Products of Ordinary Differential Operators by Evaluation and Interpolation

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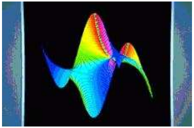

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Products of Linear Differential Operators

Van der Hoeven (2002), JSC: Skew-polynomial multiplication in bidegree (n, n) in either of the skew-polynomial settings:

- $F\langle x, \partial\rangle$ where $\partial x = x\partial + 1$, $\partial = d/dx$,
- $F\langle x, \theta\rangle$ where $\theta x = x(\theta + 1)$, $\theta = x\, d/dx$,

reduces to $c = \mathcal{O}(1)$ matrix products over $F = \mathbb{Q}, \mathbb{F}_p$ in size n.

Approach: evaluation of operators at polynomials + interpolation.

Applications:

- Multivariate integrals in physics (Maillard and others),
- Chebyshev expansions (Benoit & Salvy, in preparation),
- Van der Hoeven's algorithm for factoring linear differential operators in MEGA (2005).

Contributions

- Over $\mathbb{Q}$, explicit analyses and optimization of Van der Hoeven's algorithms.
- A new, direct multiplication algorithm, with better constant c.

Algorithm	VdH_θ	IVdH_θ	VdH_∂	IVdH_∂	MulWeyl
All block products	37	24	96	48	12
Zeros + Strassen	20	8	47	12	8

Number c of $n \times n$ block products for multiplication of skew polynomials in (x, θ), resp. (x, ∂), of bidegree (n, n).

- Matrix multiplication has same complexity as skew-polynomial product when characteristic is 0.
- A softly quadratic approach for (small) positive characteristic.

Review of Naive and Other Existing Algorithms

- Naive expansion by Leibniz's formula and expansion of $\partial^j x^u$:

$$BA = \sum_{i,j,u,v=0}^{n} b_{i,j} a_{u,v} x^i \underbrace{\left(\partial^j x^u\right)}_{\leq n \text{ terms}} \partial^v \quad \to \mathcal{O}\left(n^5\right)$$

- Iterative scheme by derivations of the right-hand factor:

$$BA = \sum_{i=0}^{n} b_i(x) \underbrace{\left(\partial^i A\right)}_{\substack{\text{degree } \leq 2n \text{ in } \partial \\ \text{degree } \leq n \text{ in } x}} \quad \text{by } \partial T = T\partial + \frac{dT}{dx} \quad \to \mathcal{O}\left(\mathsf{M}(n)\, n^2\right)$$

- Takayama's iterative scheme by derivations of both factors:

$$BA = \sum_{k=0}^{n} \frac{1}{k!} \underbrace{\left[\frac{d^k B}{d\partial^k} \frac{d^k A}{dx^k}\right]}_{\substack{\text{bivariate commutative product} \\ \text{in bidegree } (n,\, n)}} \quad \to \mathcal{O}\left(\mathsf{M}(n^2)\, n\right)$$

Sketch of Van der Hoeven's 2002 Algorithms

$$\begin{array}{c} A(x,\theta) \text{ and } B(x,\theta) \\ \text{of bidegree } (n,n) \end{array} \;\longrightarrow\; C = BA = \sum_{i=0}^{2n} x^i C_i(\theta),\ \deg C_i \le 2n.$$

$$\theta^j(x^k) = k^j x^k \;\longrightarrow\; C(x^k) = \sum_{i=0}^{2n} C_i(k)\, x^{i+k}.$$

By Lagrange interpolation: $\big(C_i(k)\big)_{0\le i,k\le 2n} \to \big(C_i(\theta)\big)_{0\le i\le 2n}$.

$$K[x]_{\le 2n} \xrightarrow{A\cdot} K[x]_{\le 3n} \xrightarrow{B\cdot} K[x]_{\le 4n}.$$

Matrix of size $(4n+1) \times (3n+1)$ for B, $(3n+1) \times (2n+1)$ for A.

Complexity: $\mathrm{SkewM}(n,n) \subset \mathcal{O}\big(\mathrm{MM}(n)\big)$.

- Composition: matrices of the differential operators.
- Evaluation/interpolation: Vandermonde matrix and inverse.
- Conversion $\partial \leftrightarrow \theta$: matrix of Stirling numbers and inverse.

