_N\, D_s^i D_t^j(\sigma(f))$. It constructs a non-zero operator

$$P_h := \sum_{i+j \leqslant N} p_{i,j}\, D_s^i D_{x_h}^j = \sum_{i=\alpha_h}^{\beta_h} P_{h,i}(\hat{\boldsymbol{x}}; D_{x_h}) D_s^i \in K[\hat{\boldsymbol{x}}]\langle D_s, D_{x_h} \rangle$$

satisfying $P_h(\sigma(f)) = 0$, $P_{h,\alpha_h} \neq 0$, and

$$\operatorname{tdeg}(p_{i,j}) \leqslant ND \times R(ND + 1) = O(D^4 R^3) = O(d_f^4 r_f^9). \tag{4.26}$$

Then P_{h,α_h} annihilates $\Delta_{2,1}(f)$ and satisfies the announced bounds (4.2). □

4.1.3 *Iterating primary diagonals*

We can now estimate bounds on the degree and order of an annihilating operator for the complete diagonal of f obtained by successive primary diagonals. In analogy with the definition (2.2) of the complete diagonal, we consider the partial diagonal

$$g := \Delta_{k+1,k}\Delta_{k,k-1}\cdots\Delta_{2,1}(f) \in K[[x_{k+1},\ldots,x_n]].$$

obtained after k iterations of a primary diagonal. Assume that there exists a non-zero annihilating operator for g with respective degree and order bounds

$$O\left(d_f^{u(k)} r_f^{v(k)}\right) \quad \text{and} \quad O\left(d_f^{s(k)} r_f^{t(k)}\right). \tag{4.27}$$

By Theorem 4.2 applied to $f = g$, there exists a non-zero annihilating operator for $\Delta_{k+2,k+1}(g)$, with respective degree and order bounds analogous to (4.27) for exponents $u(k+1)$, $v(k+1)$, $s(k+1)$, $t(k+1)$ given by

$$\begin{pmatrix} u(k+1) & v(k+1) \\ s(k+1) & t(k+1) \end{pmatrix} = \begin{pmatrix} 4 & 9 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} u(k) & v(k) \\ s(k) & t(k) \end{pmatrix}.$$

Here, the entries of the constant matrix are obtained as the maximums of the exponents appearing in the big O terms in (4.1) and (4.2). This sets up a recurrence that we proceed to analyze. The matrix $\left(\begin{smallmatrix} 4 & 9 \\ 1 & 3 \end{smallmatrix}\right)$ has two eigenvalues satisfying $\lambda^2 - 7\lambda + 3 = 0$, namely

$$\lambda_1 := \frac{7+\sqrt{37}}{2} \approx 6.54\ldots, \qquad \lambda_2 := \frac{7-\sqrt{37}}{2} \approx 0.46\ldots. \tag{4.28}$$

Taking initial values for s, t, u, v in to account, we get

$$\begin{aligned}
s(k) &= \frac{1}{\sqrt{37}}\lambda_1^k - \frac{1}{\sqrt{37}}\lambda_2^k && \approx (0.16\ldots)\lambda_1^k - (0.16\ldots)\lambda_2^k, \\
t(k) &= \left(\frac{1}{2} - \frac{1}{2\sqrt{37}}\right)\lambda_1^k + \left(\frac{1}{2} + \frac{1}{2\sqrt{37}}\right)\lambda_2^k && \approx (0.42\ldots)\lambda_1^k + (0.58\ldots)\lambda_2^k, \\
u(k) &= \left(\frac{1}{2} - \frac{5}{2\sqrt{37}}\right)\lambda_1^k + \left(\frac{1}{2} + \frac{5}{2\sqrt{37}}\right)\lambda_2^k && \approx (0.09\ldots)\lambda_1^k + (0.91\ldots)\lambda_2^k, \\
v(k) &= \frac{9}{\sqrt{37}}\lambda_1^k - \frac{9}{\sqrt{37}}\lambda_2^k && \approx (1.48\ldots)\lambda_1^k - (1.48\ldots)\lambda_2^k.
\end{aligned} \tag{4.29}$$

Degree and order bounds for an annihilating operator P of $\Delta(f)$ are obtained for $k = n-1$, and (4.29) leads to the respective asymptotic formulas

$$\begin{aligned}
\deg(P) &= O\left(d_f^{u(n-1)} r_f^{v(n-1)}\right) = d_f^{O(\lambda_1^n)} r_f^{O(\lambda_1^n)}, \\
\mathrm{ord}(P) &= O\left(d_f^{s(n-1)} r_f^{t(n-1)}\right) = d_f^{O(\lambda_1^n)} r_f^{O(\lambda_1^n)}.
\end{aligned}$$

when n, d_f, and r_f tend independently to infinity, and where the constants in the big O's are small (at most 1).

