POSITIVE SEQUENCES IN THE OEIS

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- Neil Sloane collects integer sequences since 1964.
- Since 1996 the database was first published online as the On-Line Encyclopedia of Integer Sequences (OEIS).
- At the moment it contains around 360 000 integer sequences.
- How many of them satisfy a linear recurrence?

Figure: Neil Sloane at the e-party to celebrate the 100,000-th sequence in 2004

Definition: C-finite

A sequence $c(n) \in \mathbb{Q}^{\mathbb{N}}$ is called C-finite if there are constants $\gamma_0, \ldots, \gamma_r \in \mathbb{Q}$, not all zero, with

$$\gamma_0\, c(n) + \cdots + \gamma_r\, c(n+r) = 0 \quad \text{for all } n \in \mathbb{N}.$$

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- Examples: Fibonacci sequence, Perrin numbers, geometric sequences, etc.
- The minimal r is called the order of c.

Definition: P-recursive

A sequence $c(n) \in \mathbb{Q}^{\mathbb{N}}$ is called *P-recursive* if there are polynomials $p_0, \ldots, p_r \in \mathbb{Q}[x]$, not all zero, such that

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- Examples: C-finite sequences, Harmonic numbers, factorials, etc.
- We call r the order of the recurrence.
- We call $\max_i \deg p_i$ the degree of the recurrence.
- A P-recursive sequence satisfies many recurrences (order/degree curve)...

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This linear equation has the nontrivial solution

$$\gamma_0 = 1, \gamma_1 = 1, \gamma_2 = -1$$

which gives rise to the recurrence

$$c(n) + c(n+1) - c(n+2) = 0.$$

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- If c is P-recursive of order r and degree d, we get a linear system

$$\sum_{i=0}^{r} \sum_{j=0}^{d} p_{i,j} n^j c(n+i) = 0 \quad \text{for } n = 0, \ldots, D-r$$

of size

$$D - r + 1 \times (r+1)(d+1)$$

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- Idea: use guessing to determine how many sequences in the OEIS are C-finite or P-recursive.

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- For about 6% of the sequences at most 10 terms are given, about 50% have at most 100 terms, about 13% have at least 10 000 terms given.

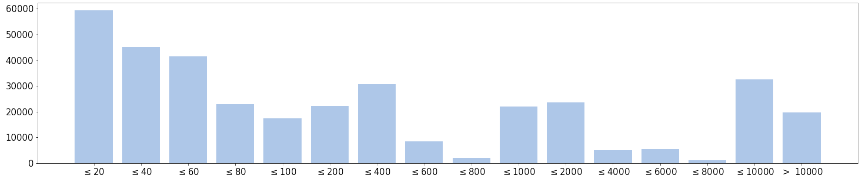


Figure: Number of sequences with number of terms specified in the OEIS

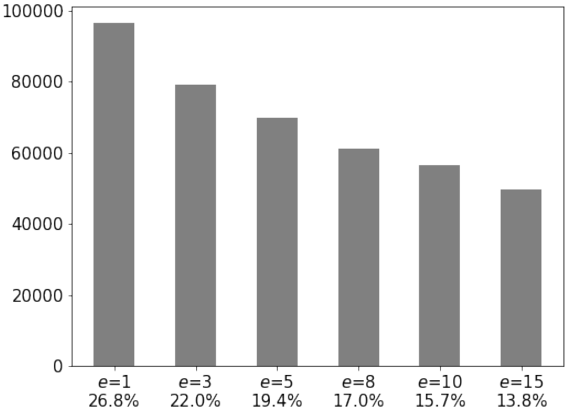
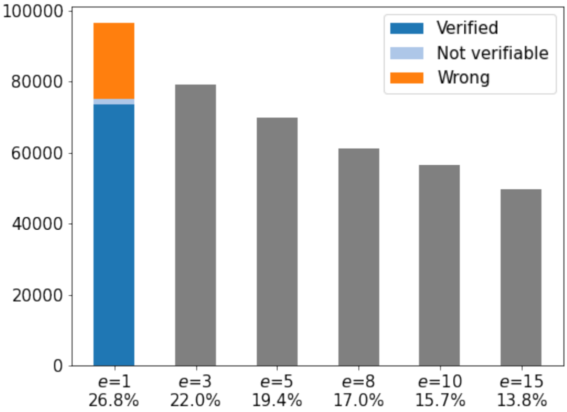


Figure: Number of C-finite sequences with order ≤ 100



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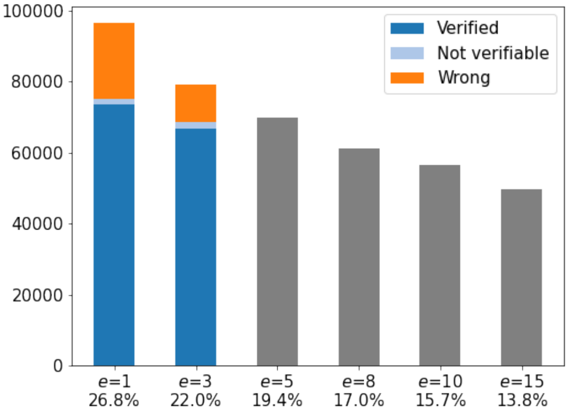

