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# A Symbolic-Numeric Validation Algorithm for Linear ODEs with Newton-Picard Method

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Computing uniform approximations with validated error bounds for solutions of various kinds of differential equations is a very common task in the community of computer-assisted proofs in mathematics [1, 2, 3, 4].

Spectral-Galerkin methods are a tool of choice for computing uniform approximations of the solution of a linear ordinary differential equation [5, 6, 7]. The variable coefficients as well as the solution are approximated by truncated series in a well-chosen chosen basis of orthogonal functions, like the Chebyshev polynomials. The idea is to rephrase the differential equation as an infinite linear system and solve a finite-dimensional truncation of it, thanks to some compactness property. It is therefore natural to consider the same truncation scheme to design a Newton-like a posteriori validation operator [8, 9, 10]. Using the Banach fixed-point theorem, one obtains a rigorous error bound associated to the approximation obtained by the numerical spectral method.

In the first part of this talk, we will show that although spectral methods are known to produce exponentially fast convergent approximations, the corresponding validation procedure may converge much slower [10]. Indeed, the truncation index for the validation operator may be much larger than the one actually used for numerical approximation in the spectral method, rapidly leading to very large matrices.

In the second part of this talk, we present an alternative validation algorithm [11] with the desired "exponential convergence" property. Inspired by the famous Picard iterations [12], the idea consists in approximating the so-called "resolvent kernel" of the inverse integral operator rather than truncating the corresponding infinite matrix. It is similar in essence to the symbolic Newton iterations on differential equations [13, 14, 15], but in a numerical setting in a well-chosen Banach space of coefficients of orthogonal functions rather than exact Taylor expansions. This complexity gap is illustrated in practice by examples involving "large" parameters.

**Keywords**
Linear Differential Equations, Validated Numerics, Spectral Methods

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