First-Order Factors of Linear Mahler Operators

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Joint work with Th. Dreyfus, Ph. Dumas, and M. Mezzarobba

In dedication to Marko Petkovšek.

A problem suggested by Th. Dreyfus

Can you solve [a certain type of non-linear functional equations] for their rational solutions? If yes, you will prove, e.g., that the two series

$$\prod_{k\geq 0}(1 + x^{2^k} + x^{2^{k+1}}) \qquad \textit{and} \qquad \prod_{k\geq 0}(1 + x^{2^{k+1}} + x^{2^{k+2}})$$

satisfy no non-linear ODE over $\mathbb{Q}[x]$.

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$\rightarrow$ solved in this work (CDDM, 2025)

Linear Mahler Operators and Mahler Function

Linear Mahler equation

$$\ell_r(x)y(x^{b^r}) + \cdots + \ell_1(x)y(x^b) + \ell_0(x)y(x) = 0 \qquad \text{(L)}$$

for a *radix* $b \in \mathbb{N}_{\geq 2}$, an *order* $r \in \mathbb{N}_{\geq 0}$, rational functions $\ell_i \in \bar{\mathbb{Q}}(x)$.

Operator notation

In the skew algebra $\bar{\mathbb{Q}}(x)\langle M \rangle$ where $Mx = x^b M$, write

$$L := \ell_r(x)M^r + \cdots + \ell_1(x)M + \ell_0(x).$$

Action: $My(x) = y(x^b)$. $\qquad$ (L) $\Leftrightarrow$ $Ly(x) = 0$.

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$\rightarrow$ Transcendence theory, Automata theory, "Divide-and-conquer" recurrences, Difference Galois theory, Computer algebra.

Mahler, Cobham, Christol, Kamae, Mendès France, Rauzy, Loxton, v. d. Poorten, Nishioka, Allouche, Shallit, Becker, Dumas, Bell, Coons, Philippon, Adamczewski, Faverjon, Dreyfus, Hardouin, Roques, Smertnig, Arreche, Zhang, …

Mahler-Hypergeometric Solutions and First-Order Factors

Mahler-Hypergeometric functions (w.r.t. a given base b)

The function y is *Mahler* if it satisfies some (L) of any order,
hypergeometric if it satisfies some (L) of order 1.

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Problem

Given some skew polynomial $L = L(x, M)$, several equivalent formulations:

- Find all hypergeometric solutions y of the linear Mahler equation

$$\ell_r(x)y(x^{b^r}) + \cdots + \ell_1(x)y(x^b) + \ell_0(x)y(x) = 0. \tag{L}$$

- Find all first-order right-hand factors $M - u$ of L for $u \in \bar{\mathbb{Q}}(x)$.
- Find all rational solutions u of the Riccati Mahler equation

$$\ell_r(x)u(x)\cdots u(x^{b^{r-1}}) + \cdots + \ell_2(x)u(x)u(x^b) + \ell_1(x)u(x) + \ell_0(x) = 0. \tag{R}$$

$u = \frac{My}{y}$. lhs of (R) = remainder in division of L by $M - u$.

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We provide an algorithm by means of structured Hermite–Padé approximants.

Examples of Hypergeometric Mahler Series

Thue–Morse sequence over the alphabet $\{+,-\}$

$$y(x) = \prod_{j\geq 0}(1 - x^{2^j})$$

fixpoint of the morphism + $\to$ +−, − $\to$ −+: (+)(−)(−+)(−++−)(−++−+−−+)$\cdots$

Stern–Brocot sequence

$$y(x) = \prod_{j\geq 0}(1 + x^{2^j} + x^{2^{j+1}})$$

explicit bijection $\mathbb{N} \simeq \mathbb{Q}_{\geq 0}$: $n \mapsto [x^n]y/[x^{n+1}]y$

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Ramified Mahler-Hypergeometric Solutions

Hypergeometric = infinite product + log-factor + a ramification order

For $b = 3$, solve:

$$L := (1 - 7x^3)M^2 + \left(2x - 14x^2 - \lambda x^3 - 2\lambda x^6\right)M + 2\lambda x^2(1 + 2x).$$

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One finds that

$$y := (\ln x)^{\log_3 \lambda} x^{1/2} \prod_{k \geq 0} \frac{1 - 7x^{3^k}}{1 + 2x^{3^k}} \qquad (b = 3)$$

is annihilated by $L = \left(M - 2x\right)\left((1 - 7x)M - \lambda x(1 + 2x)\right)$.