4.2 Complete diagonal in a single step

Following [20, Remarks, item (3)], instead of iterating primary diagonal transformations, we can get the operator that annihilates the complete diagonal of f in a single step. The goal of this subsection is indeed the construction of a specific linear differential operator annihilating $\Delta(f)$ that satisfies the bounds presented in the following theorem. These bounds are simply exponential in n, and therefore asymptotically smaller than the bounds obtained by the method by iteration, which are doubly exponential in n.

Theorem 4.12. *Let $f \in K[[\boldsymbol{x}]]$ be D-finite over $K(\boldsymbol{x})$ and let d_i, f_i, d_f, r_f be as in Definition 2.3. Then, there exists an annihilating operator $\tilde{P}$ of $\Delta(f)$ in $K[t]\langle D_t\rangle$ that satisfies, for all $\varepsilon > 0$,*

$$\deg(\tilde{P}) \leqslant N' = O((2+\varepsilon)^n n^{2n} d_f^n r_f^n), \quad \operatorname{ord}(\tilde{P}) \leqslant N = O((2+\varepsilon)^n n^{2n-1} d_f^{n-1} r_f^n), \tag{4.30}$$

when n, d_f, and r_f tend independently to infinity, and where

$$N' = (2D+1)^n \prod_{j=1}^{n} r_j, \; N = \frac{(2D+1)^n}{D} \prod_{j=1}^{n} r_j, \; \text{for } D = n\left(2 + \sum_{i=1}^{n} d_i\right). \tag{4.31}$$

To prepare for the proof, we specialize the setting introduced at the beginning of Section 4 by setting $m = n-1$, so that $t = x_n$, and we define two maps σ and τ from $\mathcal{S}$ to M by

$$\tau(h(\boldsymbol{x})) = h\left(s_1, \frac{s_2}{s_1}, \frac{s_3}{s_2}, \ldots, \frac{s_{n-1}}{s_{n-2}}, \frac{t}{s_{n-1}}\right) \quad \text{and} \quad \sigma(h(\boldsymbol{x})) = \frac{\tau(h(\boldsymbol{x}))}{s_1 \cdots s_{n-1}}, \tag{4.32}$$

which the reader will compare with (4.3). Hence, as in the previous subsection, τ is a ring morphism and the formula (4.4) holds again.

In order to generalize Lemmas 4.3 and 4.4, we introduce some convenient notation for coefficient extraction. For a series

$$g = \sum_{\boldsymbol{i},j} g_{\boldsymbol{i},j} \boldsymbol{s}^{\boldsymbol{i}} t^j \in M,$$

variables $v_1, \ldots, v_\ell$ and exponents $e_1, \ldots, e_\ell$, with $\{v_1, \ldots, v_\ell\} \subset \{\boldsymbol{s}, t\}$, we denote by

$$[v_1^{e_1} \cdots v_\ell^{e_\ell}]g$$

the sub-series of g involving only the monomials $\boldsymbol{s}^{\boldsymbol{i}} t^j$ in which v_1 has exponent exactly e_1, v_2 has exponent exactly e_2, etc. Note that this is mere notation and that $[v_1^{e_1}]g$ need not be equal to $[v_1^{e_1} v_2^0]g$ although $v_1^{e_1} = v_1^{e_1} v_2^0$ in M. We do analogously with an operator $P \in (K[t]\langle D_t\rangle)[\boldsymbol{D}_s]$ and a set of variables $\{v_1, \ldots, v_\ell\} \subset \{\boldsymbol{D}_s\}$, with the convention that coefficients are always written to the left of the monomials.

