Computing Solutions of Linear Mahler Equations

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Inria informatiques mathématiques

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Joint work with Th. Dreyfus, Ph. Dumas, and M. Mezzarobba

https://arxiv.org/abs/1612.05518

Linear Mahler Equations

Mahler operator of radix b $(b \in \mathbb{N},\ b \geq 2)$:

$$(My)(x) = y(x^b).$$

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Kurt Mahler
1903–1988

- automata theory
- combinatorics of words
- partition theory
- transcendence theory
- complexity analysis of divide-and-conquer algorithms

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- $\mathbb{K} = \bar{\mathbb{Q}}$ (Nishioka, 1985): any solution y is rational or transcendental.
- If $b = \operatorname{char} \mathbb{K}$, then $y(x^b) = y(x)^b$: all solutions y are algebraic.

Our Goal in this Work: Solutions in Closed Form When $\operatorname{char} \mathbb{K} = 0$

Targeted classes of solutions

Given L of order r, with polynomial coefficients of degree $\leq d$, find algorithms of reasonable complexity for:

- (power/Puiseux) series solutions,
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Challenge: dodge the factor b^r

- $L(x^n)$ has degree $d + b^r n$,
- after change of unknowns $y = \frac{\tilde{y}}{q}$, new equation has degree $d + b^r \deg q$.

Part I

Series Solutions

Series Solutions: Understanding the Linear System

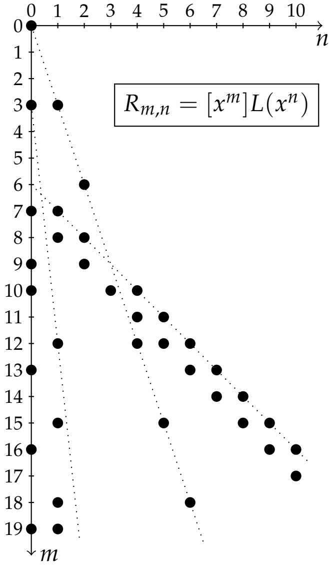
For $b = 3$, look for a solution $y = \sum_{n \geq 0} y_n x^n$ of

$$\begin{aligned} L = x^3(1 - x^3 + x^6)(1 - x^7 + x^{10})\, M^2 \\ -(1 - x^{28} - x^{31} - x^{37} - x^{40})\, M \\ + x^6(1 + x)(1 - x^{21} - x^{30}). \end{aligned}$$

For $b = 3$, look for a solution $y = \sum_{n \geq 0} y_n x^n$ of

$$L = (x^3 - x^6 + x^9 - x^{10} + 2x^{13} - 2x^{16} + x^{19})\, M^2 \\ -(1 - x^{28} - x^{31} - x^{37} - x^{40})\, M \\ +(x^6 + x^7 - x^{27} - x^{28} - x^{36} - x^{37}).$$

$L \longleftrightarrow$ infinite matrix $R = (R_{m,n})$



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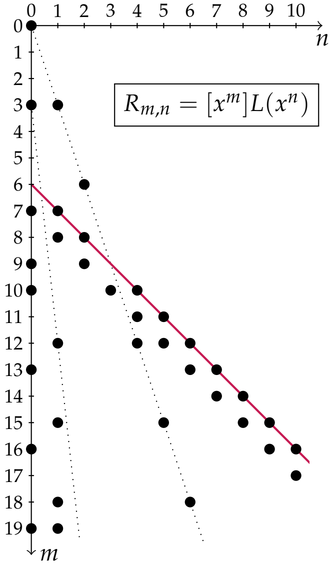
$$x^n \xrightarrow{x^6} x^{n+6}$$



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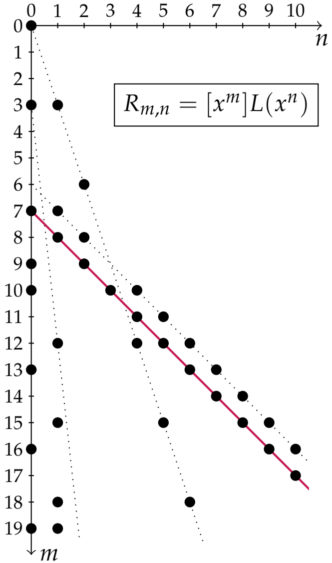
$$x^n \xrightarrow{x^7} x^{n+7}$$



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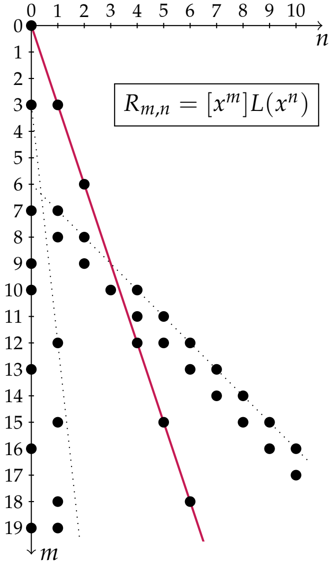
$$x^n \xrightarrow{M} x^{3n}$$



Series Solutions: Understanding the Linear System

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$$L = ({\color{purple}x^3} - x^6 + x^9 - x^{10} + 2x^{13} - 2x^{16} + x^{19})\, {\color{purple}M^2}$$
$$-(1 - x^{28} - x^{31} - x^{37} - x^{40})\, M$$
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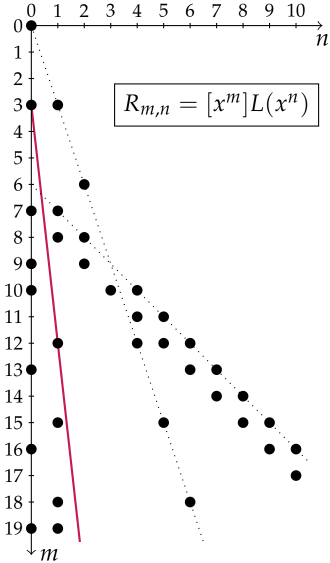
$$x^n \xrightarrow{{\color{purple}x^3 M^2}} x^{\color{purple}9n+3}$$



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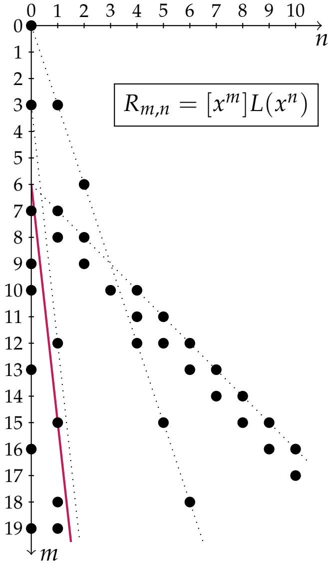
$$x^n \xrightarrow{x^6 M^2} x^{9n+6}$$



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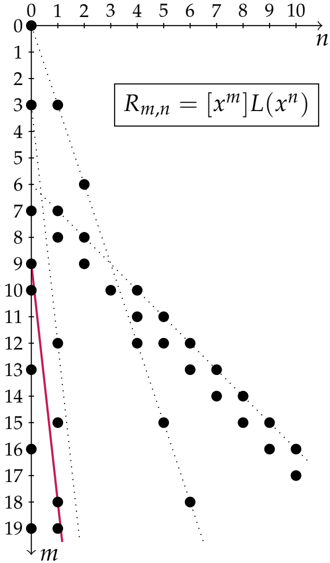

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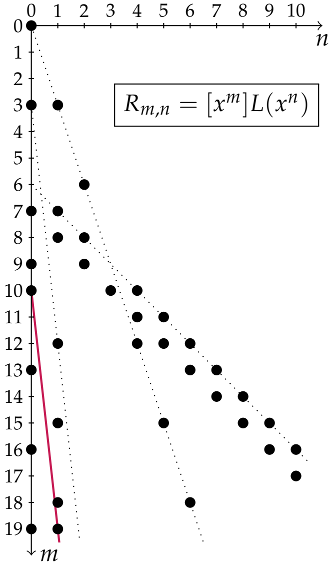
$$x^n \xrightarrow{x^{10} M^2} x^{9n+10}$$



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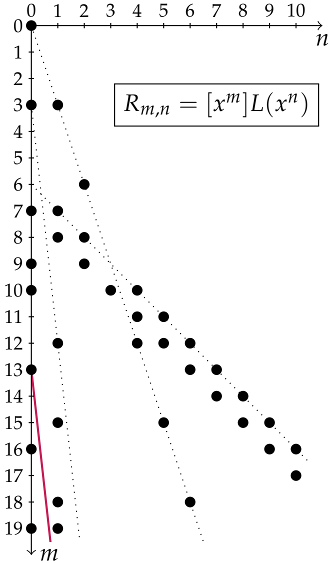

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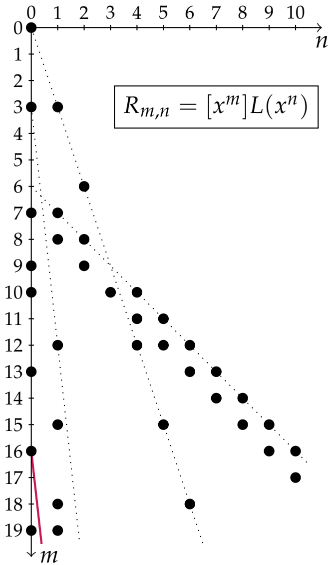

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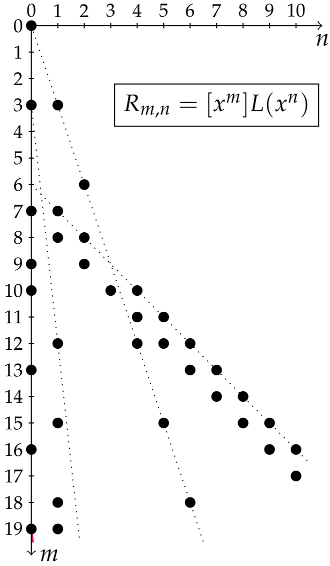
$$x^n \xrightarrow{x^{19} M^2} x^{9n+19}$$



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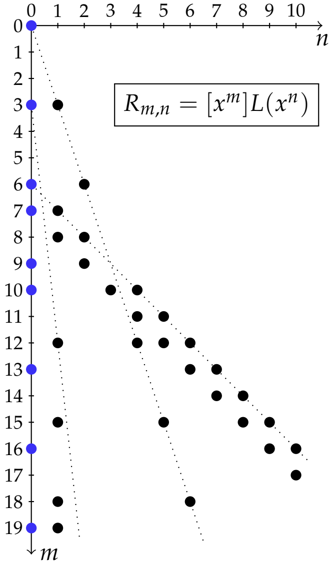
$x^0 \xrightarrow{L}$ span of the •'s



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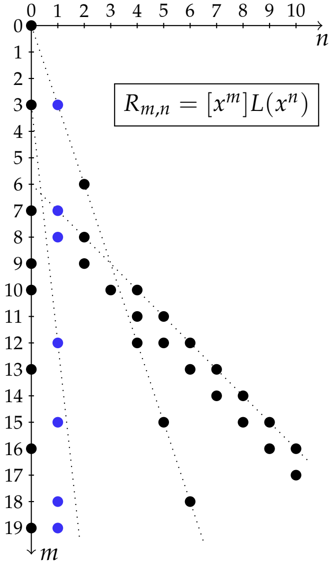

