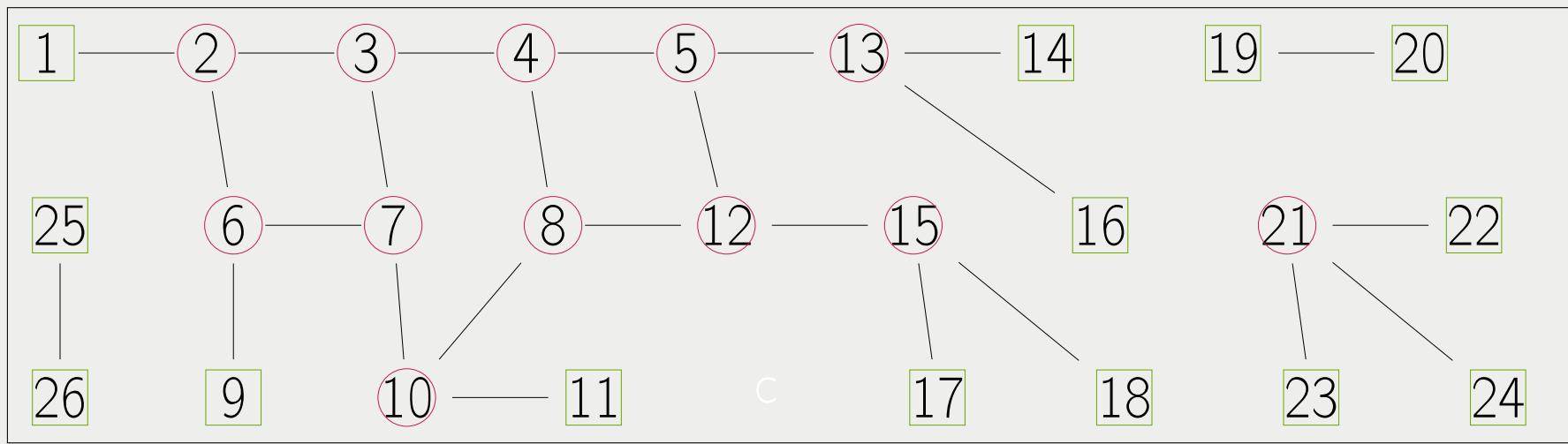




Conjecture 16 in (Kauers, Koutschan, 2023)

Sequence A339987 of the OEIS, which counts vertex-labelled (3,1)-regular graphs having one more vertex than edges by half the number of vertices, satisfies an explicit, guessed recurrence relation of order 5, valid for all $k \geq 0$.



26 vertices, 25 edges, all degrees in $\{1, 3\}$

$a_0 = 0, \quad a_1 = 1, \quad a_2 = 4, \quad a_3 = 90, \quad a_4 = 8400, \quad a_5 = 1426950,$
 $a_6 = 366153480, \quad a_7 = 134292027870$

$$32(328k^3 + 3300k^2 + 10844k + 11589) \\ \times (k + 1)(k + 2)(2k + 1)(2k + 3)(2k + 5)(2k + 7)(2k + 9) a_k \\ \vdots \quad \quad \quad \vdots \quad \quad \quad \vdots \\ + 3(328k^3 + 2316k^2 + 5228k + 3717)(k + 4)(k + 6) a_{k+5} = 0$$

An Application of Gessel’s Theory (1987, 1990)

Scalar product (= standard pairing on symmetric functions):

$$\langle p_1^{r_1} p_2^{r_2} p_3^{r_3}, p_1^{s_1} p_2^{s_2} p_3^{s_3} \rangle = \begin{cases} 1^{r_1} r_1! 2^{r_2} r_2! 3^{r_3} r_3! & \text{if } r = s, \\ 0 & \text{otherwise.} \end{cases}$$

Defined on $\mathbb{Q}[p_1, p_2, p_3]$, then extended by bilinearity to $\mathbb{Q}[p_1, p_2, p_3][[q, t]]$.