Lemma 4.13. *Let $P \in (K[t]\langle D_t\rangle)[\boldsymbol{D}_s]$ be a non-zero operator viewed with coefficients in $K[t]\langle D_t\rangle$. Consider any lexicographical order $\succ$ on the commutative monoid generated by $\{D_{s_1}, \ldots, D_{s_{n-1}}\}$, e.g., the lexicographical order for which $D_{s_1} \succ D_{s_2} \succ \cdots \succ D_{s_{n-1}}$. Let $D_{s_1}^{\alpha_1} \cdots D_{s_{n-1}}^{\alpha_{n-1}}$ be the minimal monomial in P with respect to this order, so that*

$$P = \tilde{P}(t; D_t) D_{s_1}^{\alpha_1} \cdots D_{s_{n-1}}^{\alpha_{n-1}} + \textit{terms with higher monomials} \tag{4.33}$$

for some non-zero $\tilde{P} \in K[t]\langle D_t\rangle$. Additionally, let

$$g = \sum_{\boldsymbol{i},j} g_{\boldsymbol{i},j} \boldsymbol{s}^{\boldsymbol{i}} t^j$$

be any series in M. Then,

$$[s_1^{-(\alpha_1+1)} \cdots s_{n-1}^{-(\alpha_{n-1}+1)}]P(g) = (-1)^{|\boldsymbol{\alpha}|} \alpha_1! \cdots \alpha_{n-1}! \, \tilde{P}([s_1^{-1} \cdots s_{n-1}^{-1}]g).$$

Proof. For the proof, we fix the lexicographical order $\succ$ to satisfy $D_{s_1} \succ D_{s_2} \succ \cdots \succ D_{s_{n-1}}$. Any other lexicographical order would be dealt with by obvious modifications. We claim that, for any i, after writing

$$P = \bar{P}(t; D_t, D_{s_{i+1}}, \ldots, D_{s_{n-1}}) D_{s_1}^{\alpha_1} \cdots D_{s_i}^{\alpha_i} + Q$$

for some non-zero $\bar{P} \in K[t]\langle D_t, D_{s_{i+1}}, \ldots, D_{s_{n-1}}\rangle$ and some operator Q whose monomials $\boldsymbol{D}_s^{\boldsymbol{\beta}}$ are all such that $(\beta_1, \ldots, \beta_i)$ is lexicographically higher than $(\alpha_1, \ldots, \alpha_i)$, we have

$$[s_1^{-(\alpha_1+1)} \cdots s_i^{-(\alpha_i+1)}]P(g) = (-1)^{\alpha_1+\cdots+\alpha_i} \alpha_1! \cdots \alpha_i! \, \bar{P}([s_1^{-1} \cdots s_i^{-1}]g). \tag{4.34}$$

The proof is by induction on $i \in \{0, \dots, n-1\}$. The base case $i = 0$ corresponds to no coefficient extraction and $\bar{P} = P$, so that (4.34) is the tautology $P(g) = 1 \times \bar{P}(g)$. Fix $i \geqslant 1$ and, in order to prove (4.34), assume the analog of (4.34) at $i-1$, that is,

$$[s_1^{-(\alpha_1+1)} \cdots s_{i-1}^{-(\alpha_{i-1}+1)}]P(g) = (-1)^{\alpha_1+\dots+\alpha_{i-1}} \alpha_1! \cdots \alpha_{i-1}! \, \hat{P}([s_1^{-1} \cdots s_{i-1}^{-1}]g), \tag{4.35}$$

for some non-zero

$$\hat{P} = \sum_{j \geqslant \alpha_i} \hat{P}_j(t; D_t, D_{s_{i+1}}, \dots, D_{s_{n-1}}) D_{s_i}^j \in K[t]\langle D_t, D_{s_i}, \dots, D_{s_{n-1}} \rangle.$$

Consider a series $c \in M$ involving only $t, s_{i+1}, \dots, s_{n-1}$, as well as some integer $u \in \mathbb{Z}$, to compute

$$[s_i^{-(\alpha_i+1)}]\hat{P}(cs_i^u) = \sum_{j \geqslant \alpha_i} \hat{P}_j(c) \, u(u-1) \cdots (u-j+1) \, [s_i^{-(\alpha_i+1)}] s_i^{u-j}.$$