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Linear equations with no ramification can need ramification to be solved.
A ramified y with unramified $u = My/y$ is possible.

Disproving Hypergeometricity

Missing digit in ternary expansion (OEIS A005836)

$L := 3(1+x^2)^2M^2 - (1+3x+4x^2)M + x$ for $b = 2$ annihilates

$$y(x) := \sum_{n\geq 0} (n\text{-th positive integer written without 2 in base 3})\, x^n$$
$$= 1x^1 + 3x^2 + 4x^3 + 9x^4 + 10x^5 + \cdots .$$

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Unique monic right-hand first-order factor is $M - \dfrac{1}{3(1 + x)}$

$\Rightarrow$ all hypergeometric solutions in $\bar{\mathbb{Q}} \dfrac{(\ln x)^{\log_2(1/3)}}{1 - x}$

$\Rightarrow$ $y(x)$ is not hypergeometric.

Parametrized Mahler-Hypergeometric Solutions

Infinitely-many factorization occur

$$\mathrm{LCLM}(M-1, M-x^{b-1}) = M^2 - \frac{x^{b^2}-x}{x^b-x}M + \frac{x^{b^2}-x^b}{x^b-x} = \operatorname*{LCLM}_{(g_1:g_2)}\left(M - \frac{g_1x^b+g_2}{g_1x+g_2}\right)$$

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Parities of digit repetitions in ternary expansion

After (Adamczewski and Faverjon, 2017), introduce $y(x) := \sum_{n \in S} x^n$ for

$$S := \Big\{ n \text{ whose ternary expansion has an even number of 1s and an odd number of 2s} \Big\}.$$

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$\rightarrow$ linear Mahler equation of order 4 and degree 258.
$\rightarrow$ hypergeometric solutions correspond to a ratio u among

$$\frac{1}{1-x-x^2}, \quad \frac{1}{1+x-x^2}, \quad \frac{g_1+g_2x^3}{g_1+g_2x}\,\frac{1}{1+x^2+x^4} \text{ for } (g_1:g_2)\in\mathbb{P}^1(\bar{\mathbb{Q}}).$$

So, y is not hypergeometric.

Related Work

Algorithms for solving for various kinds of solutions

- rational solutions
  - (Bell, Coons, 2017)
  - (CDDM, 2018)
- formal power/Laurent series
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- infinite products and hypergeometric solutions
  - (Roques, 2018)
  - (CDDM, 2025)
- Hahn series
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Difference theory of Mahler equations

- local structure (Roques, 2020)
- regular singular systems (Roques, 2018)
- Frobenius method (Roques, 2024)

Classical Algorithms by Gosper–Petkovšek Forms

shift $x \mapsto x + 1$ (Petkovšek, 1992)

$$u(x) = \eta \frac{C(x+1)}{C(x)} \frac{A(x)}{B(x)} \qquad + \text{coprimality constraints}$$

q-shift $x \mapsto qx$ (Abramov, Paule, Petkovšek, 1998)

$$u(x) = \eta \frac{C(qx)}{C(x)} \frac{A(x)}{B(x)} \qquad + \text{coprimality constraints}$$

Mahler (order 2) (Roques, 2018)

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All those algorithms:

- iterate on factors of A of ℓ_0 and B of ℓ_r (or slight variations),
- determine a polynomial equation on η + a degree bound on C,
- solve an auxiliary linear functional equation for C.