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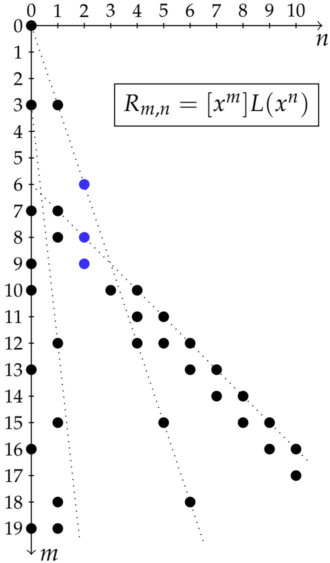
$x^2 \xrightarrow{L}$ span of the •'s



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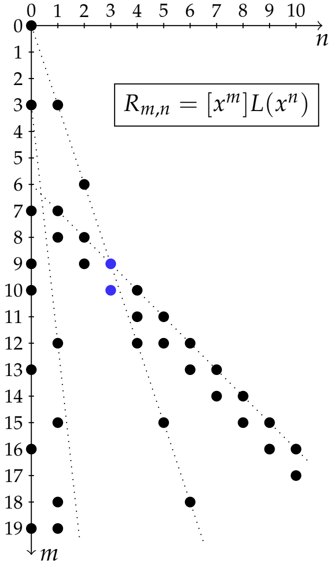
$x^3 \xrightarrow{L}$ span of the •'s



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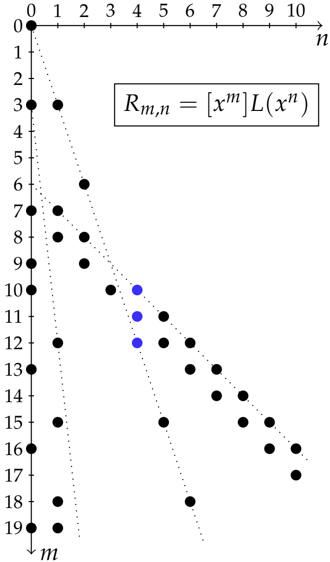

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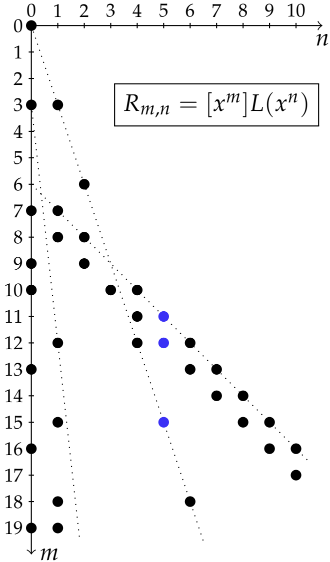
$x^5 \xrightarrow{L}$ span of the $\bullet$'s



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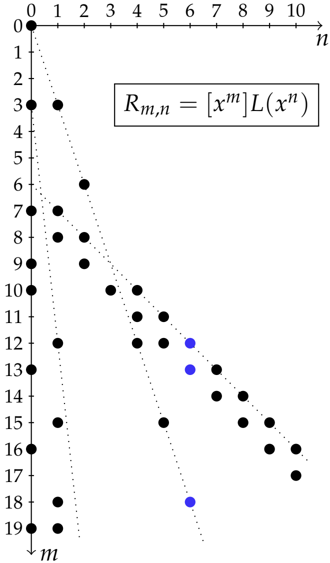
$x^6 \xrightarrow{L}$ span of the •'s



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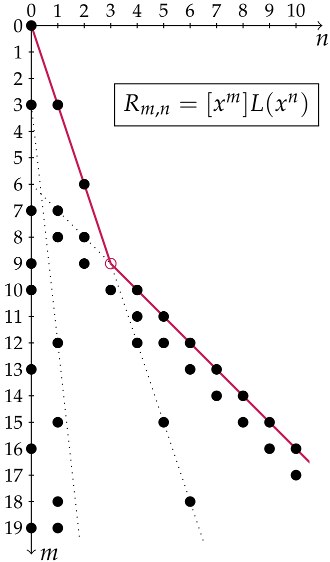
$$x^n \xrightarrow{L} * \, x^{\min(n+6,3n)} + \cdots$$



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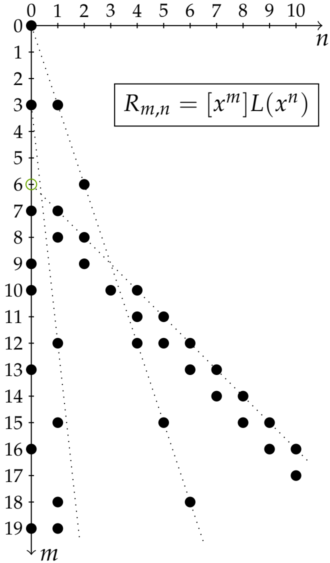
$$x^0 \xrightarrow{L} x^{\neq 6}$$



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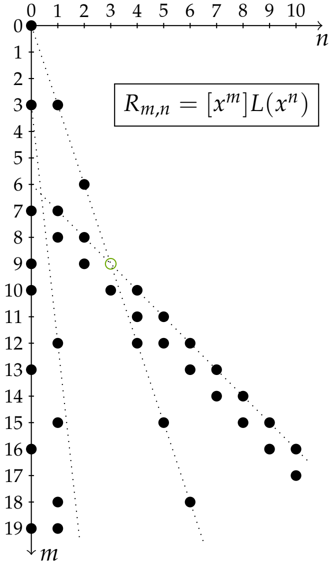
$$x^3 \xrightarrow{L} x^{>9}$$



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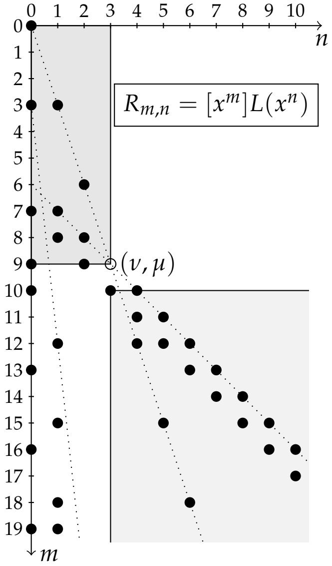

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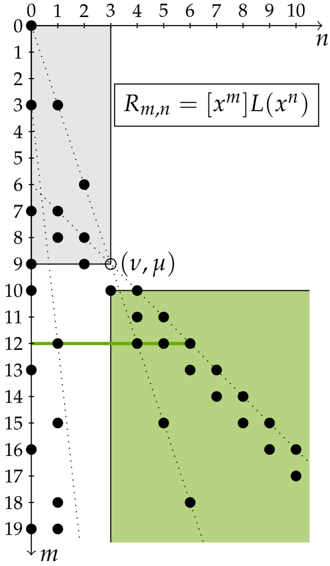
$m = 12 \longrightarrow y_6$ in terms of y_5, y_4, y_1



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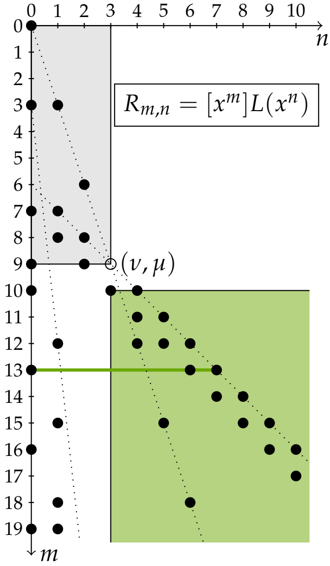
$m = 13 \longrightarrow y_7$ in terms of y_6, y_5, y_0



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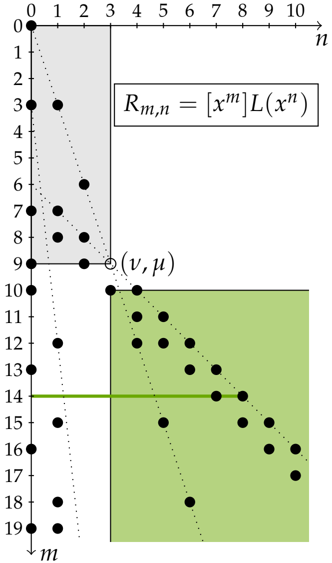
$m = 14 \longrightarrow y_8$ in terms of y_7



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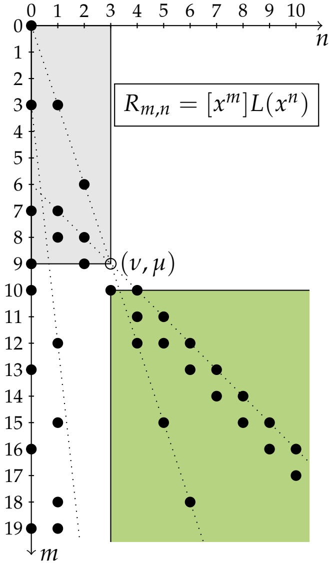
$m \geq 10 \longrightarrow y_{m-6}$ in terms of lower indices



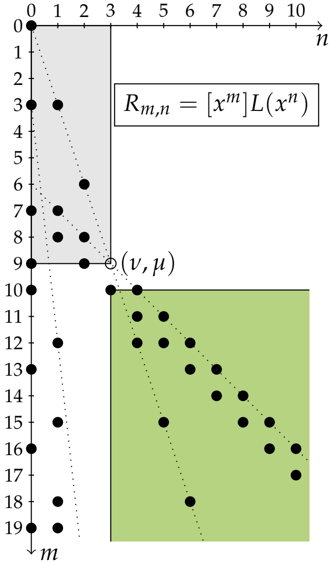
For $b = 3$, look for a solution $y = \sum_{n \geq 0} y_n x^n$ of

$$L = (x^3 - x^6 + x^9 - x^{10} + 2x^{13} - 2x^{16} + x^{19})\, M^2 \\ -(1 - x^{28} - x^{31} - x^{37} - x^{40})\, M \\ +(x^6 + x^7 - x^{27} - x^{28} - x^{36} - x^{37}).$$

For $m \geq 10$, y_{m-6} can be determined uniquely by

$$\left(y_{\frac{m-3}{9}} - y_{\frac{m-6}{9}} + y_{\frac{m-9}{9}} - y_{\frac{m-10}{9}} + 2y_{\frac{m-13}{9}} - 2y_{\frac{m-16}{9}} - y_{\frac{m-19}{9}}\right) \\ - \left(y_{\frac{m}{3}} - y_{\frac{m-28}{3}} - y_{\frac{m-31}{3}} - y_{\frac{m-37}{3}} - y_{\frac{m-40}{3}}\right) \\ + \left(y_{m-6} + y_{m-7} - y_{m-27} - y_{m-28} - y_{m-36} - y_{m-37}\right) = 0$$

where $y_s = 0$ if $s \notin \mathbb{N}$.