The last term $[s_i^{-(\alpha_i+1)}] s_i^{u-j}$ is equal to 1 if and only if $j = u + \alpha_i + 1$, and is zero otherwise. So the sum reduces to $\hat{P}_{u+\alpha_i+1}(c) \, u(u-1) \cdots (-\alpha_i)$. This is zero if $u \geqslant 0$ because of the polynomial in u, but also if $u \leqslant -2$ because $\hat{P}_j = 0$ if $j < \alpha_i$. The only possibly non-zero case is therefore for $u = -1$, making the sum equal to $(-1)^{\alpha_i} \alpha_i! \, \hat{P}_{\alpha_i}(c)$. By linearity, we obtain

$$[s_i^{-(\alpha_i+1)}]\hat{P}([s_1^{-1} \cdots s_{i-1}^{-1}]g) = (-1)^{\alpha_i} \alpha_i! \, \hat{P}_{\alpha_i}([s_i^{-1}] \, [s_1^{-1} \cdots s_{i-1}^{-1}]g). \tag{4.36}$$

Applying $[s_i^{-(\alpha_i+1)}]$ to (4.35), combining with (4.36), and setting $\bar{P} = \hat{P}_{\alpha_i}$, we thus obtain (4.34). The case $i = n-1$ proves the lemma by providing $\tilde{P} = \bar{P}$. □

Consider again a non-necessarily D-finite series f as in (4.7). By the definition (2.2) of the complete diagonal $\Delta(f)$, and by the definition (4.32) of τ and σ, this complete diagonal $\Delta(f)$ is $[s_1^{-1} \cdots s_{n-1}^{-1}]\sigma(f)$. We will now derive the following analogue of Lemma 4.4.

Lemma 4.14. *Let f be as in (4.7). Fix any lexicographical order $\succ$ on the commutative monoid generated by $\{D_{s_1}, \dots, D_{s_{n-1}}\}$. If $P(\sigma(f)) = 0$ for P and $\tilde{P} \neq 0$ as in (4.33), then $\tilde{P}$ annihilates $\Delta(f)$.*

Proof. Lemma 4.13 and the equality $[s_1^{-1} \cdots s_{n-1}^{-1}]\sigma(f) = \Delta(f)$ imply

$$(-1)^{|\boldsymbol{\alpha}|} \alpha_1! \dots \alpha_{n-1}! \, \tilde{P}(\Delta(f)) = [s_1^{-(\alpha_1+1)} \cdots s_{n-1}^{-(\alpha_{n-1}+1)}]P(\sigma(f)) = 0.$$

Hence, $\tilde{P}(\Delta(f)) = 0$. □

We will now construct an operator P. Henceforth, it will be convenient to write w in place of $s_1 \cdots s_{n-1}$ and D_i in place of D_{x_i}, for $i = 1, \dots, n$. Define

$$\mathcal{G}_m := \frac{K[\boldsymbol{s}, t]_{\leqslant 2nm}}{w^{2m}} \sigma \left(\operatorname{span}_K \{ \boldsymbol{D}_x^{\boldsymbol{\alpha}} f \mid |\boldsymbol{\alpha}| \leqslant m \} \right).$$

For convenience, write $s_0 := 1, s_n := t$. By the chain rule, for all $g \in \mathcal{S}$ and each $i = 1, 2, \dots, n-1$, we have

$$D_{s_i}(\sigma(g)) = -\frac{1}{s_i}\sigma(g) + \frac{1}{s_{i-1}}\sigma(D_i(g)) - \frac{s_{i+1}}{s_i^2}\sigma(D_{i+1}(g)), \tag{4.37}$$

and

$$D_t(\sigma(g)) = \frac{1}{s_{n-1}}\sigma(D_n(g)). \tag{4.38}$$

For all $|\boldsymbol{\alpha}| \leqslant m$, and all $p(\boldsymbol{s}, t) \in K[\boldsymbol{s}, t]_{\leqslant 2nm}$, the chain rule implies that if $1 \leqslant i \leqslant n-1$, then

$$D_{s_i}\left(\frac{p(\boldsymbol{s}, t)}{w^{2m}} \sigma(\boldsymbol{D}_x^{\boldsymbol{\alpha}} f) \right) = D_{s_i}\left(\frac{1}{w^{2m}} \right) p \, \sigma(\boldsymbol{D}_x^{\boldsymbol{\alpha}} f) + \frac{D_{s_i}(p)}{w^{2m}} \sigma(\boldsymbol{D}_x^{\boldsymbol{\alpha}} f) + \frac{p}{w^{2m}} D_{s_i}(\sigma(\boldsymbol{D}_x^{\boldsymbol{\alpha}} f)).$$

