Reduction Based Creative Telescoping for Definite Summation of P-recursive Sequences:

the integral basis approach

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Examples.

$$\sum_{0 \le k < n} k = \frac{n(n-1)}{2}, \qquad \sum_{0 \le k < n} k^2 = \frac{n(n-1)(2n-1)}{6}$$

$$\sum_{1 \le k \le n} \frac{1}{k(k+1)} = \frac{n}{n+1}, \qquad \sum_{0 \le k \le n} \frac{\binom{2k}{k}^2}{(k+1)4^{2k}} = \frac{(n+1)\binom{2n+2}{n+1}^2}{4^{2n+1}}$$

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Additive Decomposition Problem. Given $f \in A$, compute $g, r \in A$ s.t.

$$f = \Delta_k(g) + r$$

with the following two properties:

- (minimality) r is minimal in some sense,
- (summability) f is summable in A $\quad \Leftrightarrow \quad r = 0$.

Creative telescoping

Creative Telescoping Problem. If $f \in A$ depends on n and k, find $g \in A$ and a nonzero linear recurrence operator $L(n, S_n)$ s.t.

$$L(n, S_n)(\ f\ ) = \Delta_k(\ g\ )$$

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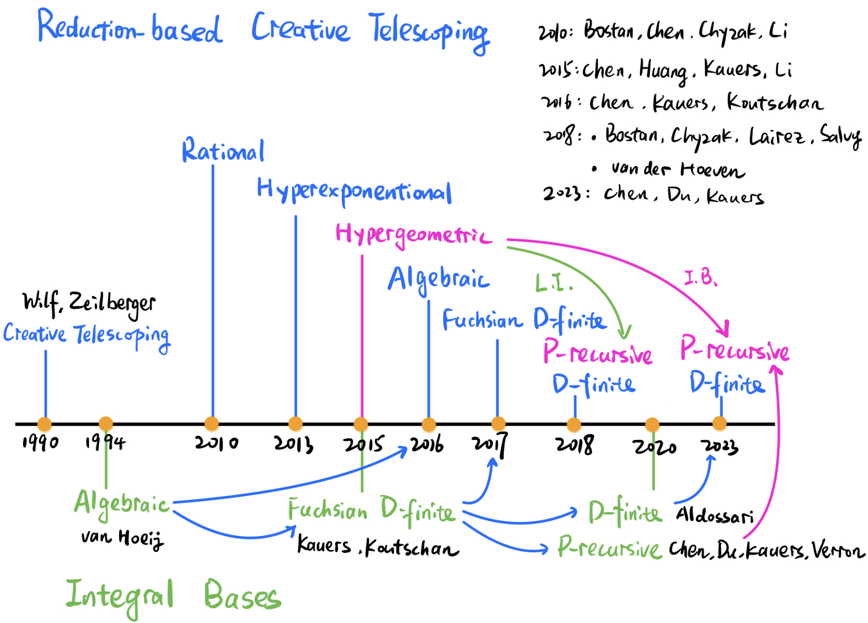
Example. Let $f(n,k) = \binom{n}{k}$. Then f has a telescoper:

$$L = S_n - 2 \quad \text{and} \quad g = \frac{k\binom{n}{k}}{k-n-1}$$

Proving Identities.

$$\sum_{0 \le k \le n} \binom{n}{k} = 2^n, \qquad \sum_{0 \le k \le n} \binom{n}{k}^2 = \binom{2n}{n}$$


Reduction-based Creative Telescoping
2010: Bostan, Chen, Chyzak, Li
2015: Chen, Huang, Kauers, Li
2016: Chen, Kauers, Koutschan
2018: • Bostan, Chyzak, Lairez, Salvy
• van der Hoeven
2023: Chen, Du, Kauers
Rational
Hyperexponential
Hypergeometric
Algebraic
Fuchsian D-finite
L.I.
I.B.
P-recursive
D-finite
P-recursive
D-finite
Wilf, Zeilberger
Creative Telescoping
1990
1994
2010
2013
2015
2016
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Integral Bases

Rational summation: Abramov's algorithm

Summability Problem. Given $f \in C(k)$, decide whether

$$f = \Delta_k(g) \quad \text{for some } g \in C(k).$$

If g exists, f is said to be summable in $C(k)$.

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Definition. For $p \in C[k]$, the dispersion of p in k is

$$\begin{aligned}\mathsf{disp}_k(p) &= \max\{i \in \mathbb{Z} \mid \gcd(p(k), p(k+i)) \neq 1\} \\ &= \max\{i \in \mathbb{Z} \mid \exists \alpha \in \bar{C} \text{ s.t. } p(\alpha) = p(\alpha+i) = 0\}\end{aligned}$$

Example. Let $p = k(k+3)(k-\sqrt{2})(k+\sqrt{2})$. Then $\mathsf{disp}_k(p) = 3$.