Improvements

A1. Fast multipoint evaluation/interpolation of (von zur Gathen & Gerhard, MCA) in $\mathcal{O}\big(n\,\mathrm{M}(n)\log n\big)$.

A2. Fast change between monomial and falling-factorial bases of (Gerhard, RWCA'00) in $\mathcal{O}\big(n\,\mathrm{M}(n)\log n\big)$.

$\rightarrow \mathcal{O}\big(\mathrm{MM}(n)\big)$ with the better constants given in the table.

B1. Smaller matrices are sufficient: when B, A of bidegree (n, n) in (x, ∂),

$$K[x]_{\leq 2n} \xrightarrow{A\cdot} K[x]_{\leq 3n} \xrightarrow{B\cdot} K[x]_{\leq\, 2n}.$$

Size $(2n{+}1) \times (3n+1)$ for B, $(3n+1) \times (2n+1)$ for A.

B2. Direct calculation with ∂.

Direct Evaluation & Interpolation: Generating Function

Simultaneous computations of the $P(x^k)$ by:

- factorization of generating functions,
- (formal) Laplace transform $\mathcal{L}_\alpha(\alpha^n) := n!\,\alpha^n$.

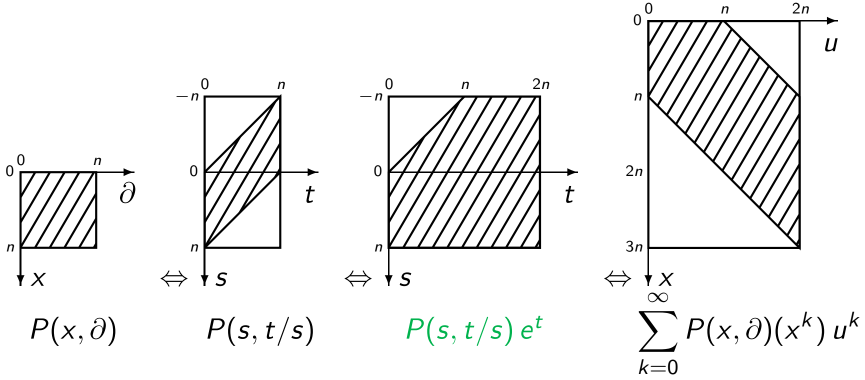
$$\sum_{k=0}^{\infty} P(x,\partial)(x^k)\,u^k = \sum_{i,k=0}^{\infty} p_i(x)\partial^i(x^k)\,u^k =$$

$$\sum_{k\geq i\geq 0} p_i(x)\frac{k!}{(k-i)!}x^{k-i}u^k = \mathcal{L}_u\left(\sum_{i,j\geq 0} p_i(x)\frac{x^j}{j!}u^{i+j}\right) =$$

$$\mathcal{L}_u\big(P(x,u)\,e^{xu}\big) = \mathcal{L}_t\left(P\left(s,\frac{t}{s}\right)e^t\right)[s=x,t=xu].$$

Bidegree (n, n) in (x, ∂) $\rightarrow$ evaluation/interpolation in $\mathcal{O}(n\,\mathsf{M}(n))$.

$$\sum_{k=0}^{\infty} P(x,\partial)(x^k)\,u^k = \mathcal{L}_t\left(P\left(s, \frac{t}{s}\right) e^t\right)[s = x, t = xu].$$



$\rightarrow$ Good constants for MulWeyl.

Equivalence $\mathrm{SkewM}(n, n) \propto \mathrm{MM}(n)$

$\mathrm{SkewM}(n, n) \subset \mathcal{O}\big(\mathrm{MM}(n)\big)$: Van der Hoeven (2002).

