Applications of Computer Algebra – ACA 2023
Warsaw, Poland, | July 17-21, 2023
Session *"D-Finite Functions and Beyond: Algorithms, Combinatorics, and Arithmetic"*

Two applications of the telescoping method

Qing-Hu Hou[1]***, Guo-Jie Li***[2]***, Na Li***[1]***, Ke Liu***[3] [qh_hou@tju.edu.cn]

[1] School of Mathematics, Tianjin University, Tianjin, China
[2] School of Science, Hainan University, Hainan, China
[3] College of Science, Chongqing University of Technology, Chongqing, China

In this talk, we will give two applications of telescoping method.

The first one focuses on series involving π. In [5], Sun derived several identities involving π by telescoping method. For example, from Bauer's series [1]

$$\sum_{k=0}^{\infty}(4k+1)\frac{\binom{2k}{k}^3}{(-64)^k}=\frac{2}{\pi}$$

and the telescoping sum

$$\sum_{k=0}^{n}\frac{(16k^3-4k^2-2k+1)\binom{2k}{k}^2}{(2k-1)^2(-64)^k}=\frac{8(2n+1)}{(-64)^n}\binom{2n}{n}^3,$$

he deduced

$$\sum_{k=0}^{\infty}\frac{k(4k-1)\binom{2k}{k}^3}{(2k-1)^2(-64)^k}=-\frac{1}{\pi}. \tag{1}$$

We aim to give a systematic method to construct series like (1). This motivates us to consider the following problem: Given a hypergeometric term t_k, for which rational functions $r(k)$ is the product $r(k)t_k$ Gosper summable?

By aid of Gosper's algorithm, we give candidates for the denominator of $r(k)$. Then by polynomial reduction [2,4], we derive an upper bound and a lower bound on the degree of the numerator of $r(k)$. Based on these results, we are able to construct several new series involving π.

Wang and Zhong [6] further extended the method of polynomial reduction to P-recursive sequences. We also give a brief introduction on their results.

The second one focuses on the congruences of partial sums of P-recursive sequences [3]. For example, we have

$$\frac{2}{n}\sum_{k=1}^{n}(2k+1)M_k^2\in\mathbb{Z}.$$

where

$$M_k = \sum_{l=0}^{k} \binom{k}{2l} \frac{\binom{2l}{l}}{l+1}$$

is the k-th Motzkin number.

Let $\{a_k^{(i)}\}_{k\geq 0}, (1 \leq i \leq m)$ be P-recursive sequences of order d_i, respectively. We aim to find non-trivial polynomials $X(k)$ and $A_{i_1,\ldots,i_m}(k)$ such that

$$X(k)a_k^{(1)} \cdots a_k^{(m)} = \Delta \left(\sum_{(i_1,\ldots,i_m)\in S} A_{i_1,\ldots,i_m}(k) a_{k-i_1}^{(1)} \cdots a_{k-i_m}^{(m)} \right).$$

Summing over k from 0 to $n-1$ and considering the congruences of boundary values, we will derive the congruence of

$$\sum_{k=0}^{n-1} X(k) a_k^{(1)} \cdots a_k^{(m)}.$$

Keywords
telescoping, Gosper's algorithm, congruence

References

[1] G. BAUER, Von den Coefficienten der Reihen von Kugelfunctionen einer Variablen. *J. Reine Angew. Math.* **56**, 101–121 (1859).
[2] S. CHEN; H. HUANG; M. KAUERS; Z. LI, A modified Abramov-Petkovšek reduction and creative telescoping for hypergeometric terms. In *Proceedings of the 2015 ACM on International Symposium on Symbolic and Algebraic Computation*, 117–124. ACM, Bath, 2015.
[3] Q.-H. HOU; K. LIU, Congruences and telescopings of P-recursive sequences, *J. Difference Equ. Appl.* **27**(5), 686–697 (2021).
[4] Q.-H. HOU; Y.-P. MU; D. ZEILBERGER, Polynomial reduction and supercongruences. *J. Symbolic Comput.* **103**, 127–140 (2021).
[5] Z.-W. SUN, New series for powers of π and related congruences, *Electron. Res. Arch.* **28**(3), 1273–1342 (2020).
[6] R.-H. WANG; M.X.X. ZHONG, Polynomial reduction for holonomic sequences and applications in π-series and congruences, arXiv: 2205.11129.