Algebraic consequences of the fundamental theorem of calculus in differential rings

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joint work with Clemens G. Raab

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Fundamental theorem of calculus

Algebraic consequences of the Leibniz rule and

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x) \quad \text{and} \quad \int_a^x f'(t)\,dt = f(x) - f(a)$$

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$$\partial fg = (\partial f)g + f\partial g \qquad\qquad \partial(fg) = \partial(f)g + f\partial(g)$$

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$$\int f \qquad \int(f)$$

Integration and evaluation in differential rings

$(\mathcal{R}, \partial)$ **differential ring**, $\partial\colon \mathcal{R} \to \mathcal{R}$ is **linear** of its **constants**

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Definition

Let $(\mathcal{R}, \partial)$ be a differential ring with constants $\mathcal{C}$. We call a $\mathcal{C}$-linear map $\int : \mathcal{R} \to \mathcal{R}$ an **integration** on $\mathcal{R}$, if

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A $\mathcal{C}$-linear functional

$$e : \mathcal{R} \to \mathcal{C}$$

acting on $\mathcal{C}$ as the identity is called an **evaluation** on $\mathcal{R}$.

Integro-differential rings

Definition

Let $(\mathcal{R}, \partial)$ be a differential ring and let $\int : \mathcal{R} \longrightarrow \mathcal{R}$ be an integration on $\mathcal{R}$. We call $(\mathcal{R}, \partial, \int)$ a **(generalized) integro-differential ring** and we define the **(induced) evaluation** E on $\mathcal{R}$ by

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Lemma

Let $(\mathcal{R}, \partial, \int)$ be an integro-differential ring with constants $\mathcal{C}$. Then,

$$\mathrm{E}f \in \mathcal{C}, \quad \mathrm{E}\int f = 0, \quad \textit{and} \quad \mathrm{E}c = c.$$

for all $f \in \mathcal{R}$ and $c \in \mathcal{C}$.

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for all $f \in \mathcal{R}$ and $c \in \mathcal{C}$. Moreover,

$$\mathcal{R} = \mathcal{C} \oplus \int \mathcal{R}$$

as direct sum of $\mathcal{C}$-modules.

(Multiplicative) integro-differential rings and algebras

multiplicative evaluation

$$\mathrm{E}fg = (\mathrm{E}f)\mathrm{E}g$$

(Rosenkranz ’03 ’05, Rosenkranz-R ’08, Guo-R-Rosenkranz ’14, Hossein Poor-Raab-R ’18)

$\int_a^x f(t)\,dt$ and evaluation $\mathrm{E}f = f(a)$
of **continuous** functions

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Ex: (Matrices of) polynomials, smooth/analytic functions, formal power series

Motivation and application:

algebraic setting for **boundary value problems** for linear ODEs

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Differential Rota-Baxter algebras

$$(\textstyle\int f)\int g = \int f \int g + \int (\int f) g$$

(Guo-Keigher ’08)

Laurent polynomials and series

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$$\int x^k \ln(x)^n = \begin{cases} \frac{x^{k+1}}{k+1} & k \neq -1 \land n = 0 \\ \frac{x^{k+1}}{k+1} \ln(x)^n - \frac{n}{k+1} \int x^k \ln(x)^{n-1} & k \neq -1 \land n > 0 \\ \frac{\ln(x)^{n+1}}{n+1} & k = -1 \end{cases}$$

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Laurent series:

$$K((x))[\ln(x)]$$

contain rational functions $K(x)$ and hyperlogarithms (Kummer 1840)

D-finite functions

D-finite functions are closed under antiderivatives

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Define integration in terms of an evaluation and antiderivates

Lemma

Let $(\mathcal{R}, \partial)$ *be a differential ring such that* $\partial\mathcal{R} = \mathcal{R}$ *and* e *be an evaluation on* $\mathcal{R}$*.*
Define $\int_e \colon \mathcal{R} \to \mathcal{R}$ *by*

$$\int_e f = g - eg$$

where $g \in \mathcal{R}$ *is such that* $\partial g = f$*.*
Then $(\mathcal{R}, \partial, \int_e)$ *is an integro-differential ring with induced evaluation* $\mathrm{E} = e$*.*