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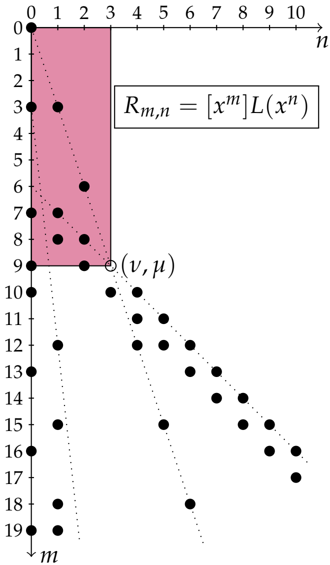
$y_0, \dots, y_3$ can be determined uniquely by

$$\left(y_{\frac{m-3}{9}} - y_{\frac{m-6}{9}} + y_{\frac{m-9}{9}} - y_{\frac{m-10}{9}} + 2y_{\frac{m-13}{9}} - 2y_{\frac{m-16}{9}} - y_{\frac{m-19}{9}}\right) \\ - \left(y_{\frac{m}{3}} - y_{\frac{m-28}{3}} - y_{\frac{m-31}{3}} - y_{\frac{m-37}{3}} - y_{\frac{m-40}{3}}\right) \\ + \left(y_{m-6} + y_{m-7} - y_{m-27} - y_{m-28} - y_{m-36} - y_{m-37}\right) = 0$$

for $0 \leq m \leq 9$, where $y_s = 0$ if $s \notin \mathbb{N}$.



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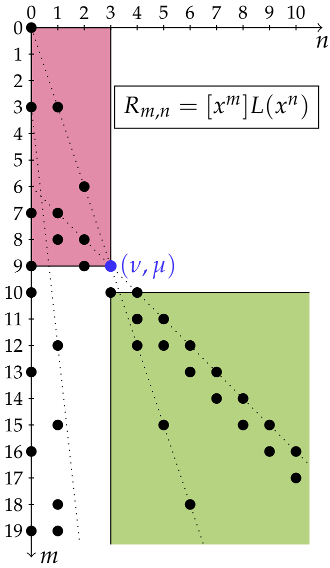
$y_0, \ldots, y_\nu \longleftrightarrow$ eqns for $0 \leq m \leq \mu$

y_ν is free $\longleftrightarrow$ coeff. at $(\nu, \mu) = 0$

y_n for $n > \nu \longleftrightarrow$ eqns for $m > \mu$

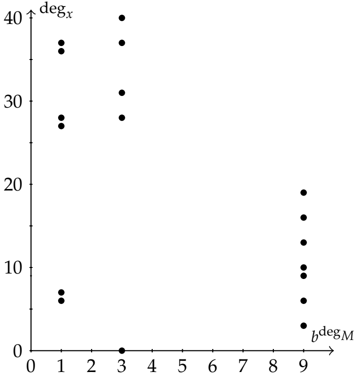


Newton Diagrams, Newton Polygons

Newton diagram

collection of points (b^k, j) associated with the monomials $x^j M^k$ of L

$$L = (x^3 - x^6 + x^9 - x^{10} + 2x^{13} - 2x^{16} + x^{19})\, M^2 \\ - (1 - x^{28} - x^{31} - x^{37} - x^{40})\, M \\ + (x^6 + x^7 - x^{27} - x^{28} - x^{36} - x^{37}).$$



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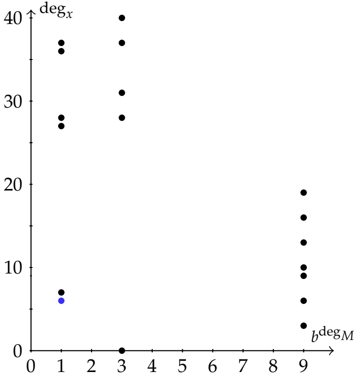

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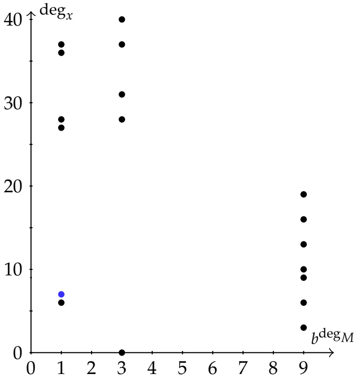

Newton Diagrams, Newton Polygons

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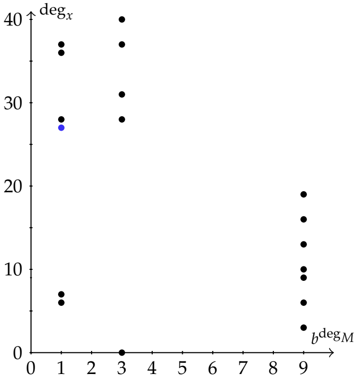

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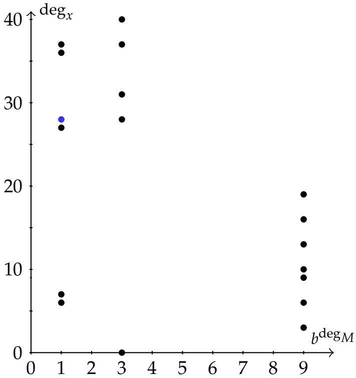
$$L = (x^3 - x^6 + x^9 - x^{10} + 2x^{13} - 2x^{16} + x^{19})\, M^2 - (1 - x^{28} - x^{31} - x^{37} - x^{40})\, M + (x^6 + x^7 - x^{27} - x^{28} - x^{36} - x^{37}).$$



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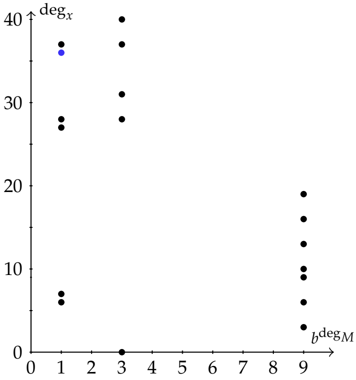

Newton Diagrams, Newton Polygons

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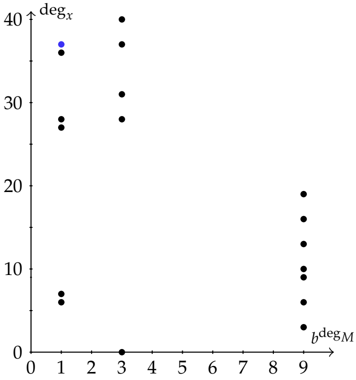

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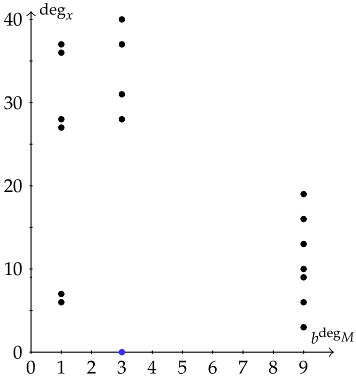

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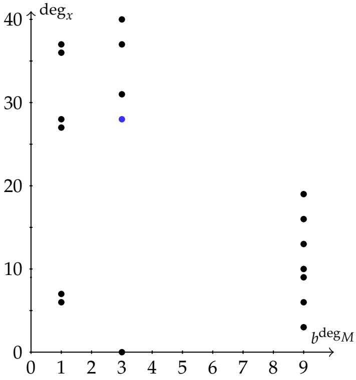

Newton Diagrams, Newton Polygons

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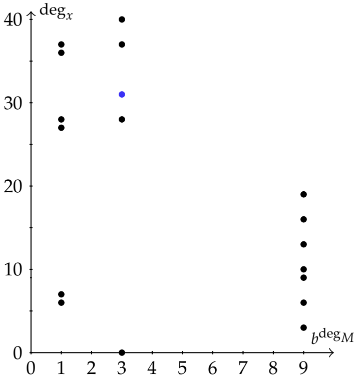

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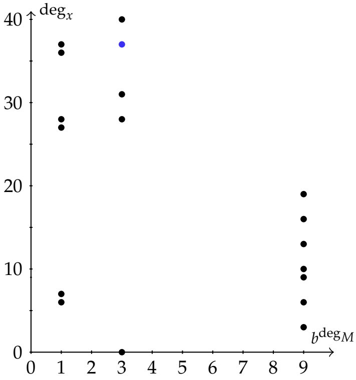

Newton Diagrams, Newton Polygons

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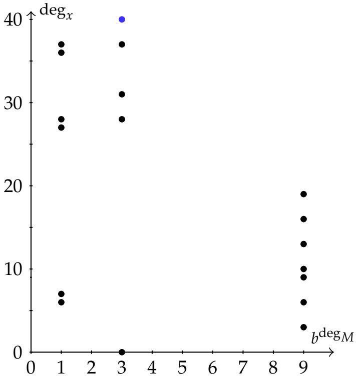
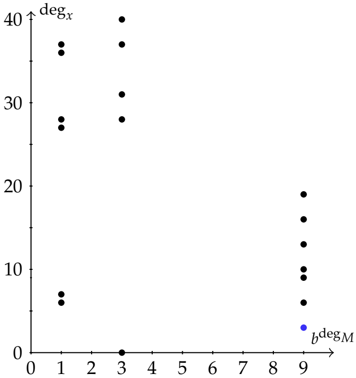


Newton diagram

collection of points (b^k, j) associated with the monomials $x^j M^k$ of L

$$
\begin{aligned}
L = {} & (x^3 - x^6 + x^9 - x^{10} + 2x^{13} - 2x^{16} + x^{19})\, M^2 \\
& - (1 - x^{28} - x^{31} - x^{37} - x^{40})\, M \\
& + (x^6 + x^7 - x^{27} - x^{28} - x^{36} - x^{37}).
\end{aligned}
$$



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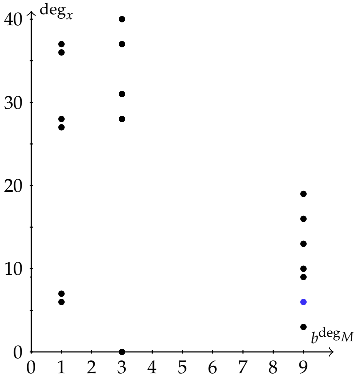

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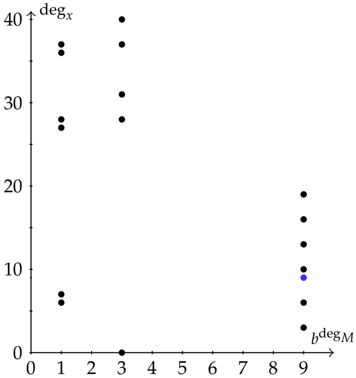

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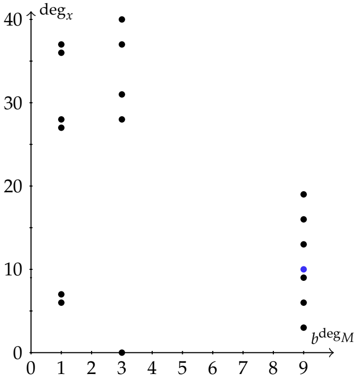