A D-finite generating function:

$r_{m,n} := \#\{(3,1)\text{-regular simple graphs on } n \text{ labelled vertices with } m \text{ edges}\}$

The EGF $\sum_{m,n=0}^{\infty} r_{m,n} q^m \frac{t^n}{n!} = \langle \exp(f(p, q)), \exp(tg(p)) \rangle,$

where

$f(p, q) = \frac{1}{2}q(p_1^2 - p_2) - \frac{1}{4}q^2p_2^2 + \frac{1}{6}q^3p_3^2, \quad g(p) = p_1 + \frac{1}{6}p_1^3 + \frac{1}{2}p_1p_2 + \frac{1}{3}p_3,$

is D-finite w.r.t. q and t .

Related works: Algorithms and ODEs for k -regular graphs and k -uniform tableaux in (Chyzak, Mishna, Salvy, 2005), (Chyzak, Mishna, 2026).

First approach: Fixing Creative Telescoping

Triple-sum representation:

$$a_k = r_{2k-1,2k} = \sum_{u \in \mathbb{Z}} \sum_{v \in \mathbb{Z}} \sum_{w \in \mathbb{Z}} \frac{(-1)^{u+v+w+1}}{2^{k-1-u+w} 3^{2u+w-k}} \\ \times (2k)!(2w)!(k+1)_i (u+w-k)_i u_i v_i w_i (2k-1-3u-2v-w)_i$$

Everywhere-defined factorial and “inverse factorial” functions:

$$n! = \begin{cases} n! & \text{if } n \geq 0, \\ 0 & \text{if } n < 0. \end{cases} \quad n_i = \begin{cases} 1/n! & \text{if } n \geq 0, \\ 0 & \text{if } n < 0. \end{cases}$$

Everywhere-valid recurrence relations:

$\forall n \in \mathbb{Z}, \quad n! = n(n-1)! + \llbracket n = 0 \rrbracket, \quad \forall n \in \mathbb{Z}, \quad n_i = (n+1)(n+1)_i.$

Classical creative telescoping: (Zeilberger, 1990), (Chyzak, 2000), (Koutschan, 2010) introduce lines of poles + many ill-shaped auxiliary sums + non-natural bounds $\rightarrow$ complete proof is intractable.

Satisfiability Modulo Linear Integer Arithmetic (SMT LIA):

$t := \text{poly}(k, u, v, w) \times \prod_i \ell_i(k, u, v, w)! \times \prod_i \ell'_i(k, u, v, w)_i \times \llbracket e = 0 \rrbracket$

can be simplified in a recurrence relation by observing

$t = 0 \Leftarrow \bigwedge_i (\ell_i(k, u, v, w) \geq 0) \wedge \bigwedge_i (\ell'_i(k, u, v, w) \geq 0) \text{ unsatisfiable.}$

Complete proof: experimental search of a modified multivariate CT relation + simplification by SMT LIA.

Second approach: Residue Representation

Formal Laplace transform, formal residue:

$$\mathcal{L}(p_1^{r_1} p_2^{r_2} p_3^{r_3}) = \frac{r_1!}{p_1^{r_1+1}} \frac{r_2!}{p_2^{r_2+1}} \frac{r_3!}{p_3^{r_3+1}}, \quad \text{res} \left(\sum_{r \in \mathbb{Z}^3} c_r p^r \right) = c_{-1, \dots, -1}.$$

Scalar products and diagonals are residues:

$$\sum_{k=0}^{\infty} a_k \frac{t^{2k}}{(2k)!} = \text{even}_t \text{diag}_{q,t} \left\langle q \exp(f(q, p)), \exp(tg(p)) \right\rangle \\ = \text{even}_t \text{res}_{p,q} \left(e^{f(p_1, p_2, p_3, q)} \mathcal{L} \left(e^{q^{-1}tg(1p_1, 2p_2, 3p_3)} \right) \right).$$

Composition of (holonomy-preserving) operations:

- $\exp(q^{-1}t\tilde{g}(p))$: annihilated by $q\partial_{p_i} - t\frac{d\tilde{g}}{dp_i}, \quad q^2\partial_q + t\tilde{g}, \quad q\partial_t - \tilde{g}.$
- $\mathcal{L}(\dots)$: change $p_i \rightarrow -\partial_{p_i}$ and $\partial_{p_i} \rightarrow p_i$, and think modulo $\ker(\text{res}_p)$
- $\exp(f(p)) \times \dots$: change $\partial_{p_i} \rightarrow \partial_{p_i} - \frac{df}{dp_i}$
- $\text{res}_{p,q}(\dots)$: compute the integral of a module

Integration in D-modules: (Brochet, Chyzak, Lairez, 2026)

$$\begin{array}{ccc} \text{holonomic system in} & & \\ \mathbb{Q}(t)[p_1, p_2, p_3, q] \langle \partial_{p_1}, \partial_{p_2}, \partial_{p_3}, \partial_q \rangle & \longrightarrow & \boxed{\begin{array}{c} M \\ \hline \frac{\partial M}{\partial t} \\ + \text{action of } \partial_t \end{array}} \longrightarrow \text{ODE in } \mathbb{Q}(t) \langle \partial_t \rangle \\ + \text{action of } \partial_t & & \end{array}$$

Hadrien Brochet’s MultivariateCreativeTelescoping.jl

Usual creative telescoping unsuitable: system contains $p_3q - t$.

Third approach: A Combinatorial Recurrence Reminiscent of (Read, 1970) and (Wormald, 1979)

Four interrelated combinatorial classes:

- $\mathcal{G}_0$: vertices of degree 3 or 1 only
- $\mathcal{G}_1$: vertices of degree at most 3, including exactly one of degree 2
- $\mathcal{G}_2$: vertices of degree at most 3, including exactly two of degree 2, which are possibly adjacent
- $\tilde{\mathcal{G}}_2$: vertices of degree at most 3, including exactly two of degree 2, which are not adjacent

Four generating functions:

$g_{a,b,c} = \# \text{ of labelled graphs with } a, b, c \text{ vertices of degree } 1, 2, 3, \quad \tilde{g}_{a,2,c} = \dots$

$$G_b(q, t) = \sum_{a,c \geq 0} g_{a,b,c} q^{\frac{a+2b+3c}{2}} \frac{t^{a+b+c}}{(a+b+c)!}, \quad 0 \leq b \leq 2, \quad \tilde{G}_2(q, t) = \dots$$

One of five decompositions:

$\text{element of } \mathcal{G}_0 \text{ with a marked edge} \rightarrow \text{element of } \mathcal{G}_1 \text{ or element of } \tilde{\mathcal{G}}_2$

$$\frac{1}{2}(a+3c)g_{a,0,c} = (a+c)g_{a-1,1,c-1} + \tilde{g}_{a,2,c-2} \\ q\partial_q \cdot G_0(q, t) = qtG_1(q, t) + q\tilde{G}_2(q, t)$$

Algebraic calculations:

- elimination by module Gröbner basis over $\mathbb{Q}(q, t) \langle \partial_q, \partial_t \rangle \rightarrow$ system for G_0 in $\mathbb{Q}(q, t) \langle \partial_q, \partial_t \rangle$
- diagonal by creative telescoping for D-finite functions (Bostan, Chyzak, Lairez, Salvy, 2018), (van der Hoeven, 2021) $\rightarrow$ differential equation for $\sum_{k \geq 0} a_k \frac{t^{2k}}{(2k)!}$

Conclusions

Creative telescoping in computational proofs:

- triple sum**: CT for hypergeometric multi-summation
traditionally: implicit assumption make proofs incomplete
here: fix whose generality has to be explored
- residues**: CT in D-modules for residues
generally: ensuring a holonomic input system is difficult
here: polynomial exponential make it simple
- combinatorial recursion**: CT for the diagonal is very robust

Role of computation in our proofs:

- triple sum**: right formulation by a lot of mathematical experimentation
- residues**: straightforward algorithmic calculation
- recursion**: computer enumeration used to fix proof by cases