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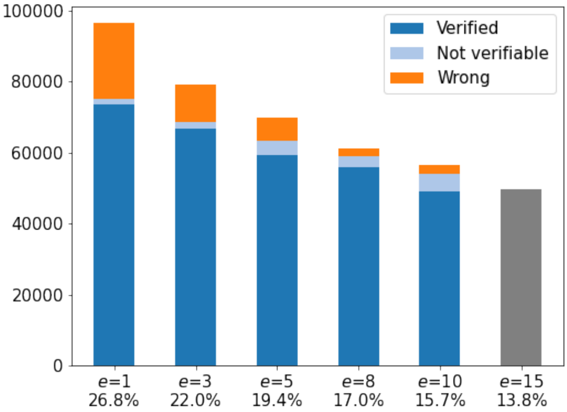

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C-finite sequences orders

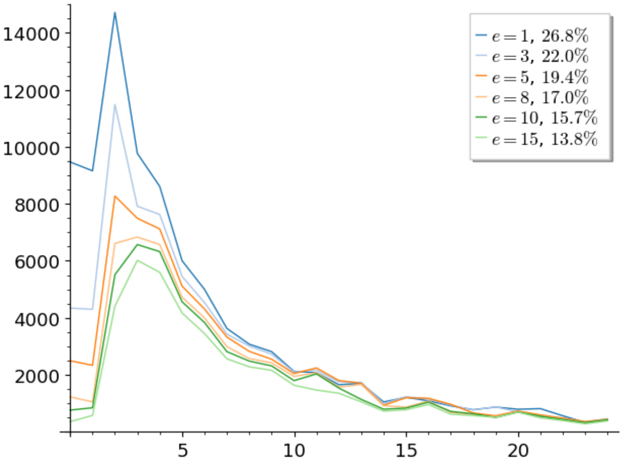

Figure: Number of C-finite sequences of given order

Let $a(n)$ be the sequence A000122 with initial values

$$1, 2, 0, 0, 2, 0, 0, 0, 0, 2, 0, 0, 0, 0, 0, 0, 2, 0, 0, 0, \ldots$$

Guessing yields the recurrence (if $e = 1$)

$$n(n-1)\,a(n) = 0.$$

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Guessing yields the recurrence (if $e = 5$)

$$n(n-1)(n-4)\, a(n) = 0.$$

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Guessing yields the recurrence (if $e = 50$)

$$n(n-1)(n-4)(n-9)(n-16)(n-25)(n-36)(n-49)\,a(n) = 0.$$

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Idea: If sequence has many zeros, make sure that the system is even more overdetermined (Kauers and Verron 2019).

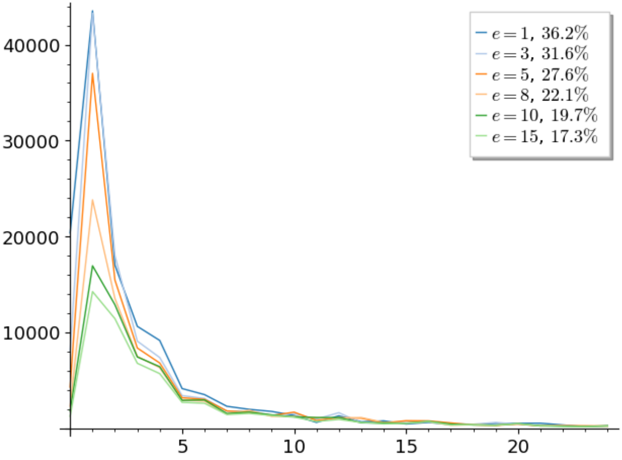


Figure: Number of P-recursive sequences of given order

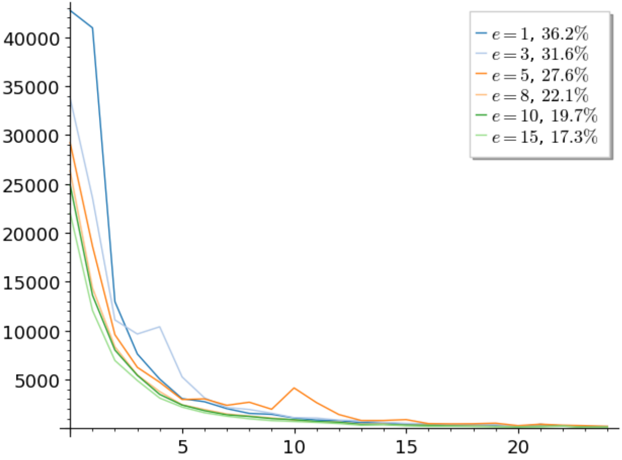
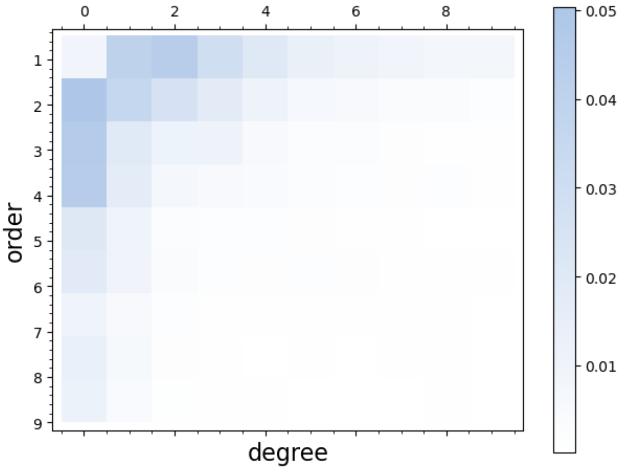


