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Separating Variables in Bivariate Polynomial Ideals: the Local Case

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We report on work in progress on the problem of finding all elements of $\mathbb{K}(x)+\mathbb{K}(y)$ that are of the form qp for a given irreducible polynomial $p \in \mathbb{K}[x,y]$ and some non-zero rational function $q \in \mathbb{K}(x,y)$ whose denominator is not divisible by p.

Let p be an irreducible polynomial of $\mathbb{K}[x,y]$ that is not an element of $\mathbb{K}[x]\cup\mathbb{K}[y]$. We define the *local ring* of $\mathbb{K}[x,y]$ at p by

$$\mathbb{K}[x,y]_p := \{r \in \mathbb{K}(x,y) : p \nmid \operatorname{denom}(r)\}\,.$$

and denote by $\langle p\rangle$ the set of multiples of p in $\mathbb{K}[x,y]_p$. The set of *separated multiples* of p is

$$\langle p\rangle \cap (\mathbb{K}(x)+\mathbb{K}(y))$$

and can be described by

$$\mathrm{F}(p) := \{(f,g) \in \mathbb{K}(x)\times\mathbb{K}(y) : f-g \in \langle p\rangle\}\,.$$

The latter is a field with respect to component-wise addition and multiplication and referred to as the *field of separated multiples* of p. By Lüroth's theorem there is a pair $(f,g) \in \mathbb{K}(x)\times\mathbb{K}(y)$ of rational functions such that

$$\mathrm{F}(p) = \mathbb{K}((f,g)).$$

If there are $f \in \mathbb{K}(x)$ and $g \in \mathbb{K}(y)$ such that $qp = f-g$ for some $q \in \mathbb{K}(x,y)\setminus\{0\}$, then it is enough to know the singularities of f and g and their multiplicities to find them. The latter essentially means to know the denominators of f, g and q and the degrees of their numerators. The unknown f, g and q can then be determined by making an ansatz for their numerators, clearing denominators in $qp = f-g$, comparing coefficients and solving a system of linear equations for them. We present a heuristic to determine the singularities of f and g and their multiplicities by inspecting the leading parts of p with respect to different gradings. Based on previous work [1] we also explain why we belief that our reasoning gives rise to an algorithm that solves the problem of determining a $(f,g) \in \mathbb{K}(x)\times\mathbb{K}(y)$ that generates $\mathrm{F}(p)$.

Keywords
Commutative algebra, elimination theory, separation of variables

References

[1] Manfred Buchacher, Manuel Kauers, Gleb Pogudin, *Separating variables in bivariate polynomial ideals.* Proceedings of the 45th International Symposium on Symbolic and Algebraic Computation, 54–61, 2020.
[2] Olivier Bernardi, Mireille Bousquet-Mélou, Kilian Raschel, *Counting quadrant walks via Tutte's invariant method.* Discrete Mathematics & Theoretical Computer Science, 2020.