Two Applications of the Telescoping Method

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The telescoping method

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Example

$$k = \binom{k+1}{2} - \binom{k}{2} \implies \sum_{k=1}^{n} k = \binom{n+1}{2} - \binom{1}{2} = \frac{n(n+1)}{2}.$$

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Example

$$\binom{k}{m} = \binom{k+1}{m+1} - \binom{k}{m+1} \implies \sum_{k=1}^{n} p(k) = q(n).$$

A series involving π

- Sun[1] showed that

$$\sum_{k=0}^{\infty} \frac{k(4k-1)}{(2k-1)^2} \frac{\binom{2k}{k}^3}{(-64)^k} = -\frac{1}{\pi}.$$

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- The key observation is

$$\left((4k+1) + \frac{2k(4k-1)}{(2k-1)^2} \right) \frac{\binom{2k}{k}^3}{(-64)^k} = \Delta \left(-8 \frac{k^3}{(2k-1)^2} \frac{\binom{2k}{k}^3}{(-64)^k} \right).$$

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- Given t_k, for what kind of rational functions $r(k)$, $r(k)t_k$ is summable?

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Reduction

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$$\alpha_k = \beta_k + \Delta\gamma_k.$$

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- Chen et al. (2015) introduced the word polynomial reduction and gave the reduction-based creative telescoping.
- Hou, Mu and Zeilberger reformulated the polynomial reduction and applied it to proving congruences.

Polynomial reduction (hypergeometric cases)

Suppose that t_k is hypergeometric. We have Gosper's representation

$$\frac{t_{k+1}}{t_k} = \frac{a(k)}{b(k)} \frac{c(k+1)}{c(k)}, \qquad \gcd(a(k), b(k+h)) = 1, \quad \forall\, h \in \mathbb{N}.$$

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For any hypergeometric term t_k, there exists a non-zero polynomial $p(k)$ such that $p(k)t_k$ is summable.

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Theorem

For any hypergeometric term t_k, there exists a non-zero polynomial $p(k)$ such that $p(k)t_k$ is summable. Moreover, the degree of $p(k)$ is not greater than

$$B = \begin{cases} d+1, & \textit{if } (a(k), b(k)) \textit{ is degenerated or } \deg u(k) < \deg a(k) - 1, \\ d, & \textit{otherwise.} \end{cases}$$

where

$$u(k) = a(k) - b(k-1), \quad d = \max\{\deg u(k), \deg a(k) - 1\}.$$

Polynomial reduction (hypergeometric cases)

By Gosper's algorithm, we show that the degree bound is also tight.

Theorem

Suppose that t_k is hypergeometric and the Gosper's representation of t_{k+1}/t_k satisfies

$$\deg u(k) = \max\{\deg a(k), \deg b(k)\} \quad \text{and} \quad c(k) = 1.$$

If $p(k)t_k$ is Gosper summable, then

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Example

Let $t_k = \binom{2k}{k}^3/(-64)^k$. We have

$$a(k) = -(2k+1)^3, \quad b(k) = 8(k+1)^3, \quad c(k) = 1,$$

and

$$(4k+1)(4k^2+2k+1)t_k = \Delta(-8k^3 t_k).$$

Polynomial reduction (holonomic cases)

Wang and Zhong extended the polynomial reduction to holonomic cases.

Theorem

Suppose that

$$p_0(n)f_n + p_1(n)f_{n+1} + \cdots + p_d(n)f_{n+d} = 0.$$

Then for any polynomial $g(n)$, $L^*g(n) \cdot f_n$ *is summable, where*

$$L^*g(n) = \sum_{i=0}^{d} p_i(n-i)g(n-i).$$

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Example

Given $P(n) \in \mathbb{Q}[n]$, *one can find* $Q(n) \in \mathbb{Q}[n]$ *with* $\deg Q(n) \leq 2$ *and* $c \in \mathbb{Q}$

$$\sum_{n=0}^{\infty} P(n)Q(n)\frac{Domb(n)}{(-32)^n} = \frac{c}{\pi}, \tag{1}$$

where $Domb(n)$ *is the n-th Domb number.*

Rational reduction

Now we consider the case when $r(k)$ is a rational function. By divisibility, we showed that