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Define an evaluation in terms of formal series solutions

(van Hoeij '97)

Integro-differential operators

Linear operators

- differential operator ∂
- integral $\int$
- evaluation $\mathrm{E} = \mathrm{id} - \int \partial$
- multiplication operators: $f \in \mathcal{R}$ acting as $g \mapsto fg$

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Linear operators with composition form a ring

$\mathcal{C}$ commutative: a $\mathcal{C}$-algebra $\mathcal{C}$ noncommutative: a $\mathcal{C}$-ring

Generators and relations

Definition

Let $(\mathcal{R}, \partial, \int)$ be an integro-differential ring with constants C.
We define the **ring of integro-differential operators** (IDO)

$$\mathcal{R}\langle \partial, \textstyle\int, \mathrm{E} \rangle$$

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A known consequence

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integration by parts holds in $\mathcal{R}$

$$\int f \partial g = fg - \mathrm{E} fg - \int (\partial f) g$$

A new consequence

From

$$\int \cdot f \cdot \partial \cdot \int$$

integration by parts and $\partial \cdot \int$,

we obtain

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Rota-Baxter identity with evaluation in $\mathcal{R}$

$$(\int f) \int g = \int f \int g + \int (\int f) g + \mathrm{E}(\int f) \int g$$

products of integrals = nested integrals **plus evaluation**

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E is multiplicative: Rota-Baxter identity (shuffle identities)

$$(\int f) \int g = \int f \int g + \int (\int f) g$$

All consequences

$\partial \cdot f = f \cdot \partial + \partial f$	$\int \cdot f \cdot \partial = f - \mathrm{E} \cdot f - \int \cdot \partial f$
$\partial \cdot \mathrm{E} = 0$	$\int \cdot f \cdot \mathrm{E} = \int f \cdot \mathrm{E}$
$\partial \cdot \int = 1$	$\int \cdot f \cdot \int = \int f \cdot \int - \int \cdot \int f - \mathrm{E} \cdot \int f \cdot \int$
$\mathrm{E} \cdot f \cdot \mathrm{E} = \mathrm{E} f \cdot \mathrm{E}$	$\int \cdot \partial = 1 - \mathrm{E}$
$\mathrm{E} \cdot \mathrm{E} = \mathrm{E}$	$\int \cdot \mathrm{E} = \int 1 \cdot \mathrm{E}$
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Table: Rewrite rules for operator expressions

All consequences

$\partial \cdot f = f \cdot \partial + \partial f$	$\int \cdot f \cdot \partial = f - \mathrm{E} \cdot f - \int \cdot \partial f$
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$\partial \cdot \int = 1$	$\int \cdot f \cdot \int = \int f \cdot \int - \int \cdot \int f - \mathrm{E} \cdot \int f \cdot \int$
$\mathrm{E} \cdot f \cdot \mathrm{E} = \mathrm{E}f \cdot \mathrm{E}$	$\int \cdot \partial = 1 - \mathrm{E}$
$\mathrm{E} \cdot \mathrm{E} = \mathrm{E}$	$\int \cdot \mathrm{E} = \int 1 \cdot \mathrm{E}$
$\mathrm{E} \cdot \int = 0$	$\int \cdot \int = \int 1 \cdot \int - \int \cdot \int 1 - \mathrm{E} \cdot \int 1 \cdot \int$

Table: Rewrite rules for operator expressions

Theorem

Let $(\mathcal{R}, \partial, \int)$ be an integro-differential ring. Then, by repeatedly applying the rewrite rules above in any order, every element of the ring $\mathcal{R}\langle \partial, \int, \mathrm{E} \rangle$ can be written as a sum of expressions of the form

$$f \cdot \partial^j, \quad f \cdot \int \cdot g, \quad f \cdot \mathrm{E} \cdot g \cdot \partial^j, \quad \text{or} \quad f \cdot \mathrm{E} \cdot h \cdot \int \cdot g$$

where $j \in \mathbb{N}_0$, $f, g \in \mathcal{R}$, and $h \in \int \mathcal{R}$.