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Mahler (general order) (CDDM, 2025)

$$u(x^{b^{r-1}}) = \eta \frac{C(x^b)}{C(x)} \frac{A(x^{b^{r-1}})}{B(x)} \qquad + \text{coprimality constraints (technical generalization)}$$

Where to Look for Solutions of the Linear Equation?

Inspired by (Roques, 2018, 2020), we introduce:

- the field of Puiseux series, $\bar{\mathbb{Q}}((x^{1/*})) := \bigcup_{q \in \mathbb{N}_{\neq 0}} \bar{\mathbb{Q}}((x^{1/q}))$,
- solutions $e_\lambda := (\ln x)^{\log_b \lambda}$ of $M e_\lambda = \lambda\, e_\lambda$ satisfying $e_\lambda e_{\lambda'} = e_{\lambda\lambda'}$,
- the difference algebra

$$\mathfrak{D} := \bar{\mathbb{Q}}((x^{1/q}))\left[(e_\lambda)_{\lambda \in \bar{\mathbb{Q}}_{\neq 0}}\right] = \bigoplus_{\lambda \in \bar{\mathbb{Q}}_{\neq 0}} (\ln x)^{\log_b \lambda}\, \bar{\mathbb{Q}}((x^{1/*})).$$

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Theorem (CDDM, 2025)

The ramified rational solutions of (R) can all be obtained from the hypergeometric solutions of (L) in $\mathfrak{D}$.

Similarity Classes of Hypergeometric Solutions

Similarity, hypergeometricity

- y_1 and y_2 are similar if $\exists q \in \mathbb{Q}(x)_{\neq 0},\ y_2 = q y_1$.
- y is hypergeometric if $\exists u \in \mathbb{Q}(x),\ My = uy$.

Partitioning hypergeometric solutions into similarity classes

$$\{\text{hypergeometric solutions of (L) in } \mathfrak{D}\} = \{0\} \sqcup \coprod_{j=1}^{m} (\mathfrak{H}_j)_{\neq 0}$$

where:

- Each $(\mathfrak{H}_j)_{\neq 0}$ is a similarity class of hypergeometric solutions.
- The vector spaces $\mathfrak{H}_j$ are in direct sum in $\mathfrak{D}$.
- The sum of the dim $\mathfrak{H}_j$ add up to at most the order of L.
- $\mathfrak{H}_j \subset (\ln x)^{\log_b \lambda_j}\, \bar{\mathbb{Q}}((x^{1/*}))$ for a suitable λ_j.

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- $\mathfrak{H}_j \subset (\ln x)^{\log_b \lambda_j} \bar{\mathbb{Q}}((x^{1/*}))$ for a suitable λ_j.

This generalizes to allow ramifications in q and u.

Rational Solutions to the Riccati Equation

$$\rho : (\ln x)^{\log_b \lambda}\,\bar{\mathbb{Q}}((x^{1/*}))_{\neq 0} \to \bar{\mathbb{Q}}((x^{1/*})) \quad \text{is well-defined for each } \lambda.$$
$$y \mapsto My/y$$

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$$(\ln x)^{\log_b \lambda}\, x^v (1 + \cdots) \mapsto \lambda\, x^{(b-1)v}(1 + \cdots)$$

Partitioning (ramified) rational solutions

$$\{\text{ramified rational solutions of (R) in } \bar{\mathbb{Q}}((x^{1/*}))\} = \coprod_{j=1}^{m} \mathfrak{R}_j$$

where:

- $\mathfrak{R}_j := \rho((\mathfrak{H}_j)_{\neq 0})$
- ρ induces a one-to-one parametrization of $\mathfrak{R}_j$ by $\mathbb{P}(\mathfrak{H}_j) \simeq \mathbb{P}^{d-1}(\bar{\mathbb{Q}})$ for $d = \dim \mathfrak{H}_j$.