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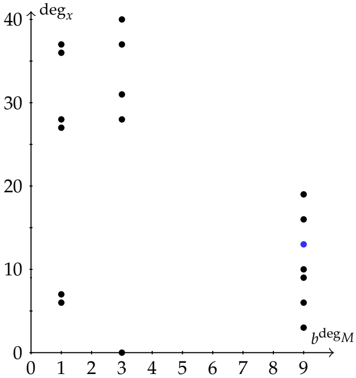

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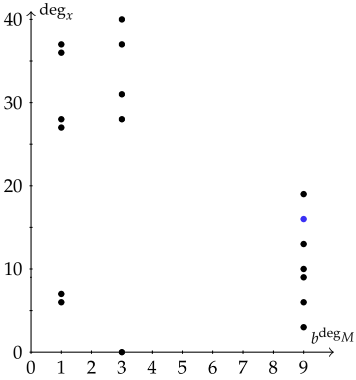

Newton Diagrams, Newton Polygons

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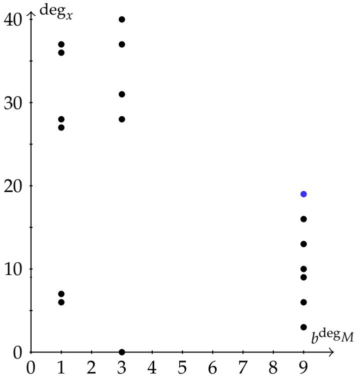

Newton Diagrams, Newton Polygons

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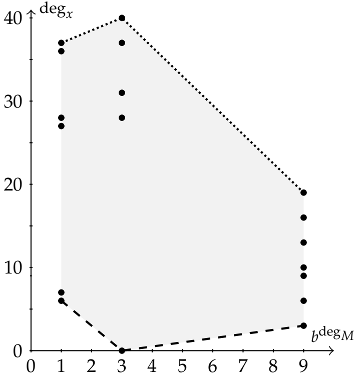

Newton polygons = boundary of convex hull

- lower Newton polygon $\longleftrightarrow$ valuations,
- upper Newton polygon $\longleftrightarrow$ degrees.

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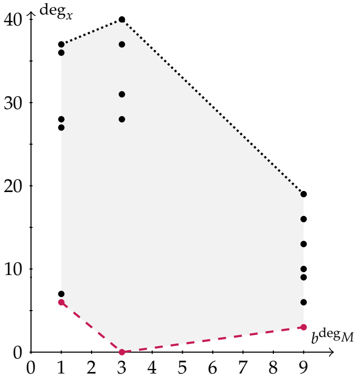

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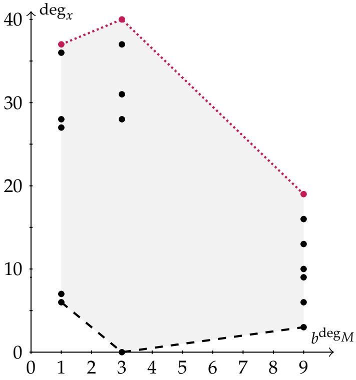

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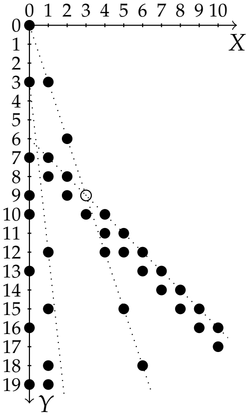
Link between the Linear System and the Newton Polygon

Linear system



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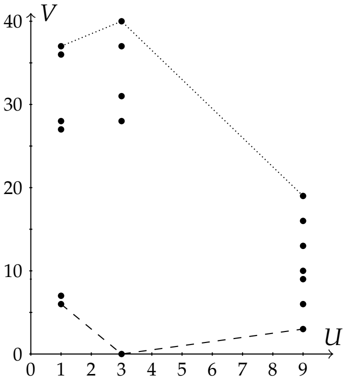
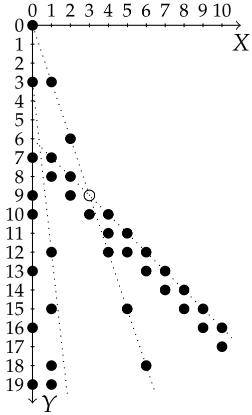


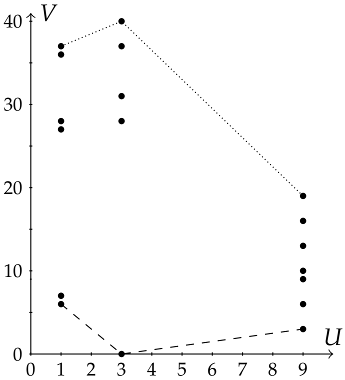
$\boxed{\text{Linear system}}$

line $Y = j + b^k X$



$\boxed{\text{Newton diagram}}$

point (b^k, j)



Link between the Linear System and the Newton Polygon

Linear system

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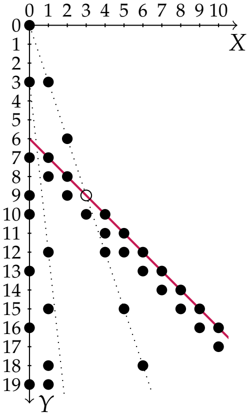

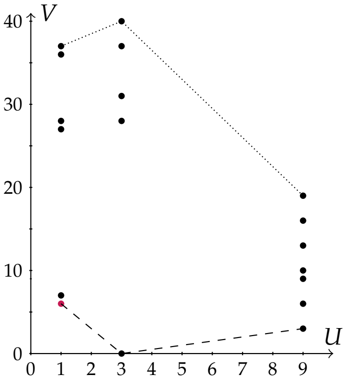

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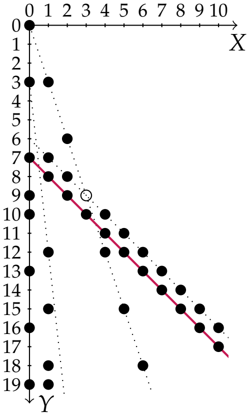

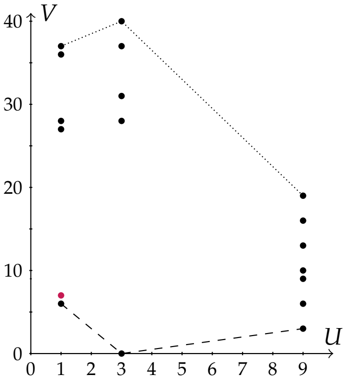

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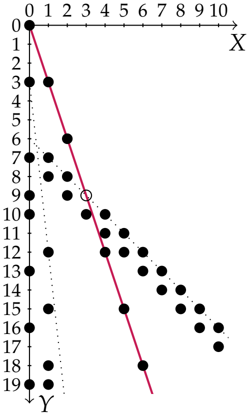

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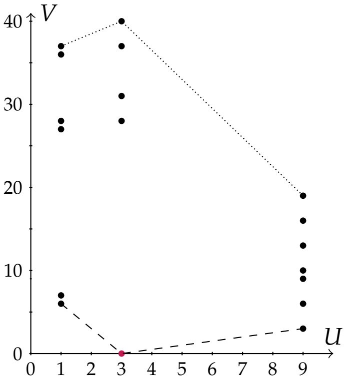
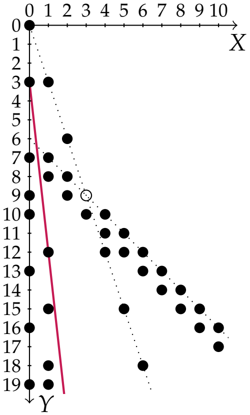


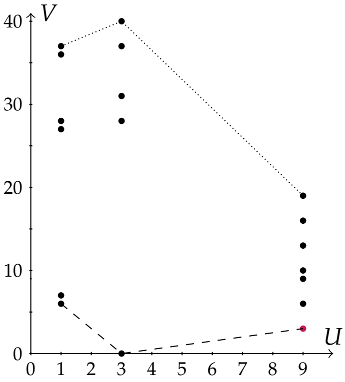
Linear system

line $Y = j + b^k X$



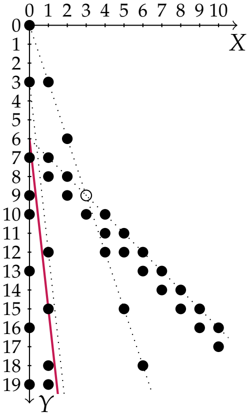
Newton diagram

point (b^k, j)



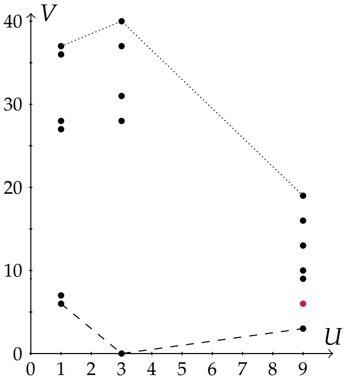
Linear system

line $Y = j + b^k X$



Newton diagram

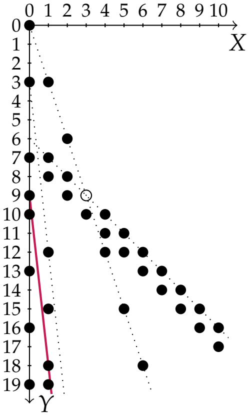
point (b^k, j)



Link between the Linear System and the Newton Polygon

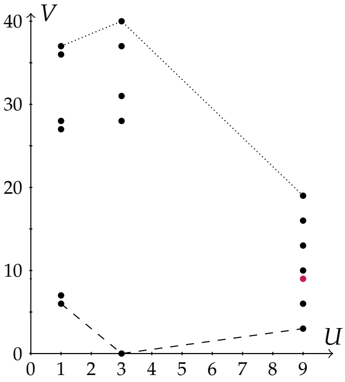
Linear system

line $Y = j + b^k X$



Newton diagram

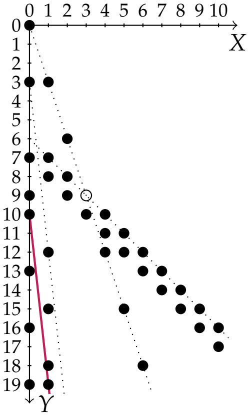
point (b^k, j)



Link between the Linear System and the Newton Polygon

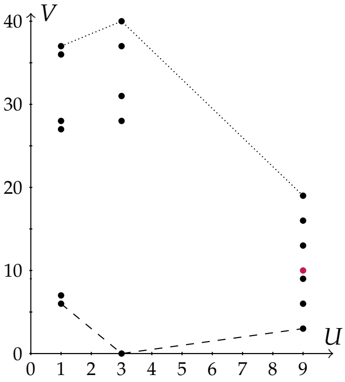
Linear system

line $Y = j + b^k X$



Newton diagram

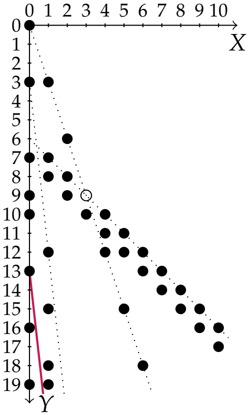
point (b^k, j)