Rewriting the first two terms of the right-hand side over the denominator $w^{2(m+1)}$ shows that they are both in $\mathcal{G}_{m+1}$. Similarly, making $g = \boldsymbol{D}_x^{\boldsymbol{\alpha}} f$ in (4.37) and rewriting over the denominator $w^{2(m+1)}$ shows that the third term is also in $\mathcal{G}_{m+1}$. Therefore, $D_{s_i}\,\mathcal{G}_m \subseteq \mathcal{G}_{m+1}$. A similar proof, using (4.38), also shows $D_t\,\mathcal{G}_m \subseteq \mathcal{G}_{m+1}$. Since $1 \in \mathcal{G}_0$, we get by induction that for all $\boldsymbol{i} \in \mathbb{N}^{n-1}$ and $j \in \mathbb{N}$,

$$D_t^j\, \boldsymbol{D}_s^{\boldsymbol{i}}(\sigma(f)) \in \mathcal{G}_{j+|\boldsymbol{i}|}.$$

Also note that $\mathcal{G}_m \subseteq \mathcal{G}_{m'}$ if $m \leqslant m'$. Now, if $k \leqslant N'$, $j + |\boldsymbol{i}| \leqslant N$, then

$$t^k D_t^j \boldsymbol{D}_s^{\boldsymbol{i}}(\sigma(f)) \in \frac{K[\boldsymbol{s},t]_{\leqslant 2nN+N'}}{w^{2N}}\, \sigma\left(\operatorname{span}_K \{\boldsymbol{D}_x^{\boldsymbol{\alpha}} f \mid |\boldsymbol{\alpha}| \leqslant N\}\right). \tag{4.39}$$

Using Definition 3.9 when $S = \{1, \ldots, n\}$ fixes $B = \prod_{i=1}^n \{0, 1, \ldots, r_i - 1\}$, the polynomial C, and $d_C = \sum_{j=1}^n d_j \leqslant n d_f$. Then by Lemma 3.13, with $u = N, v = |\boldsymbol{\alpha}|, t = 0$, we have

$$\boldsymbol{D}_x^{\boldsymbol{\alpha}} \in \frac{1}{C^N}\mathcal{H}_{N d_C, 0} + \frac{1}{C^N} J.$$

Applying to f, then applying σ, yields:

$$\sigma(\boldsymbol{D}_x^{\boldsymbol{\alpha}} f) \in \frac{1}{\tau(C^N)}\sigma(\mathcal{H}_{N d_C,0}\,(f)) \subseteq \frac{K[\boldsymbol{s},t]_{\leqslant nNd_C}}{w^{d_C N}\tau(C^N)} \bigoplus_{\boldsymbol{\beta}\in B} K\, \sigma(\boldsymbol{D}_x^{\boldsymbol{\beta}} f). \tag{4.40}$$

Therefore, by (4.39) and (4.40), and for D defined as in (4.31), we have

$$t^k D_t^j \boldsymbol{D}_s^{\boldsymbol{i}}\, \sigma(f) \in \frac{K[\boldsymbol{s},t]_{\leqslant DN+N'}}{w^{(2+d_C)N}\tau(C^N)} \bigoplus_{\boldsymbol{\beta}\in B} K\, \sigma(\boldsymbol{D}_x^{\boldsymbol{\beta}} f). \tag{4.41}$$

Denote $A_N(\boldsymbol{s},t) := w^{(2+d_C)N}\tau(C^N) \in K[\boldsymbol{s},t]$. For any given N' and N, define

$$V_{N,N'} = A_N(\boldsymbol{s},t)\ \operatorname{span}_K\{t^k D_t^j \boldsymbol{D}_s^{\boldsymbol{i}} \mid k \leqslant N',\ j + |\boldsymbol{i}| \leqslant N\}$$

and

$$W_{N,N'} = \sum_{\boldsymbol{\beta}\in B} K[\boldsymbol{s},t]_{\leqslant DN+N'}\sigma(\boldsymbol{D}_x^{\boldsymbol{\beta}} f).$$