Definition. $p \in C[k]$ is shift-free in k if $\mathsf{disp}_k(p) = 0$.

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If g exists, f is said to be summable in $C(k)$.

Additive Decomposition. Let $f \in C(k)$. Then

$$f = \Delta_k(g) + \frac{a}{b},$$

where $g \in C(k)$ and $a, b \in C[k]$ with $\deg_k(a) < \deg_k(b)$ and b being shift-free in k. Moreover

$$f \text{ is summable in } C(k) \quad \Leftrightarrow \quad a = 0$$

Hypergeometric summation

Definition. $H(k)$ is hypergeometric over $C(k)$ if

$$\frac{H(k+1)}{H(k)} \triangleq \frac{S_k(H)}{H} \in C(k).$$

Examples.

$$1/(1+k), \quad 2^k, \quad k!, \quad \Gamma(2k+1), \ldots$$

Summability Problem. For a hypergeom. $H(k)$, decide whether

$$H = \Delta_k(G) \quad \text{for hypergeom. } G \text{ over } \mathbb{F}(k).$$

If G exists, H is said to be hypergeom. summable.

Hypergeometric summation

Let T be hypergeometric w.r.t. k with $f = S_k(T)/T \in C(k)$.

$$f = \frac{S_k(r)}{r} \cdot K \quad \leftrightsquigarrow \quad T = r \cdot H \quad \text{with } \frac{S_k(H)}{H} = K,$$

where $K = m/e$ satisfies $\gcd(m, S_k^i(e)) = 1$ for all $i \in \mathbb{Z}$.

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Modified Abramov-Petkovšek's Reduction (CHKL 2015):

$$T = \Delta_k(\cdots) + \left(\frac{p}{d} + \frac{q}{e}\right) \cdot H,$$

where $p, q, d \in C[k]$ with $\deg_k(p) < \deg_k(d)$, d shift-free, strongly prime with K and q in a f.d. vector space N_K over C.

Proposition.

$$T = \Delta_k(T') \quad \Leftrightarrow \quad p = q = 0$$

P-recursive sequences

Definition. A sequence $f(n)$ is called P-recursive over $C[n]$ if

$$p_r(n)\, f(n+r) + \cdots + p_1(n)\, f(n+1) + p_0(n)\, f(n) = 0,$$

where $p_r \ldots, p_0 \in C[n]$ (not all zero).

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Examples. The harmonic sequence $f(n) := \sum_{k=1}^{n}\frac{1}{k}$ satisfying

$$(n+2)\,f(n+2)-(2n+3)\,f(n+1)+(n+1)\,f(n)=0$$

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Setting.

- $L=p_r(n)S^r+\cdots+p_1(n)S+p_0(n)\in C[n][S]$ with $p_rp_0\neq 0$.
- $A=C(n)[S]/\langle L\rangle$, $Sn=(n+1)S$.
- $1\in A$ represents a solution y of L. Indeed, $L\cdot 1=L=0$ in A.

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$$f = S(g) - g \triangleq \Delta_n(g) \qquad \text{for some} \quad g \in A.$$

If g exists, f is said to be summable in A.

Integral bases: three cases

Algebraic case

- $A = C(x)[y]/\langle m\rangle$, where $m \in C(x)[y]$ is irreducible
- $f \in A$ is integral iff its minimal polynomial is monic. (e.g. $\sqrt{x}$)
- The integral elements of A form a free $C[x]$-module.
- Computation: van Hoeij 1994, etc.

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D-finite case

- $A = C(x)[D]/\langle L\rangle$, $Dx = xD + 1$, where $L \in C(x)[D]$.
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P-recursive case

- $A = C(n)[S]/\langle L\rangle$, $Sn = (n+1)S$, where $L \in C(n)[S]$
- The integral elements of V at z form a free $C(n)_{n-z}$-module.
- Computation: Chen-Du-Kauers-Verron 2020.

Suitable bases

Let W be a $C(x)$-vector space basis of $A = C(n)[S]/\langle L \rangle$. Then

$$SW = \frac{1}{e} MW, \quad \text{where} \quad e \in C[n],\ M \in C[n]^{r \times r}$$

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Theorem (Chen-Du-Kauers-Wang 2023+). A suitable basis always exists and can be computed via integral bases.

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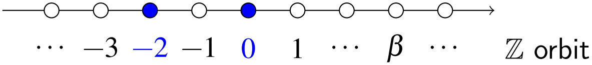
Example 1. Let $L = n^2(n+2)S - (n+1)^4 \in \mathbb{C}(n)[S]$ with one solution $y = \frac{n^2 n!}{n+1}$.



For $U = \{1\}$,

$$SU = \underbrace{\frac{1}{n^2(n+2)}}_{1/e} \underbrace{(n+1)^4}_{M} U$$

Then U is not a suitable basis since e has two roots -2, 0 in $\mathbb{Z}$.