Link between the Linear System and the Newton Polygon

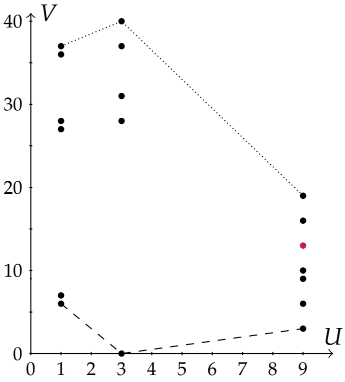
$\boxed{\text{Linear system}}$

line $Y = j + b^k X$



$\boxed{\text{Newton diagram}}$

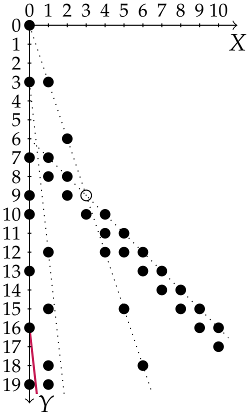
point (b^k, j)



Link between the Linear System and the Newton Polygon

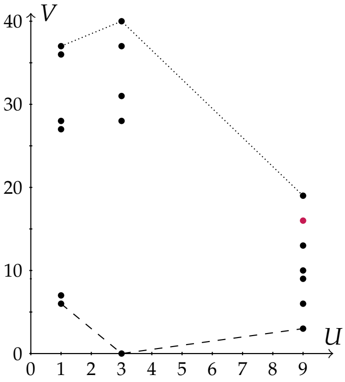
Linear system

line $Y = j + b^k X$



Newton diagram

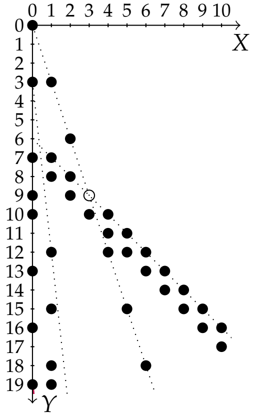
point (b^k, j)



Link between the Linear System and the Newton Polygon

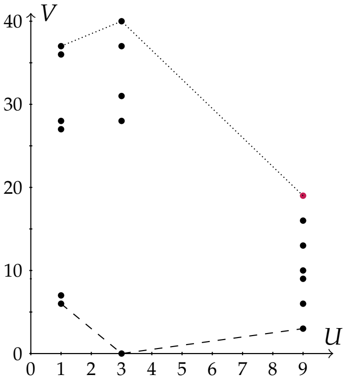
Linear system

line $Y = j + b^k X$



Newton diagram

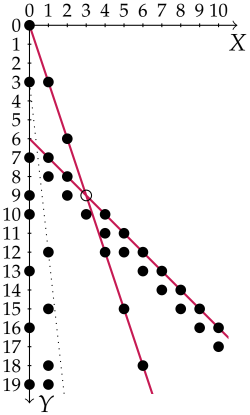
point (b^k, j)



Link between the Linear System and the Newton Polygon

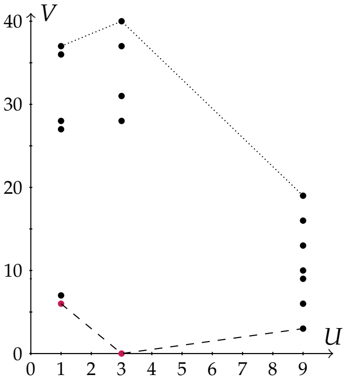
$\boxed{\text{Linear system}}$

line $Y = j + b^k X$



$\boxed{\text{Newton diagram}}$

point (b^k, j)

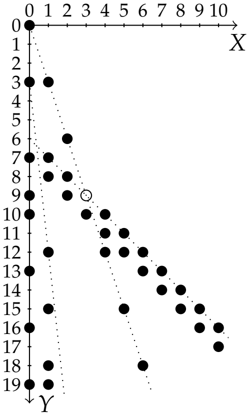


Link between the Linear System and the Newton Polygon

$\boxed{\text{Linear system}}$

line $Y = j + b^k X$

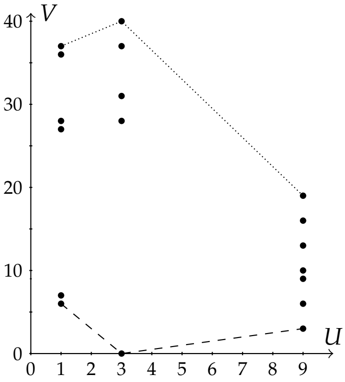
point (n, m)



$\boxed{\text{Newton diagram}}$

point (b^k, j)

line $V = m - nU$

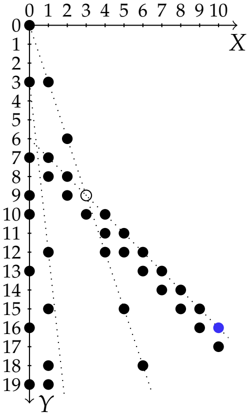


Link between the Linear System and the Newton Polygon

$\boxed{\text{Linear system}}$

line $Y = j + b^k X$

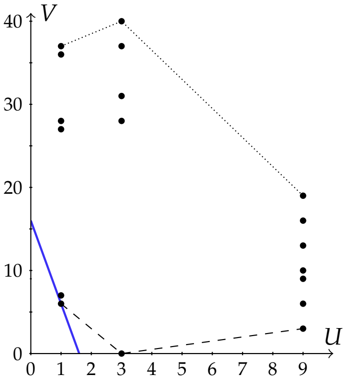
point (n, m)



$\boxed{\text{Newton diagram}}$

point (b^k, j)

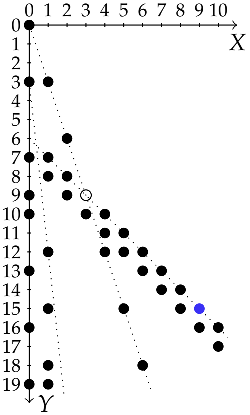
line $V = m - nU$



Linear system

line $Y = j + b^k X$

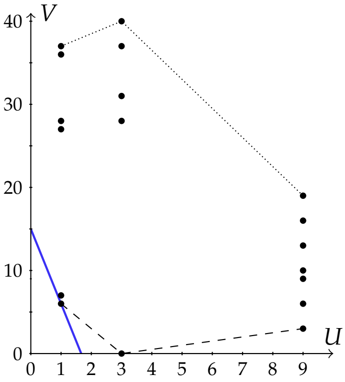
point (n, m)



Newton diagram

point (b^k, j)

line $V = m - nU$



Linear system

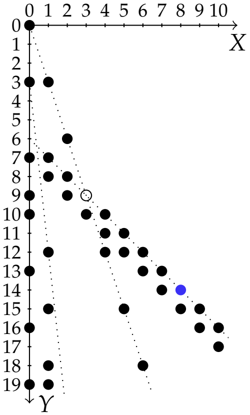
line $Y = j + b^k X$

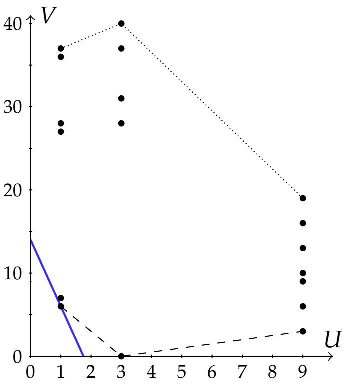
point (n, m)

Newton diagram

point (b^k, j)

line $V = m - nU$





Linear system

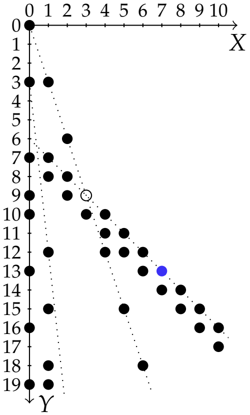
line $Y = j + b^k X$

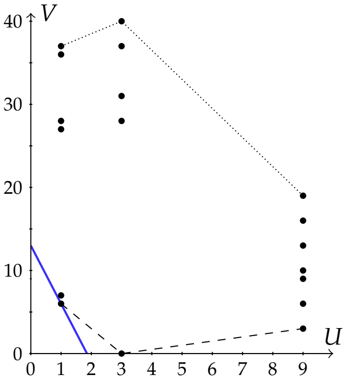
point (n, m)

Newton diagram

point (b^k, j)

line $V = m - nU$



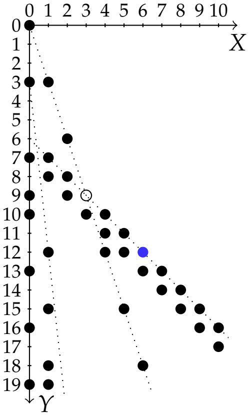


Link between the Linear System and the Newton Polygon

$\boxed{\text{Linear system}}$

line $Y = j + b^k X$

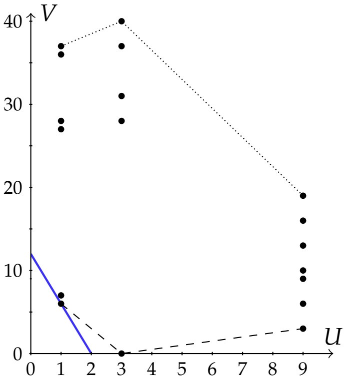
point (n, m)



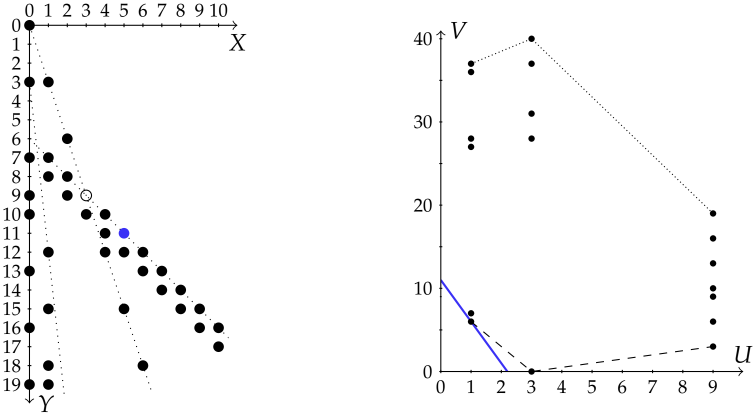
$\boxed{\text{Newton diagram}}$

point (b^k, j)

line $V = m - nU$

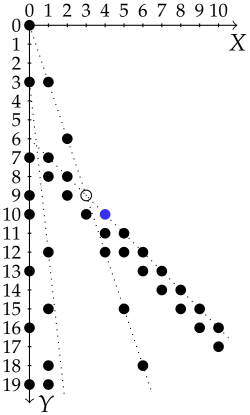


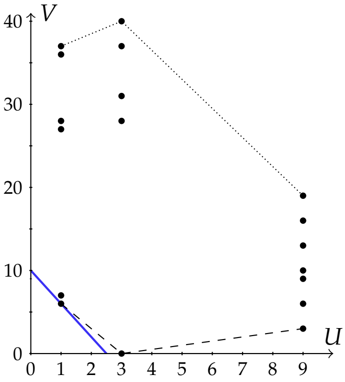
Linear system	Newton diagram
line $Y = j + b^k X$	point (b^k, j)
point (n, m)	line $V = m - nU$