Rings of linear operators via tensor reduction systems

Constructive and algorithmic approach to rings of linear operators
via **tensor reduction systems** for tensor algebras and rings

(Bergman ’78, Hossein Poor-Raab-R ’16, Hossein Poor-Raab-R ’18, Raab-R ’23)

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- families of relations can be represented as one homomorphism

$$\partial \otimes f \mapsto f \otimes \partial + \partial f$$

- tensor over the ring of constants to deal with linearity
- construction via quotients of tensor rings

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Mathematica package `TenRes`

Proving variation of constants

$$x'(t) + A(t)x(t) = f(t) \qquad x_0(t) = \Phi(t) \int_{t_0}^{t} \Phi^{-1}(s) f(s) ds$$

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$(\mathcal{R}, \partial, \int)$ and

$$L = \partial + a$$

“fundamental matrix” $z \in \mathcal{R}$, $\qquad \partial z + az = 0,$

Proving variation of constants

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Rewrite rules and normal forms: equational prover in calculus,
discover identities by ansatz, basics of linear ODEs with initial conditions

Generalized Taylor formula

Integro-differential subring generated by 1 and constants contains "monomials"

$$x_i = \int^i 1$$

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Let $(\mathcal{R}, \partial, \int)$ be an integro-differential ring such that E is multiplicative on the integro-differential subring generated by 1. Then, for all $n \in \mathbb{N}$, we have

$$1 = \sum_{k=0}^{n} x_k \cdot \mathrm{E} \cdot \partial^k + \sum_{k=0}^{n} (-1)^{n-k} x_k \cdot \int \cdot x_{n-k} \cdot \partial^{n+1} - \sum_{k=0}^{n-1} \sum_{j=1}^{n-k} (-1)^{n-k-j} x_k \cdot \mathrm{E} \cdot x_j \cdot \int \cdot x_{n-k-j} \cdot \partial^{n+1}$$

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$\mathbb{Q} \subseteq \mathcal{R}$:

$$f = \sum_{k=0}^{n} \frac{x_1^k}{k!} \mathrm{E} \partial^k f + \sum_{k=0}^{n} \frac{(-1)^{n-k}}{k!(n-k)!} x_1^k \int x_1^{n-k} \partial^{n+1} - \sum_{k=0}^{n-1} \sum_{j=1}^{n-k} \frac{(-1)^{n-k-j}}{k!j!(n-k-j)!} x_1^k \mathrm{E} x_1^j \int x_1^{n-k-j} \partial^{n+1} f$$

Generalized shuffle relations

$\mathcal{R}$ commutative integro-differential ring

$\mathcal{C}\langle\mathcal{R}\rangle = \bigoplus_{n=0}^{\infty} \mathcal{R}^{\otimes n}$ with **shuffle product** $⧢$, homomorphism

$$\varphi(a_1 \otimes \ldots \otimes a_n) = \int a_1 \int a_2 \ldots \int a_n \in \mathcal{R} \quad \text{and} \quad a_i^j = a_i \otimes a_{i+1} \otimes \ldots \otimes a_j$$

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$$\varphi(f)\varphi(g) = \varphi(f \sqcup\!\sqcup g) + \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} e(f_{i+1}^m, g_{j+1}^n)\varphi(f_1^i \sqcup\!\sqcup g_1^j) \in \mathcal{R}$$

with constants $e(f_{i+1}^m, g_{j+1}^n) = \mathrm{E}\varphi(f_{i+1}^m)\varphi(g_{j+1}^n) \in C$.

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(Ree '58)

Outlook

- Integro-differential rings over integral domains
- Tensor reduction systems

- Other operator rings (with linear substitutions, discrete analogs, . . .)
- Free integro-differential rings (integro-differential polynomials)

Clemens G. Raab, R., The fundamental theorem of calculus in differential rings. arXiv:2301.13134 [math.RA] (2023)

Clemens G. Raab, R., The free commutative generalized integro-differential ring. (2023) In preparation.