Theorem

Let t_k be hypergeometric and $a(k), b(k), c(k)$ be the Gosper representation of t_{k+1}/t_k. Suppose that for any $h \in \mathbb{N}$,

$$\gcd(v(k), a(k-1-h)) = \gcd(v(k), b(k+h)) = \gcd(v(k), c(k)) = 1,$$

and

$$\gcd(v(k), v(k+1+h)) = 1.$$

If $u(k)t_k/v(k)$ is Gosper summable, then

$$v(k) \mid u(k).$$

An application

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$$u_k = \frac{1}{v(k)} \frac{t_k}{c(k)}.$$

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- By applying the extended Zeilberger's algorithm to

$$\texttt{Ext\_Zeil}\ ([t,\ u,\ k \cdot u,\ k^2 \cdot u,\ \ldots],\ k)$$

we get a polynomial $p(k)$ such that $p(k)u_k - t_k$ is summable.

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we get a polynomial $p(k)$ such that $p(k)u_k - t_k$ is summable.
- Hence
$$\sum_{k=0}^{\infty} p(k)u_k = C.$$

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Examples

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$$\sum_{k=1}^{\infty} \frac{21k-8}{k^3\binom{2k}{k}^3} = \frac{\pi^2}{6},$$

we derived

$$\sum_{k=1}^{\infty} \frac{28k^2+31k+8}{(2k+1)^2} \frac{1}{k^3\binom{2k}{k}^3} = \frac{\pi^2}{2} - 4,$$

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and

$$\sum_{k=2}^{\infty} \frac{(3k+1)(21k+1)}{k(2k+1)(k-1)} \frac{1}{\binom{2k}{k}^3} = \frac{163-16\pi^2}{36}.$$

Combinatorial congruences

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$$\sum_{k=0}^{n-1} a_k.$$
- For example,
$$\sum_{k=0}^{n-1}(-1)^k(2k+1) \equiv 0 \pmod{n}$$
$$\sum_{k=0}^{n-1}(2k+1) \equiv 0 \pmod{n^2}.$$

Methods of proving congruences

- Combinatorial identities

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- Symbolic computation

  Y.-P. Mu and Z.-W. Sun, Telescoping method and congruences for double sums, *Int. J. Number Theory* **14 (2018)**.

An example

- The k-th Delannoy is given by

$$D_k = \sum_{i=0}^{k} \binom{k}{i} \binom{k+i}{i}.$$

It satisfies

$$(2+n)D_{n+2} + (-9-6n)D_{n+1} + (1+n)D_n = 0.$$

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- Sun showed that

$$\sum_{k=0}^{n-1}(4k+2)D_k \equiv 0 \pmod{n}.$$

- Wang noticed that

$$\sum_{k=0}^{n-1}(4k+2)D_k = n(D_n - D_{n-1}), \quad \forall\, n \geq 1.$$

The general case

- Given a P-recursive sequence $\{a_n\}$ and polynomials $f(n)$ with $f(0) = 0$, we try to construct polynomials $X(n)$ and $A_i(k)$ with integral coefficients such that

$$X(k)a_k = \Delta \left(f(k) \sum_{i=0}^{d-1} A_i(k) a_{k-i} \right).$$

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- Summing over k from 0 to $n-1$, we derive that

$$\sum_{k=0}^{n-1} X(k)a_k = f(n) \sum_{i=0}^{d-1} A_i(n) a_{n-i} \equiv 0 \pmod{f(n)}.$$