We have proved by (4.41) that there is a K-linear map ϕ from $V_{N,N'}$ to $W_{N,N'}$ defined by $\phi(P) := P(\sigma(f))$. Note that

$$\dim_K V_{N,N'} = (N'+1)\binom{N+n}{n}, \quad \dim_K W_{N,N'} \leqslant R\binom{DN+N'+n}{n}, \tag{4.42}$$

where $R := r_1 \cdots r_n = O(r_f^n)$. Fix N and N' as in (4.31) (D has already been defined as there), so that $N' = DN$, $N > R\frac{1+2nD}{D} > 2n > n$, and

$$N^n N' = R(2ND + N)^n > R(2ND + n)^n = R(DN + N' + n)^n,$$

from which follows, with the help of (4.42),

$$\begin{aligned}\dim_K V_{N,N'} &> N'\frac{N^n}{n!}\\ &> R\frac{(DN+N'+n)^n}{n!} > R\binom{DN+N'+n}{n} \geqslant \dim_K W_{N,N'}.\end{aligned}$$

We therefore obtain $\dim_K V_{N,N'} - \dim_K W_{N,N'} > 0$, so that ϕ is non-injective. Consider any non-zero kernel element Z, that is, any family of constants $c_{\boldsymbol{i},j,k} \in K$ indexed by $\boldsymbol{i}, j, k$ with $|\boldsymbol{i}|+j \leqslant N$ and $k \leqslant N'$, and such that $\phi(Z) = 0$ for

$$Z = \sum_{\boldsymbol{i}+j\leqslant N,\ k\leqslant N'} c_{\boldsymbol{i},j,k} A_N\, t^k D_t^j \boldsymbol{D}_s^{\boldsymbol{i}}. \tag{4.43}$$

Then, the operator $P := A_N^{-1} Z = \sum c_{\boldsymbol{i},j,k}\, t^k D_t^j \boldsymbol{D}_s^{\boldsymbol{i}}$ satisfies $P(\sigma(f)) = 0$ as well. From (4.43) it follows that

$$\deg(P) \leqslant N', \quad \operatorname{ord}(P) \leqslant N. \tag{4.44}$$

Finally, P can be written

$$P = \tilde{P}(t; D_t) D_{s_1}^{\boldsymbol{\alpha}_1} \dots D_{s_{n-1}}^{\boldsymbol{\alpha}_{n-1}} + \text{higher terms}$$

with $\tilde{P}(t; D_t) \neq 0$, and the operator $\tilde{P}$ annihilates $\Delta(f)$ by Lemma 4.14 and satisfies the announced bounds (4.30) because of (4.44).

Finishing the proof of Theorem 4.12 only requires to validate the asymptotic estimates in (4.30). Set $S := \sum_{i=1}^n d_i$, which goes to infinity because $d_f \leqslant S \leqslant n d_f$. Fix $\varepsilon > 0$. From the value of D in (4.31) follow, at least for $n \geqslant 1/(4\varepsilon)$,

$$D = nS\left(1 + \frac{2}{S}\right) \leqslant nS\left(1 + \frac{2}{d_f}\right) = O(nS),$$
$$2D + 1 = 2nS\left(1 + \frac{2}{S} + \frac{1}{4nS}\right) \leqslant 2nS\left(1 + \frac{2}{d_f} + \frac{1}{4nd_f}\right) \leqslant 2nS\left(1 + \frac{2+\varepsilon}{d_f}\right),$$

and then, at least for $n \geqslant 1/(4\varepsilon)$ and $d_f \geqslant 2$,

$$(2D+1)^n \leqslant 2^n n^n S^n (1 + \varepsilon/2)^n \leqslant (2+\varepsilon)^n n^{2n} d_f^n,$$
$$\frac{(2D+1)^n}{D} \leqslant \frac{(2+\varepsilon)^n n^{2n} d_f^n}{n(2+S)} \leqslant \frac{(2+\varepsilon)^n n^{2n} d_f^n}{n(2+d_f)} \leqslant (2+\varepsilon)^n n^{2n-1} d_f^{n-1}.$$

Combining with (4.31) yields (4.30).

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