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Given a basis $(y_1, \dots, y_d)$ of $\mathfrak{H}_j$:

$$(g_1 : \cdots : g_d) \mapsto \frac{g_1 M y_1 + \cdots + g_d M y_d}{g_1 y_1 + \cdots + g_d y_d}.$$

Useful Bounds

$$L \in \bar{\mathbb{Q}}[x]\langle M\rangle \qquad \deg_x L = d \quad \deg_M L = r$$

Ramification order of Puiseux series solutions (CDDM, 2018)

Each $(\ln x)^{\log_b \lambda}$ implies some $\bar{\mathbb{Q}}((x^{1/q_\lambda}))$ for q_λ read on a Newton polygon.

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Degree bounds for rational solutions of the Riccati equation (CDDM, 2025)

	numerators	denominators	both
$b = 2$	$\left(1 + 2^{-r}\right)(2d)$	$2d$	$O(d)$
$b \geq 3$	$\left(1 + b^{-1}\right)\dfrac{d}{b^{r-2}}$	$\dfrac{d}{b^{r-2}}$	$O(d/b^r)$

Reformulation of the problem as structured syzygies

Parametrization of the search space

For each λ, using any ramification order q:

$$\rho : (\ln x)^{\log_b \lambda}\,\bar{\mathbb{Q}}((x^{1/q})) \to \bar{\mathbb{Q}}((x^{1/q}))$$
$$y \mapsto \frac{My}{y}$$

Reformulation of the problem as structured syzygies

Parametrization of the search space

For each λ, using the ramification order $q = b - 1$:

$$\begin{array}{rcl} \rho : \{y \in (\ln x)^{\log_b \lambda}\, \bar{\mathbb{Q}}((x^{1/q})) \mid Ly = 0\} & \to & \bar{\mathbb{Q}}((x^{1/q})) \\ \cup & & \cup \\ \mathfrak{H}_j & \to & \mathfrak{R}_j \subset \bar{\mathbb{Q}}(x) \end{array}$$

Reformulation of the problem as structured syzygies

Parametrization of the search space

For each λ, using the ramification order $q = b - 1$:

$$\rho : \operatorname{span}\left((\ln x)^{\log_b \lambda} z_1, \dots, (\ln x)^{\log_b \lambda} z_d\right) \to \bar{\mathbb{Q}}((x^{1/q}))$$

$$\cup \quad \cup$$

$$\mathfrak{H}_j \to \mathfrak{R}_j \subset \bar{\mathbb{Q}}(x)$$

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Reformulation of the problem as structured syzygies

Parametrization of the search space

For each λ, using the ramification order $q = b - 1$:

$$\rho : \bar{\mathbb{Q}}^t \to \bar{\mathbb{Q}}((x^{1/q}))$$
$$(a_1, \dots, a_t) \mapsto \lambda \frac{a_1 M z_1 + \dots + a_t M z_t}{a_1 z_1 + \dots + a_t z_t} \in \bar{\mathbb{Q}}(x) \,?$$

Reformulation of the problem as structured syzygies

Parametrization of the search space

For each λ, using the ramification order $q = q_\lambda$:

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Other formulation, after renormalizing L and so that $\lambda = 1$ and $z_i \in \bar{\mathbb{Q}}[[x]]$

Describe $(a_1, \dots, a_t) \neq 0$ such that $\exists P/Q \in \bar{\mathbb{Q}}(x)_{\neq 0}$,

$$(-a_1 P)\, z_1 + \cdots + (-a_t P)\, z_t + (a_1 Q)\, M z_1 + \cdots + (a_t Q)\, M z_t = 0.$$

Relaxation of the problem

Two-stage relaxation

Solutions

$$(-a_1 P)\, z_1 + \cdots + (-a_t P)\, z_t + (a_1 Q)\, M z_1 + \cdots + (a_t Q)\, M z_t = 0$$

are structured instances of the syzygies

$$P_1\, z_1 + \cdots + P_t\, z_t + Q_1\, M z_1 + \cdots + Q_t\, M z_t = 0,$$

which are approximated by approximate syzygies

$$P_1\, z_1 + \cdots + P_t\, z_t + Q_1\, M z_1 + \cdots + Q_t\, M z_t = O(x^\sigma).$$

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Motivation

1. For $\sigma \gg 1$, approximate syzygies of "low" degree are exact syzygies.
2. Structured syzygies are linear combinations of syzygies.