Link between the Linear System and the Newton Polygon

Linear system	Newton diagram
line $Y = j + b^k X$	point (b^k, j)
point (n, m)	line $V = m - nU$



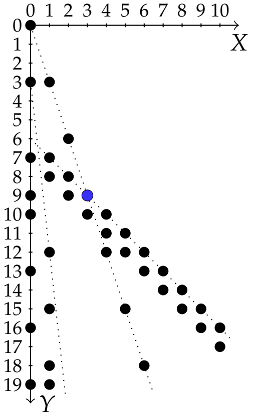


Link between the Linear System and the Newton Polygon

Linear system

line $Y = j + b^k X$

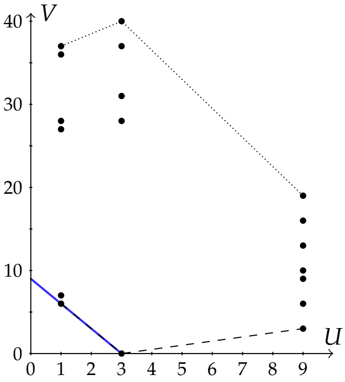
point (n, m)



Newton diagram

point (b^k, j)

line $V = m - nU$

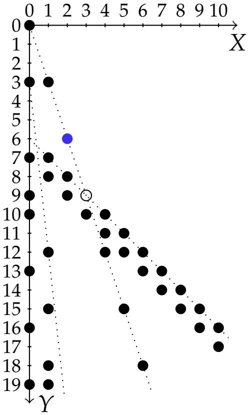


Link between the Linear System and the Newton Polygon

$\boxed{\text{Linear system}}$

line $Y = j + b^k X$

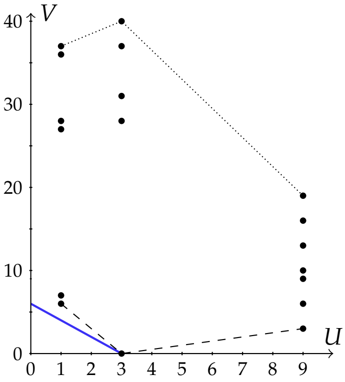
point (n, m)



$\boxed{\text{Newton diagram}}$

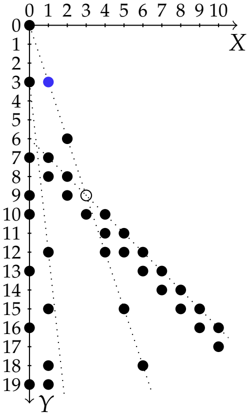
point (b^k, j)

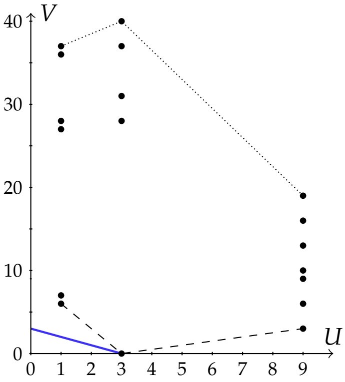
line $V = m - nU$



Link between the Linear System and the Newton Polygon

Linear system	Newton diagram
line $Y = j + b^k X$	point (b^k, j)
point (n, m)	line $V = m - nU$





Link between the Linear System and the Newton Polygon

Linear system	Newton diagram
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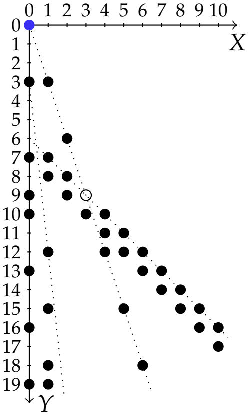

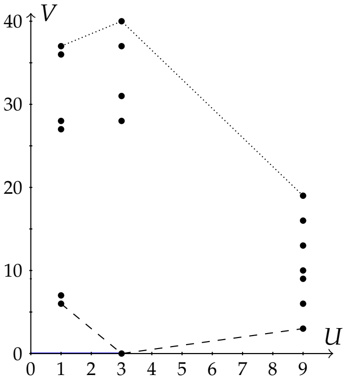

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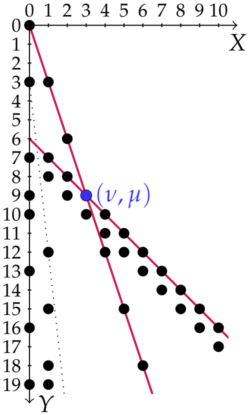
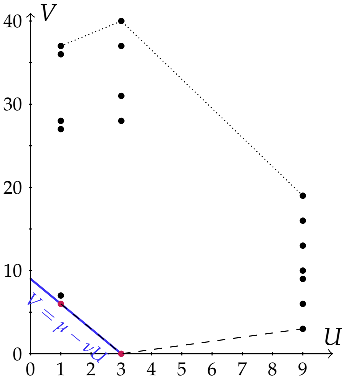




Bounds for Series and Polynomial Solutions

$$L = \ell_r(x)M^r + \cdots + \ell_0(x), \quad \ell_0\ell_r \neq 0, \quad 0 \le v_k = \operatorname{val} \ell_k, \quad d_k = \deg \ell_k \le d.$$

Size of singular system ("approximate series solutions")

$$\nu = \max_{k\ge 1} \frac{v_0 - v_k}{b^k - 1}, \qquad \mu = v_0 + \nu.$$

Any series solution to order $\mathrm{O}(x^\nu + 1)$ can be prolonged uniquely.

Bound on valuation v of a series solution

$$-\frac{v_r}{b^{r-1}(b-1)} \le v = -\frac{v_{k_2} - v_{k_1}}{b^{k_2} - b^{k_1}} \le \frac{v_0}{b-1},$$

where $v = -$ slope of edge $\left[(b^{k_1}, v_{k_1}), (b^{k_2}, v_{k_2})\right]$ in lower Newton polygon.

Bound on degree δ of any polynomial solution

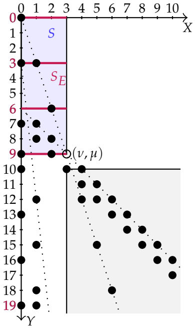
$$\delta = -\frac{d_{k_1} - d_{k_2}}{b^{k_1} - b^{k_2}} \le \frac{d}{b^{r-1}(b-1)},$$

where $\delta = -$ slope of edge $\left[(b^{k_1}, v_{k_1}), (b^{k_2}, v_{k_2})\right]$ in upper Newton polygon.

Solving the Singular System. Series and Polynomial Solutions

Algorithm: find candidate solutions, then recombine

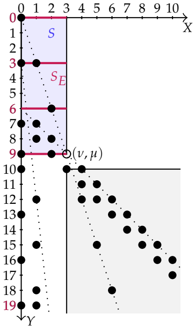
1. $h := \lfloor \mu \rfloor + 1$, $w := \lfloor \nu \rfloor + 1$, $E := \left(\min_k (v_k + nb^k)\right)_{0 \le n < w}$.
2. $S :=$ upper left subsystem $\in \mathbb{K}^{h \times w}$.
3. $S_E :=$ submatrix of S given by the rows of index in E.
4. $G := \ker S_E \in \mathbb{K}^{w \times \rho}$ with $\rho = \dim \ker S_E \le r$.
   (forward substitution)
5. For $1 \le j \le \rho$,
   1. $g_j := G_{0,j} + G_{1,j}x + \cdots + G_{w-1,j}x^{w-1} \in \mathbb{K}[x]$,
   2. $\sum_{0 \le i < h} s'_{i,j} x^i := L g_j(x) \bmod x^h$.
6. $S' := (s'_{i,j}) \in \mathbb{K}^{h \times \rho}$.
7. $K := \ker S' \in \mathbb{K}^{\rho \times \sigma}$ with $\sigma = \dim \ker S' \le r$.
   (Ibarra, Moran, Hui, 1982)
8. $F := GK \in \mathbb{K}^{w \times \sigma}$.
9. Return $(f_1, \ldots, f_\sigma)$ where $f_j = F_{0,j} + \cdots + F_{w-1,j}x^{w-1}$.



Solving the Singular System. Series and Polynomial Solutions

Algorithm: find candidate solutions, then recombine

1. $h := \lfloor \mu \rfloor + 1$, $w := \lfloor \nu \rfloor + 1$, $E := \left(\min_k(v_k + nb^k)\right)_{0 \le n < w}$.
2. $S :=$ upper left subsystem $\in \mathbb{K}^{h \times w}$. $\quad$ $O(rwh)$
3. $S_E :=$ submatrix of S given by the rows of index in E.
4. $G := \ker S_E \in \mathbb{K}^{w \times \rho}$ with $\rho = \dim \ker S_E \le r$. (forward substitution) $\quad$ $O(rw^2)$
5. For $1 \le j \le \rho$, $\quad$ $O(r^2\,\mathrm{M}(h))$
   1. $g_j := G_{0,j} + G_{1,j}x + \cdots + G_{w-1,j}x^{w-1} \in \mathbb{K}[x]$,
   2. $\sum_{0 \le i < h} s'_{i,j} x^i := Lg_j(x) \bmod x^h$.
6. $S' := (s'_{i,j}) \in \mathbb{K}^{h \times \rho}$.
7. $K := \ker S' \in \mathbb{K}^{\rho \times \sigma}$ with $\sigma = \dim \ker S' \le r$. (Ibarra, Moran, Hui, 1982) $\quad$ $O(r^{\omega-1}h)$
8. $F := GK \in \mathbb{K}^{w \times \sigma}$. $\quad$ $O(r^{\omega-1}w)$
9. Return $(f_1, \ldots, f_\sigma)$ where $f_j = F_{0,j} + \cdots + F_{w-1,j}x^{w-1}$.



Theorem (correctness and complexity) $\quad$ [cf. naive $O(v_0^\omega)$]

Returns a basis of approximate series solutions in $O(rv_0^2 + r^2\,\mathrm{M}(v_0))$ ops.

Solving the Singular System. Series and Polynomial Solutions

Algorithm: find candidate solutions, then recombine

1. $h := 3d + 1,\ w := \lfloor \frac{d}{b^r - b^{r-1}} \rfloor + 1,\ E := \left(\max_k(d_k + nb^k)\right)_{0 \le n < w}$.
2. $S :=$ upper left subsystem $\in \mathbb{K}^{h \times w}$. $\quad O(rwh)$
3. $S_E :=$ submatrix of S given by the rows of index in E.
4. $G := \ker S_E \in \mathbb{K}^{w \times \rho}$ with $\rho = \dim \ker S_E \le r$.
   (backward substitution) $\quad O(rw^2)$
5. For $1 \le j \le \rho$, $\quad O(r^2\,M(h))$
   1. $g_j := G_{0,j} + G_{1,j}x + \cdots + G_{w-1,j}x^{w-1} \in \mathbb{K}[x]$,
   2. $\sum_{0 \le i < h} s'_{i,j} x^i := Lg_j(x) \bmod x^h$.
6. $S' := (s'_{i,j}) \in \mathbb{K}^{h \times \rho}$.
7. $K := \ker S' \in \mathbb{K}^{\rho \times \sigma}$ with $\sigma = \dim \ker S' \le r$.
   (Ibarra, Moran, Hui, 1982) $\quad O(r^{\omega-1}h)$
8. $F := GK \in \mathbb{K}^{w \times \sigma}$. $\quad O(r^{\omega-1}w)$
9. Return $(f_1, \ldots, f_\sigma)$ where $f_j = F_{0,j} + \cdots + F_{w-1,j}x^{w-1}$.