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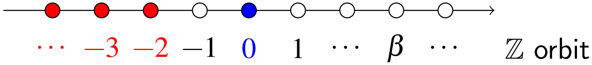

For $U = \{1\}$,

$$SU = \underbrace{\frac{1}{n^2(n+2)}}_{1/e} \underbrace{(n+1)^4}_{M} U$$

Fact. U is an integral basis at $(-\infty, -2] \cap \mathbb{Z}$.

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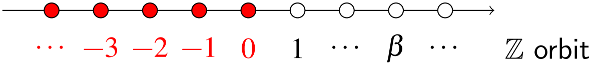

An integral basis at $\{-1,0\}$ is $W = \{(n+1)n^{-3}\}$.

$$SW = nW$$

Then W is a suitable basis since $e = 1$ is shift-free.

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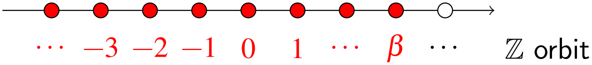

An integral basis at $\{-1, 0, \ldots, \beta\}$ is $W = \{(n+1)\prod_{i=0}^{\beta}(n-i)^{-3}\}$.

$$SW = (n-\beta)W$$

Then W is a suitable basis since $e = 1$ is shift-free.

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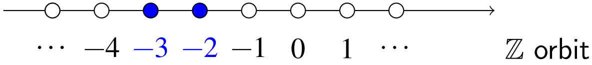
Example 2. Let $L = (n+2)(n+3)S^2 - 2(n+2)S + 1 \in \mathbb{C}(n)[S]$ with two solutions $y_1 = \frac{1}{n!}$ and $y_2 = \frac{1}{(n+1)!}$.



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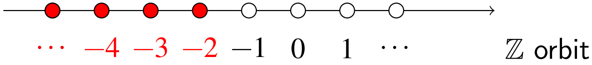

An integral basis at $\{-2\}$ is $W = \{1, (n+2)S\}$, which is suitable:

$$S \begin{pmatrix} 1 \\ (n+2)S \end{pmatrix} = \underbrace{\frac{1}{(n+2)}}_{1/e} \underbrace{\begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}}_{M} \begin{pmatrix} 1 \\ (n+2)S \end{pmatrix}$$

Reduce the dispersion

Let W be a $C(x)$-vector space basis:

$$SW = \frac{1}{e} MW$$

Theorem (Chen-Du-Kauers-Wang 2023+). There exists a suitable basis W of $A = C(n)[S]/\langle L \rangle$ such that for any $f \in A$,

$$f = \Delta_n(g) + \frac{1}{d} PW + \frac{1}{e} RW,$$

where $g \in A$, $d \in C[n]$ and $P, R \in C[n]^r$ satisfying

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where $g \in A$, $d \in C[n]$ and $P, R \in C[n]^r$ satisfying

- de is shift-free and $\deg_n(P) < \deg_n(d)$;
- $f = \Delta_n(h) \quad \Rightarrow \quad P = 0$ and $h = bW$ with $b \in C[n]^r$.

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Example. Let $L = (n+2)(n+3)S^2 - 2(n+2)S + 1 \in \mathbb{C}(n)[S]$ and $W = \{1, (n+2)S\}$. For $f = \frac{1}{(n+1)(n+2)} + \frac{n}{n+1} S$,

$$f = \Delta_n\Big(\underbrace{\frac{(-1,1)}{n+1} W}_{g}\Big) + \underbrace{\frac{1}{(n+2)^2}}_{1/d} \underbrace{(1,-1)}_{P} W + \underbrace{\frac{1}{(n+2)}}_{1/e} \underbrace{(-1,2)}_{R} W.$$

Then f is not summable because $P \neq 0$.

Additive decomposition

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Theorem (Chen-Du-Kauers-Wang 2023+). There exist a suitable basis W and an integral basis at infinity V such that for any $f \in A$,

$$f = \Delta_n(g) + \frac{1}{d}PW + \frac{1}{a}QV,$$

where $g \in A$, $d \in C[n]$, $P \in C[n]^r$, $Q \in C[n, n^{-1}]^r$ satisfying

- de is shift-free and $\deg_n(P) < \deg_n(d)$;
- $Q \in N_V$, a finite-dimensional C-vector space;
- f is summable in $A \quad \Leftrightarrow \quad P = Q = 0$.

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Creative telescoping. For $f \in A = C(k,n)[S_k, S_n]/I$, find a nonzero $T \in C(k)[S_k]$ and $g \in A$ such that $T(k, S_k)(f) = \Delta_n(g)$.

Summary

Main results.

- reduction for univariate P-recursive sequences
- telescoping algorithm for bivariate P-recursive sequences

New tools.

- find a suitable basis to reduce the dispersion
- find an integral basis at infinity to reduce the degree

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