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$$P_1\, z_1 + \cdots + P_t\, z_t + Q_1\, M z_1 + \cdots + Q_t\, M z_t = O(x^{\sigma}).$$

Motivation

1. For $\sigma \gg 1$, approximate syzygies of “low” degree are exact syzygies.
2. Structured syzygies are linear combinations of syzygies.

We search for structured syzygies as recombinations of approximate syzygies.

Structure and computation of approximate syzygies

Minimal basis of approximate syzygies

Algorithms find a basis of the module of approximate syzygies to order σ:

$$\begin{pmatrix} P_{1,1},\dots,P_{1,t} & Q_{1,1},\dots,Q_{1,t} \\ \vdots & \vdots \\ P_{t,1},\dots,P_{t,t} & Q_{t,1},\dots,Q_{t,t} \\ P_{t+1,1},\dots,P_{t+1,t} & Q_{t+1,1},\dots,Q_{t+1,t} \\ \vdots & \vdots \\ P_{2t,1},\dots,P_{2t,t} & Q_{2t,1},\dots,Q_{2t,t} \end{pmatrix} \begin{pmatrix} z_1 \\ \vdots \\ z_t \\ Mz_1 \\ \vdots \\ Mz_t \end{pmatrix} = \begin{pmatrix} O(x^\sigma) \\ \vdots \\ O(x^\sigma) \\ O(x^\sigma) \\ \vdots \\ O(x^\sigma) \end{pmatrix}$$

(Derksen, 1994), (Beckermann, Labahn, 1994, 2000), Neiger (2016).

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Properties (module)

The module of the rows: *(i)* has rank $2t$ for all σ; *(ii)* is ultimately decreasing with σ; *(iii)* has the module of (exact) syzygies as a limit (with rank $< 2t$).

Reduction to a polynomial system

Properties (vector space)

The vector space of the rows of “low” degree: *(i)* is non-increasing; *(ii)* has the vector space of exact syzygies of “low” degree as a limit.

W := submatrix of (independent) rows of “low” degree.
ρ := rank of the module of rows generated by W.

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Search for structured approximate syzygies, hoping that they are exact

Given $a := (a_1, \dots, a_t) \neq 0$, the following are equivalent:

- $\exists P/Q \in \bar{\mathbb{Q}}(x)_{\neq 0}$ such that $(-aP, aQ)$ is in the module $\bar{\mathbb{Q}}[x]^{1\times\rho}W$,

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- a is a solution of the quadratic homogeneous polynomial system

$$\Sigma := \left\{ \text{coefficients w.r.t } x \text{ of the minors of size } \rho + 2 \text{ of } W_a \right\} \subset \bar{\mathbb{Q}}[a].$$

A Polynomial System with a Linear Variety of Solutions

$$V(\Sigma) = \bigcup_j I_j \qquad (I_j = \text{irreducible component})$$

Properties (CDDM, 2025)

When σ increases, $V(\Sigma)$ stabilizes. At the limit:

- each I_j is a linear subspace of $\bar{\mathbb{Q}}^t$,
- the I_j are in direct sum,
- each I_j parametrizes a subset of rational solutions of (R),
- the images of the I_j form a partition of the rational solutions of (R).

Adjust the precision σ to be able to solve

- Primary decomposition: obtain Gröbner bases for prime ideals $\mathfrak{p}_j$ s.t.
$$\sqrt{(\Sigma)} = \bigcap_j \mathfrak{p}_j \subset \bar{\mathbb{Q}}[a_1, \dots, a_t]$$
(Gianni, Trager, Zacharias, 1988).
- If this contains a non-linear element, σ is too small (CDDM, 2025).