Theorem (correctness and complexity) [cf. naive $O(d^\omega)$]

Returns a basis of polynomial solutions in $\tilde{O}(d^2/b^r + M(d))$ ops.

Part II

Rational Solutions

Rational Solutions: Initiating the Quest for Denominator Bounds

$$\sum_{k=0}^{r} \ell_k \, M^k \left(\frac{p}{q}\right) = 0, \qquad p \wedge q = 1, \; q(0) \neq 0.$$

Rational Solutions: Initiating the Quest for Denominator Bounds

$$\sum_{k=0}^{r} \ell_k \, \frac{M^k p}{M^k q} = 0, \qquad p \wedge q = 1, \; q(0) \neq 0.$$

$$\sum_{k=0}^{r-1} \ell_k \, (M^r q) \left((M^k q)^{-1} \bigvee_{i=0}^{r-1} M^i q \right) (M^k p) + \ell_r \left(\bigvee_{i=0}^{r-1} M^i q \right) (M^r p) = 0.$$

$$\sum_{k=0}^{r} \ell_k \, \frac{M^k p}{M^k q} = 0, \qquad p \wedge q = 1, \; q(0) \neq 0.$$

$$\sum_{k=0}^{r-1} \ell_k \, (M^r q) \underbrace{\left((M^k q)^{-1} \bigvee_{i=0}^{r-1} M^i q \right)}_{\text{polynomial}} (M^k p) + \ell_r \left(\bigvee_{i=0}^{r-1} M^i q \right) (M^r p) = 0.$$

Rational Solutions: Initiating the Quest for Denominator Bounds

$$\sum_{k=0}^{r} \ell_k \, \frac{M^k p}{M^k q} = 0, \qquad p \wedge q = 1, \; q(0) \neq 0.$$

$$\sum_{k=0}^{r-1} \ell_k \, (M^r q) \underbrace{\left((M^k q)^{-1} \bigvee_{i=0}^{r-1} M^i q \right)}_{\text{polynomial}} (M^k p) + \ell_r \underbrace{\left(\bigvee_{i=0}^{r-1} M^i q \right)}_{\text{dividable by } M^r q} (M^r p) = 0.$$

Rational Solutions: Initiating the Quest for Denominator Bounds

$$\sum_{k=0}^{r} \ell_k \, \frac{M^k p}{M^k q} = 0, \qquad p \wedge q = 1,\ q(0) \neq 0.$$

$$\sum_{k=0}^{r-1} \ell_k \, (M^r q) \underbrace{\left((M^k q)^{-1} \bigvee_{i=0}^{r-1} M^i q \right)}_{\text{polynomial}} (M^k p) + \ell_r \underbrace{\left(\bigvee_{i=0}^{r-1} M^i q \right)}_{\text{dividable by } M^r q} (M^r p) = 0.$$

$$M^r q \mid \ell_r \bigvee_{i=0}^{r-1} M^i q$$

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$$q(\alpha) = 0 \text{ and } \beta^{b^r} = \alpha \implies \ell_r(\beta) = 0 \text{ or } \left(q(\alpha') = 0 \text{ for } \alpha' = \beta^{b^i}, \; i < r \right)$$

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Geometric intuition

If α, not a root of unity, is a root of q, it generates:

- a finite sequence $\alpha_0 = \alpha, \alpha_1, \ldots, \alpha_n$ of two by two distinct roots of q,
- then, a b^rth root β of α_n that is a root of ℓ_r.

The Graeffe Operator: a Quasi-Inverse for the Mahler Operator

$$\begin{array}{ccc}
\gamma = \alpha^b & (Gp)(\gamma) = 0 & Gp = \mathrm{Res}_y(y^b - x, p(y)) \\
\uparrow & \uparrow & \uparrow \\
\alpha \in \mathbb{C} & p(\alpha) = 0 & p \in \mathbb{K}[x] \\
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Key lemma (= algebraic interpretation of geometric intuition)

Given $\ell \in \mathbb{K}[x]$:

$$\begin{cases} q \in \mathbb{K}[x] \setminus \mathbb{K} \\ x \nmid q \\ M^r q \mid \ell \bigvee_{i=0}^{r-1} M^i q \end{cases} \implies \exists u \in \mathbb{K}[x] \setminus \mathbb{K}, \quad M^r u \mid \ell \quad \text{or} \quad \begin{cases} M^{r-1} u \mid \ell \\ q \mid Gu \end{cases}$$

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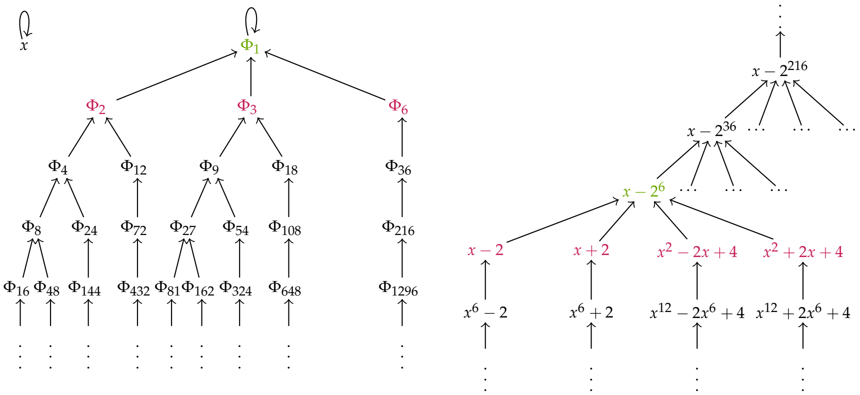
Remark: left case impossible $\implies$ q is a product of cyclotomic polynomials.

Graph of (Radical of) Graeffe Operator for $b = 6$ in $\mathbb{Q}[x]$

$$Mp = \prod_{q \text{ s.t. } \sqrt{G}(q) = p} q$$

$$M(x - 2^6) = (x-2)(x+2)(x^2-2x+4)(x^2+2x+4)$$

$$M\Phi_1 = \Phi_1 \Phi_2 \; \Phi_3 \; \Phi_6$$



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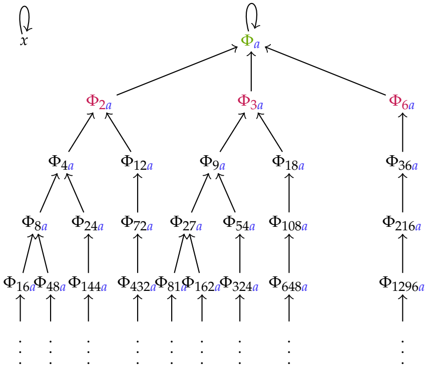
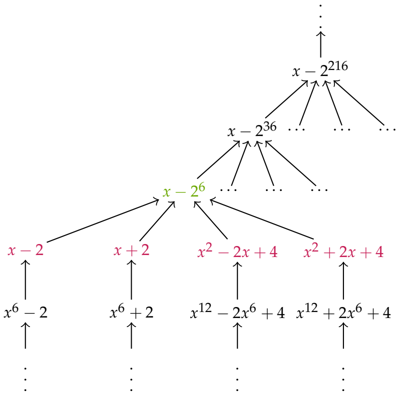
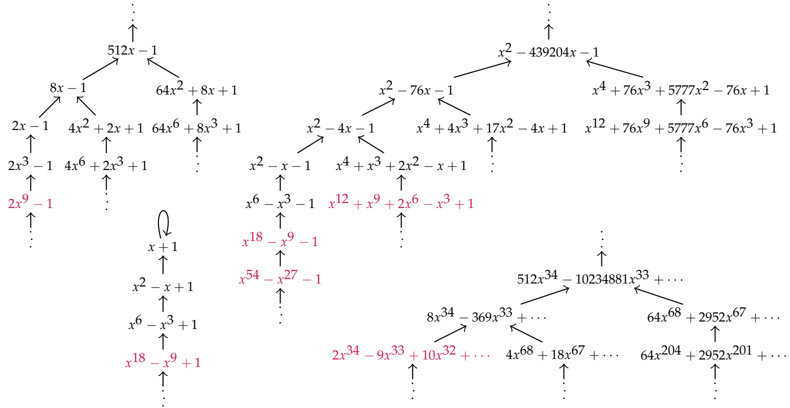




Looking for a Denominator: an Example When $b = 3$

$$L = (2x^9 - 1)(x^{18} - x^9 - 1)(x^{12} + x^9 + 2x^6 - x^3 + 1)(x^{18} - x^9 + 1)(2x^{34} - 9x^{33} + 10x^{32} + \cdots)(x^{54} - x^{27} - 1)\, M^2 + \cdots$$

$$L\left(\frac{p}{q}\right) = 0, \qquad M^2 q \mid \ell_2\,(q \vee Mq).$$



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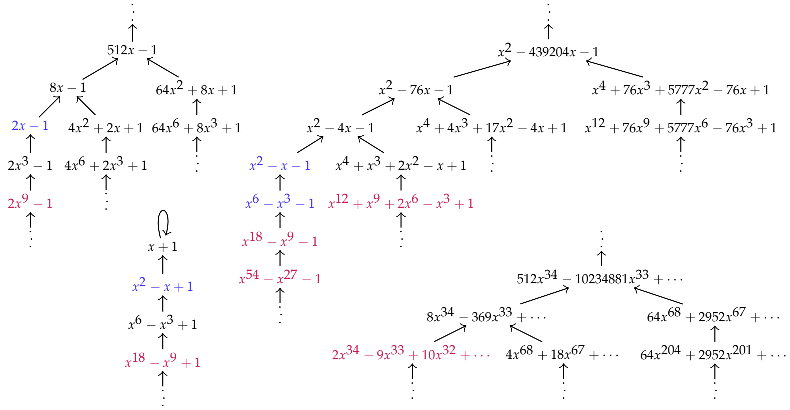

$$u_1 = (2x-1)(x^2-x+1)(x^2-x-1)(x^6-x^3-1) \implies M^2 u_1 \mid \ell_2$$

Looking for a Denominator: an Example When $b = 3$

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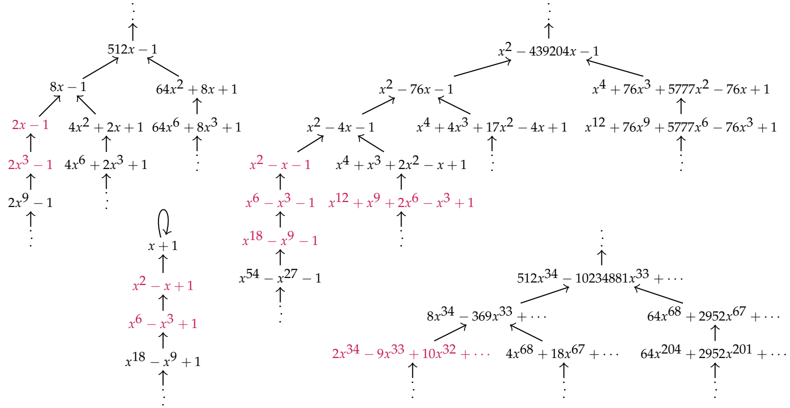