$\mathcal{O}\big(\mathrm{SkewM}(n, n)\big) \supset \mathrm{MM}(n)$:

1. $\mathrm{MM}(n) \subset \mathrm{LTMM}\big(\mathcal{O}(n)\big)$: $\begin{bmatrix} I_n & 0 & 0 \\ M & I_n & 0 \\ 0 & N & I_n \end{bmatrix}^2 = \begin{bmatrix} I_n & 0 & 0 \\ 2M & I_n & 0 \\ NM & 2N & I_n \end{bmatrix}$.
2. $\mathrm{LTMM}(n) \subset \mathcal{O}\big(\mathrm{SkewM}(n)\big)$:

$$\begin{bmatrix} m_{0,0} & & 0 & & 0 \\ \vdots & \ddots & & \ddots & \\ m_{i,0} & & m_{i,i} & & 0 \\ \vdots & \ddots & & \ddots & \\ m_{n,0} & \cdots & m_{n,n-i} & \cdots & m_{n,n} \end{bmatrix} \xleftrightarrow[\mathcal{O}\left(n\,\mathrm{M}(n)\log n\right)]{m_{i,j}=A_{i-j}(j)} \underbrace{\sum_{\ell=0}^{n} x^{\ell} A_{\ell}(\theta)}_{\text{bidegree } (n,\, n)}.$$

Positive Characteristic p: Product in $\tilde{\mathcal{O}}(pn^2)$

Using Euler's operator $\theta = x\partial$:

- $\theta x^p = x^p\theta + x\left(px^{p-1}\right) = x^p\theta$,
- $x^v f(\theta) = f(\theta - v)\, x^v$ in complexity $\mathcal{O}\big(\mathrm{M}(\deg f)\big)$.

$$\left(\sum_{u=0}^{p-1} x^u B_u(x^p, \theta)\right)\left(\sum_{v=0}^{p-1} A_v(x^p, \theta) x^v\right) = \sum_{u,v=0}^{p-1} x^u \underbrace{\Big[(B_u A_v)(x^p, \theta)\Big]}_{\substack{\text{commutative bivariate} \\ \text{product in bidegree } (n/p,\, n)}} x^v.$$

$$\left.\begin{array}{l} \text{Products } \mathcal{O}\big(p^2\,\mathrm{M}(n^2/p)\big) \subset \mathcal{O}\big(p\,\mathrm{M}(n^2)\big) \\ \text{Conversions } x \leftrightarrow x^p\text{: } \mathcal{O}\big(pn\,\mathrm{M}(n)\log n\big) \\ \text{Conversions } \partial \leftrightarrow \theta\text{: } \mathcal{O}\big(n\,\mathrm{M}(n)\log n\big) \end{array}\right\} \to \tilde{\mathcal{O}}(pn^2).$$

Implementations and Timings

Comparison to other implementations (32-bit int. inputs): GOOD!

char. 0	Dense				$x^n\partial^n \times$ Dense			
	Magma		Maple		Magma		Maple	
n	native	new	OT	OA	native	ours	OT	OA
40	1.17s	0.97s	8.51s	250s	0.47s	0.00s	0.22s	0.10s
80	11.3s	9.93s	178s	$>$ 1h	3.15s	0.00s	4.12s	0.58s
160	swap	128s	$>$ 1h	$>$ 1h	43.5s	0.01s	45.0s	2.87s

(OT = OreTools, OA = Ore_algebra.)

Our Magma implementation suffers from interpretation overhead:

characteristic	65521		4294967291			0		
n	80	1280	80	320	1280	80	160	320
total time	0.25	107	0.50	12.2	1961	9.93	128	2164
matrix prod. $\propto$ $\mathrm{MM}(n)$	4%	13%	17%	16%	39%	36%	41%	52%
polyn'l prod. $\propto$ $n\,\mathrm{M}(n)$	13%	25%	23%	23%	18%	36%	33%	24%
other matrix ops. $\propto$ n^2	38%	36%	30%	27%	11%	7%	6%	5%
other interp'd ops. $\propto$ n^2	46%	27%	30%	33%	32%	21%	20%	19%

About the Long Version & Future Work

- Refining complexity by products of triangular matrices.
- Sparse models by several morphisms and change of canonical forms.
- Analysis of unbalanced products shows non-optimality of the linear-algebra approach.
- *q*-Analogues mimic the case of Euler's operator and usual derivation.
- Rational-function coefficients $\to \mathcal{O}\big(\mathsf{MM}(n)\mathsf{M}(n^2)\big)$.
- Partial differential operators: $\partial_1, \dots, \partial_\ell \to \mathcal{O}\big(n^{2\ell} + \mathsf{MM}(n^\ell)\big)$.
- Mathemagix prototype (C++).

- Matrix Multiplication as a non-commutative yardstick?