The existence

Theorem

Suppose that $\{a_n\}_{n\geq 0}$ *is a P-recursive sequence of order d. For any polynomial* $f(n)$*, there exists polynomials* $X(n)$ *and* $A_{i_1,\ldots,i_m}(n)$*, not all zeros, such that*

$$X(n)a_n^m = \Delta\left(f(n)\sum_{0\leq i_1\leq\cdots\leq i_m\leq d-1} A_{i_1,\ldots,i_m}(n)a_{n-i_1}a_{n-i_2}\cdots a_{n-i_m}\right).$$

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Suppose that $\{a_n\}_{n\geq 0}$ is a P-recursive sequence of order d. For any polynomial $f(n)$, there exists polynomials $X(n)$ and $A_{i_1,\ldots,i_m}(n)$, not all zeros, such that

$$X(n)a_n^m = \Delta\left(f(n)\sum_{0\leq i_1\leq\cdots\leq i_m\leq d-1} A_{i_1,\ldots,i_m}(n)a_{n-i_1}a_{n-i_2}\cdots a_{n-i_m}\right).$$

Proof. Expanding the right hand side and substituting a_{n+1} with $a_n, a_{n-1}, \ldots, a_{n-d+1}$, we get a system of difference equations

$$c_{i_1,\ldots,i_m}(n)A_{i_1,\ldots,i_m}(n+1) + d_{i_1,\ldots,i_m}(n)A_{i_1,\ldots,i_m}(n) + e_{i_1,\ldots,i_m}(n)X(n) = 0.$$

Comparing the number of unknowns and equations leads to the result. ∎

Software implement

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$$X(n) = \sum_{i=0}^{t} c_i n^i.$$

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- With this method, we gave simple proofs of several congruences and found some new ones.

Examples

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- In fact, we have

$$3\sum_{k=0}^{n-1}(3k+2)(-1)^k f_k = n^2(a_n + 8a_{n-1}).$$

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$$\sum_{k=0}^{n-1}(4k+2)D_k = n(D_n - D_{n-1}).$$

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Hence

$$\sum_{k=0}^{n-1}(16k+8)D_k^2 \equiv 0 \pmod{n^2}.$$

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- Sun defined

$$T_n(b, c) = [x^n](x^2 + bx + c)^n = \sum \binom{n}{k}\binom{n-k}{k} b^{n-2k} c^k.$$

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- We found that

$$\sum_{k=0}^{n-1}(A_2k^2+A_1k+A_0)\,T_k(b,c)^2 = n^2\,(m+1)^2\,T_n(b,c)^2 + 2bmn\,(-2\,(m+1)\,n+(m-1))\,T_n(b,c)\,T_{n-1}(b,c) + n^2m^2\,(m+1)^2\,T_{n-1}(b,c)^2,$$

where $m = b^2 - 4c$,

$$A_2 = \left(-4\,b^2m^2 + m^4 + 2\,m^3 + 4\,b^2 - 2\,m - 1\right),$$
$$A_1 = \left(2\,m^4 + 4\,m^3 + \left(-4\,b^2+2\right)m^2 - 8\,b^2m + 4\,b^2\right)$$
$$A_0 = m^4 + 2\,m^3 + \left(-b^2+1\right)m^2 - 4\,b^2m + b^2.$$

Examples

- Suppose that $b^2 - 4c = 1$. Let

$$G_k(b,c) = \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} \frac{b-1}{2(2k-1)}.$$

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$$\sum_{k=0}^{n-1}(2b^2-b-1)T_k(b,c)G_k(b,c) \equiv 0 \pmod{n}.$$

Examples

We found that

$$\sum_{k=0}^{n-1}(7k^2 - 8k - 6)\binom{2k}{k} D_k = n\binom{2n}{n}\left(nD_n - (4n+2)D_{n-1}\right).$$

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$$n\binom{2n}{n}\,\Big|\,\sum_{k=0}^{n-1}(7k^2 - 8k - 6)\binom{2k}{k}D_k.$$

Thanks for your attention!