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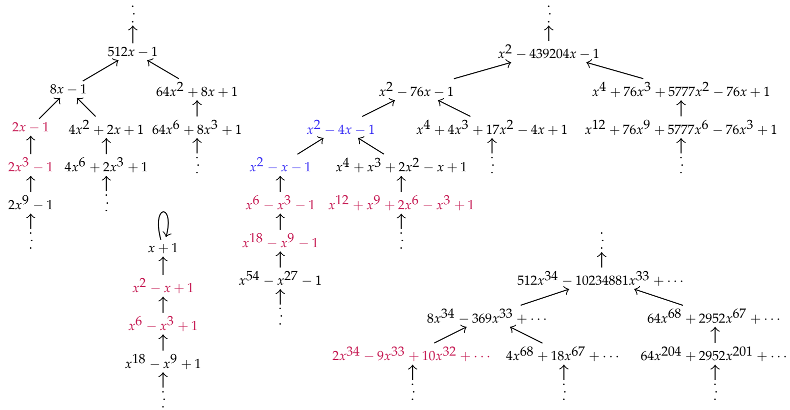

$$u_2 = (x^2-x-1)(x^2-4x-1) \implies M^2 u_2 \mid \ell_2$$

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$L[u_1 u_2] = (2x-1)(x^2-x+1)(x^2-x-1)^2(x^2-4x-1)(2x^3-1)(x^4+x^3+2x^2-x+1)(x^6-x^3+1)(x^6-x^3-1)(2x^{34}-9x^{33}+\cdots)\,M^2 + \cdots$

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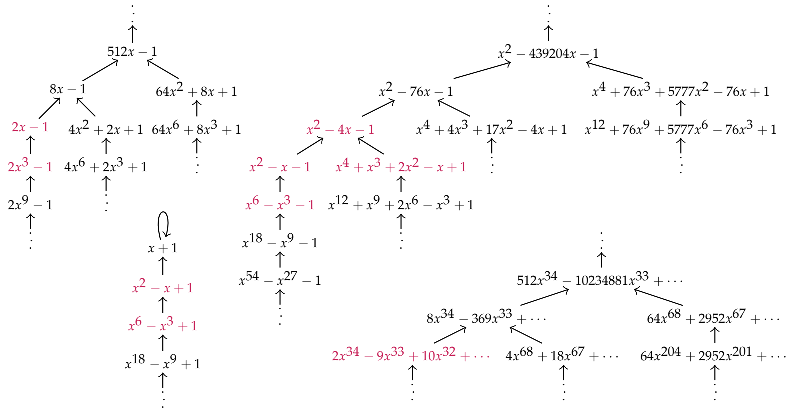

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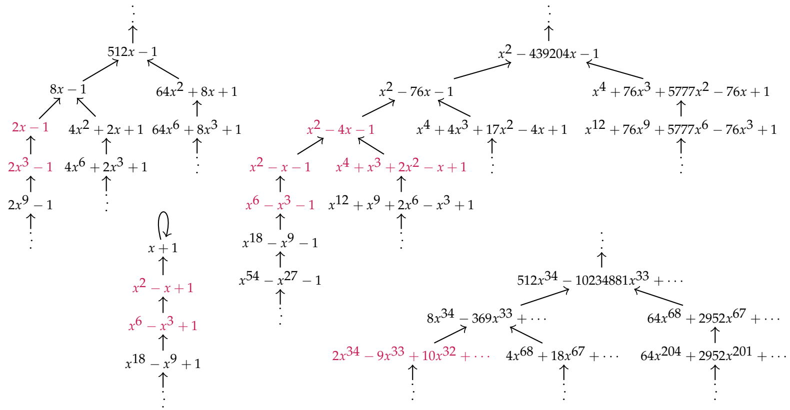

$u_3 = 1$

Looking for a Denominator: an Example When $b = 3$

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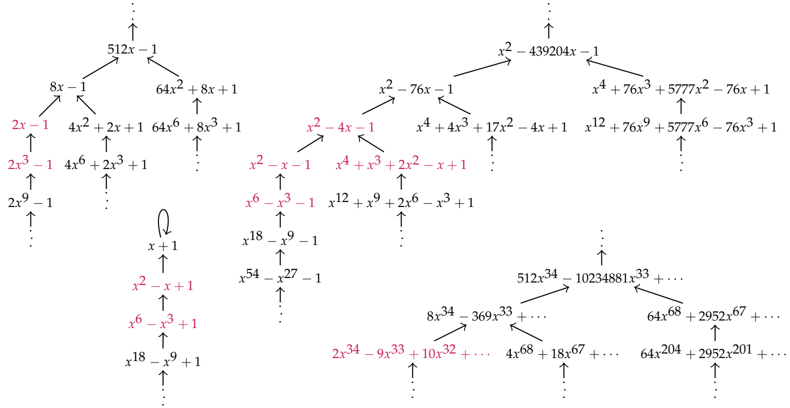

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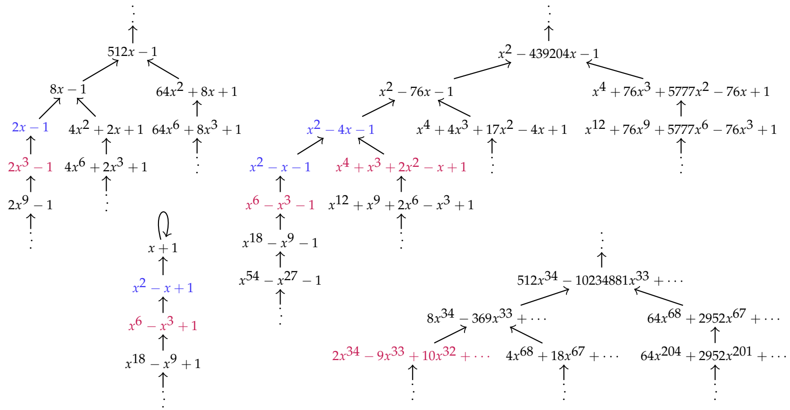

$$\tilde{u} = (2x-1)(x^2-x+1)(x^2-x-1)(x^2-4x-1) \implies M\tilde{u} \mid \ell_2 \text{ and } q \mid G\tilde{u}$$

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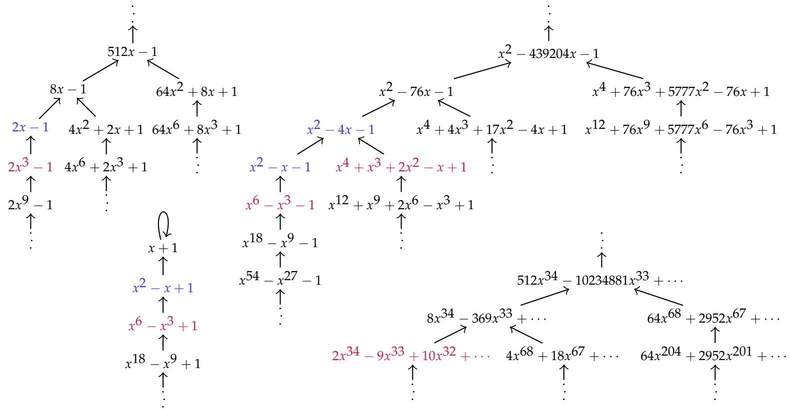

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Operator to solve:

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Denominator bound found:

$$q^\star = u_1u_2G\tilde{u} = (2x-1)(x^2-x+1)^2(x^2-x-1)^2(x^2-4x-1)(x^6-x^3-1)$$

Looking for a Denominator: an Example (Conclusion)

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Explicit rational solutions:

$$\frac{2x}{(2x-1)(x^2-x-1)} \quad \text{and} \quad \frac{x-3}{(x^2-x+1)(x^2-4x-1)(x^6-x^3-1)}$$

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Remark: $\deg \ell_2 = 145 > \deg \ell_2[u_1] = 84 > \deg \ell_2[u_1u_2] = 62.$

Section Operators

ith Section in radix b, for $0 \leq i < b$

$$S_i : \sum_{n \geq 0} c_n x^n \mapsto \sum_{n \geq 0} c_{bn+i} x^n$$

$$f = \sum_{i=0}^{b-1} x^i \left(M\, S_i(f) \right)$$

Solving divisibility problem by a gcd

$$Mu \mid \ell \iff u \mid \bigwedge_{i=0}^{b-1} S_i(\ell)$$

Section Operators

Can be used for radix b^m:

ith Section in radix b^m, for $0 \le i < b^m$

$$S_i^{(m)} : \sum_{n \ge 0} c_n x^n \mapsto \sum_{n \ge 0} c_{b^m n + i} x^n$$

$$f = \sum_{i=0}^{b^m - 1} x^i \left(M \, S_i^{(m)}(f) \right)$$

Solving divisibility problem by a gcd

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Computing Denominator Bounds

Algorithm

1. Set $\ell = \ell_r$, then repeat for $k = 1, 2, \dots$:
$$\text{set } u_k = \bigwedge_{i=0}^{b^r-1} S_i^{(r)}(\ell), \text{ then } \ell = \left(\frac{\ell}{M^r u_k}\right) \bigvee_{i=0}^{r-1} M^i u_k$$
until $\deg u_k = 0$.
2. Set $\tilde{u} = \displaystyle\bigwedge_{i=0}^{b^{r-1}-1} S_i^{(r-1)}(\ell)$.
3. Return $q^\star = u_1 \cdots u_{k-1}\, G\tilde{u}$.

Theorem (correctness)

Assume $Ly = 0$ with $y = \frac{p}{x^{\bar{v}} q}$ and $q(0) \neq 0$. Then, $q \mid q^\star$.

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Theorem (complexity and degree bounds)

- if $b = 2$: algorithm runs in $\mathrm{O}(d\ \mathrm{M}(d) \log d)$ ops and $\deg q^\star \leq d$
- if $b \geq 3$: algorithm runs in $\mathrm{O}(b^{-r} d\ \mathrm{M}(d) \log d)$ ops and $\deg q^\star \leq d/b^{r-1}$

Conclusion

Summary

- theory of Newton polygons,
- algorithms for formal series solutions, polynomial solutions, and rational-function solutions,
- algorithms for denominator bounds:
  - good: complexity is polynomial in r and d,
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Ongoing/Future work

- infinite-product solutions,
- rational solutions of Riccati-type equation.