Sketch of the algorithm (for a given λ) (CDDM, 2025)

Obtain all rational $u = \lambda x^\alpha + \cdots$ s.t. $M - u$ is a right-hand factor of L:

- Renormalize L so as to reduce the computation of the solutions of L in $(\ln x)^{\log_b \lambda}\,\bar{\mathbb{Q}}((x^{1/q_\lambda}))$ to solutions of some L_λ in $\bar{\mathbb{Q}}[[x]]$.
- Compute a basis of truncated series solutions $(z_1, \dots, z_t)$ to some initial order σ_0.
- For σ in a geometric sequence $\phi^k \sigma_0$:
  - Prolong the basis to order σ.
  - Compute a minimal basis of the module of approximate syzygies.
  - Extract the "low"-degree rows into a matrix W of rank $0 \leq \rho \leq 2t$.
  - $\rho \in \{0, 2t-1, 2t\}$ are special cases dealt with separately.
  - Compute minors of W_a, then their coefficients to obtain Σ.
  - Compute the primary decomposition $\sqrt{(\Sigma)} = \bigcap_j \mathfrak{p}_j$ over $\bar{\mathbb{Q}}$.
  - If any $\mathfrak{p}_j$ shows a non-linear polynomial, increase σ.
  - For each j:
    - Solve $\mathfrak{p}_j$ to get a matrix S and a parametrization $a = Sg$ for g in some $\bar{\mathbb{Q}}^v$.
    - Solve for the left kernel of W_a at $a = Sg$. If incompatible result, increase σ.
    - Get a candidate P/Q (with param. g) from the basis element of the kernel.
    - If degrees of $u := P/Q$ are too high, or if u does not satisfy (R), increase σ.
  - Convert all obtained u from solutions of L_λ into solutions of L.
  - Quit and return the solutions.

Benchmark (Maple implementation by Dumas, with calls to Singular)

				IP	HP						
example	b	r	d	tot	fst	dim	σ	rfn	syz	sng	tot
`Baum_Sweet`	2	2	1	0.07	0.07	(1, 1)	(6, 6)	0.03	0.03	-	0.13
`Rudin_Shapiro`	2	2	1	0.08	0.07	(1, 0)	(6, -)	0.02	0.01	-	0.10
`no_2s_in_3_exp`	2	2	4	0.12	0.08	(1, 1)	(33, 9)	0.03	0.08	-	0.21
`Stern_Brocot_b2`	2	2	4	0.12	0.07	(1)	(21)	0.01	0.02	-	0.12
`Stern_Brocot_b4`	4	2	26	5.4	0.08	(1)	(63)	0.02	0.11	-	0.22
`Dilcher_Stolarsky`	4	2	4	0.09	0.07	(2)	(27)	0.04	0.08	0.02	0.23
`Katz_Linden`	2	4	14	2.1	0.12	(0, 1, 0, 0)	(-, 69, -, -)	0.06	0.39	-	0.57
`Adamczewski_Faverjon`	3	4	258	543	0.16	(4)	(163)	0.32	1.8	0.05	2.4
`lclm_3rat_1log`	3	3	121	203	0.08	(3)	(140)	0.16	2.5	0.03	2.9
`lclm_3rat_2log`	3	3	122	215	0.09	(2, 1)	(88, 52)	0.07	0.51	-	0.71
`lclm_2rat_trunc_sl0`	2	4	56	490	0.11	(4)	(294)	2.6	12	0.05	14
`lclm_2rat_trunc_sl1`	2	4	61	828	0.12	(4)	(519)	13	104	0.05	117
`lclm_3rat_trunc_sl1`	3	5	1260	>12 hr	0.49	(3, 2)	(574, 268)	11	51	0.07	63
`lclm_4pow_b2`	2	7	107	25351	0.20	(1, 4)	(429, 739)	0.16	2.4	-	2.8
`lclm_4pow_b3`	3	6	727	>12 hr	0.56	(1, 4)	(108, 174)	0.47	0.64	-	1.7
`lclm_4pow_b4`	4	5	989	>12 hr	0.23	(4)	(223)	0.40	0.59	-	1.4
`lclm_4pow_b5`	5	5	3103	>12 hr	2.0	(1, 4)	(44, 289)	2.8	0.94	-	5.9
`lclm_5pow_b4`	4	7	17270	>60 GB	39	(1, 5)	(274, 1326)	64	6.5	-	115
`dft_Baum_Sweet`	4	2	6	0.10	0.08	(2)	(77)	0.06	0.18	0.02	0.37
`dft_Rudin_Shapiro`	4	2	7	5.8	0.06	(1, 0)	(88, -)	0.03	0.15	-	0.25
`dft_Stern_Brocot_b2`	4	2	24	3.0	0.09	(1)	(59)	0.03	0.10	-	0.22
`dft_no_2s_in_3_exp`	4	2	20	9.6	0.09	(1, 1)	(85, 33)	0.07	0.84	-	1.0
`dft_Dilcher_Stolarsky`	16	2	50	3382	0.10	(2)	(666)	0.25	3.7	-	4.1
`dft_Stern_Brocot_b4`	16	2	348	29670	0.13	(1)	(239)	0.14	2.0	-	2.4
`rmo_2_1`	2	3	19	5.3	0.07	(3)	(263)	1.1	23853	0.03	23854
`rmo_3_1`	3	3	37	14	0.07	(3)	(133)	0.22	1166	0.03	1167
`rmo_2_2`	2	3	44	15							>12 hr
`rmo_3_2`	3	3	82	39	0.08	(3)	(247)	2.6	11031	0.03	11034
`rmo_2_3`	2	3	69	26							>12 hr
`rmo_3_3`	3	3	127	70							>12 hr
`rmo_2_4`	2	3	94	41							>12 hr
`rmo_3_4`	3	3	172	109							>12 hr
`rmo_2_5`	2	3	119	58							>12 hr
`rmo_3_5`	3	3	217	166							>12 hr