Figure: Number of P-recursive sequences of given degree



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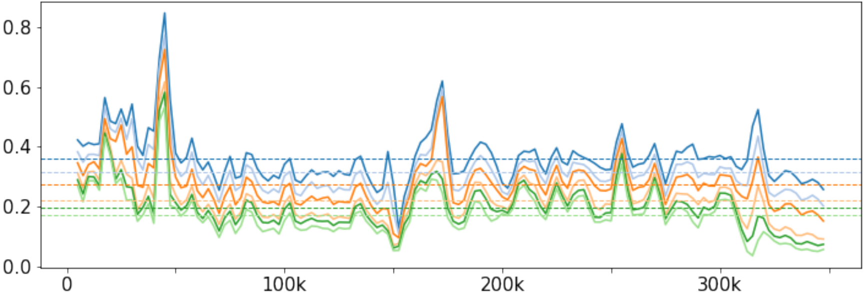

Figure: Ratio of P-recursive sequences in the OEIS

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- Only known to be decidable for sequences of small order:
  - Order 2 (Halava, Harju, and Hirvensalo 2006),
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- Can we prove positivity of sequences in practice?

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In fact, $c(n)$ is C-finite of order 8 satisfying

$$\begin{aligned} c(n) - c(n+1) - c(n+2) + c(n+3) - c(n+5) \\ + c(n+6) + c(n+7) - c(n+8) = 0 \end{aligned}$$

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Hence, c is the interlacing of positive sequences, so $c(n) > 0$ for all $n \in \mathbb{N}$.

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- The implementations can prove positivity of all 1000 sequences (N. and Pillwein 2022).
  - Decompose sequence into subsequences which have simple asymptotic behavior.

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What have we done?

- We estimated ratio of C-finite (around 15%) and P-recursive (around 20%) sequences in the OEIS.
- The results are available online. Hence, these sequences can be used for testing algorithms.
- We showed that proving positivity of C-finite sequences can usually be done in practice.

What is left?

- Can we prove positivity of P-recursive sequences in practice?

References I

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Kauers, Manuel and Thibaut Verron (2019). "Why You Should Remove Zeros from Data before Guessing". In: ACM Commun. Comput. Algebra 53.3, 126fffdfffdfffd129.

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N., Philipp and Veronika Pillwein (2022). "A Comparison of Algorithms for Proving Positivity of Linearly Recurrent Sequences". In: Computer Algebra in Scientific Computing. Vol. 13366. LNCS. Springer International Publishing, pp. 268–287.

References II

Ouaknine, Joël and James Worrell (2014). "Positivity problems for low-order linear recurrence sequences". In: SODA '14: Proceedings of the twenty-fifth annual ACM-SIAM symposium on Discrete algorithms.

Comparison

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- Some that are clearly C-finite are not in there (e.g., A272636).
- Some look very C-finite but the recurrence is only conjectured and therefore the sequence not on the Wiki page (e.g., A281605).
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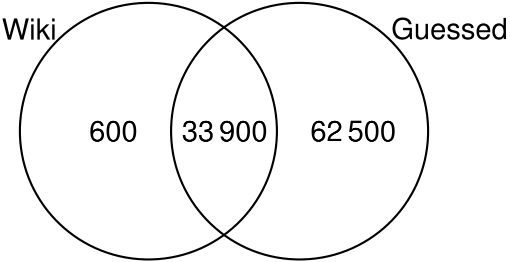

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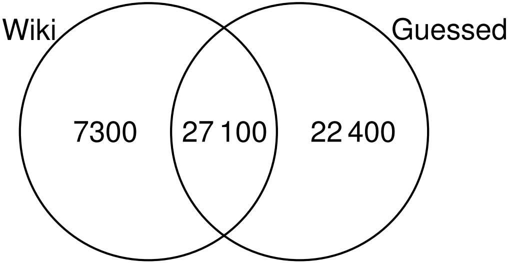
- Some look very C-finite but the recurrence is only conjectured and therefore the sequence not on the Wiki page (e.g., A281605).
- It contains about 34 000 sequences. Compare with guessed sequences with ensure=10.



Example

The Berstel sequence (A007420) is C-finite of order 3 satisfying

$$4c(n) - 4c(n+1) + 2c(n+2) - c(n+3) = 0$$

with $c(0) = c(1) = 0, c(2) = 1$.

Define $d(n) = c(n+53)^2$. Then, d looks positive. However, it does not have a unique dominant root and cannot be decomposed to obtain one.