- 'tot' is the total time for ramified rational solving using the improved Mahler analogue of Petkovšek's approach (IP) or the Hermite–Padé approach (HP).
- 'fst' is the time for a first series computation, sufficient to determine the dimensions of series-solutions spaces behind the various logarithmic parts in solutions, provided in the column 'dim'.
- 'dim' is a list, indexed by the $\lambda \in \Lambda$, of the dimension of series appearing in front of $(\ln x)^{\log_b \lambda}$ in solutions.
- 'σ' is a list with same indexing of the last value of σ used to find the hypergeometric series solutions of L_λ (or '-' when the dimension for λ is 0).
- 'rfn' is the cumulative time over λ for all refined series computations up to the corresponding final approximation orders in 'σ'.
- 'syz' is the total time for computing minimal bases.
- 'sng' is the cumulative time over λ for all prime decompositions computed by calling Singular, or '-' if no prime decomposition was needed for the operator L.

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$f \in \mathbb{C}((x))$ is hypertranscendental over $\mathbb{C}(x)$ $\quad\Leftrightarrow$
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By (Adamczewski, Dreyfus, and Hardouin, 2021), if $f \notin \mathbb{C}(x)$ then f is hypertranscendental.

Corollary of a criterion (Roques, 2018) on the difference Galois group of L

Assume:

- $y(x^{b^2}) + A(x)y(x^b) + B(x)y(x) = 0$ admits a non-zero solution $f \in \bar{\mathbb{Q}}[[x]]$.
- No rational function $u(x)$ is solution of one of the Riccati equations

$$u(x)u(x^b) + A(x)u(x) + B(x) = 0,$$

$$u(x)u(x^{b^2}) + \left(\frac{B(x^{b^2})}{A(x^{b^2})} - A(x^b) + \frac{B(x^b)}{A(x)}\right) u(x) + \frac{B(x)B(x^b)}{A(x)^2} = 0.$$

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Independence for the six examples of order 2!