Examples for redct

demonstrates an implementation of the algorithm based on generalized Hermite reduction

```
> restart;
> libname:=".",libname: # Mgfun is needed
> read "redct.mpl";
```

The syntax is
redct(Int(...,var=...),[list of variables with their natures])
the output is list of telescopers for the integrand wrt these variables.
This list forms a Gröbner basis of the telescoping ideal for tdeg.

Example from our article:

```
> F:=exp(-p*x)*ChebyshevT(n,x)/sqrt(1-x^2);
```

$$F := \frac{e^{-p\,x}\,\mathrm{ChebyshevT}(n,x)}{\sqrt{-x^2+1}} \tag{1}$$

```
> redct(Int(F,x=-1..1),[n::shift,p::diff]);
```

$$\left[p\,\mathrm{D}_n + \mathrm{D}_p\,p - n,\ p\,\mathrm{D}_n^2 - 2\,n\,\mathrm{D}_n - p - 2\,\mathrm{D}_n\right] \tag{2}$$

Several of the operators computed in this session are big. Configure the interface so that it does not display too much:

```
> interface(elisiontermsafter=2,elisiontermsbefore=2,
  elisiontermsthreshold=5,termelisionthreshold=10,
  elisiondigitsafter=3,elisiondigitsbefore=3,
  elisiondigitsthreshold=14):
```

For bigger examples, we use the following piece of code to display input, output and time:

```
> bigredct:=proc(f,vars) local res, tt;
     print(f);
     tt:=time(assign('res'=redct(args)));
     if nops(res)=1 then res:=op(res) fi;
     print(res);
     printf("time = %g sec.\n",tt);NULL
  end:
```

Examples from C. Koutschan's Examples11.nb

Easy ones

```
> st:=time():
> redct(Int((LegendreP(2*k+1,x)/x)^2,x=-1..1),[k::shift]);
```

$$\left[\mathrm{D}_k - 1\right] \tag{1.1.1}$$

```
> redct(Int((1-m*u^2)^(j-1/2)/sqrt(1-u^2),u=0..1),[j::shift]);
```

$$\left[2\,j\,m\,\mathrm{D}_j + 2\,\mathrm{D}_j^2\,j - 2\,m\,j - 4\,j\,\mathrm{D}_j + 2\,m\,\mathrm{D}_j + 3\,\mathrm{D}_j^2 + 2\,j - m - 4\,\mathrm{D}_j + 1\right] \tag{1.1.2}$$

```
> redct(Int(arccos(x/sqrt((a+b)*x-a*b)),x=a..b),[a::diff,
  b::diff]);
```

$$\left[a^2\,\mathrm{D}_a - b^2\,\mathrm{D}_a - a - 3\,b,\ a^2\,\mathrm{D}_b - b^2\,\mathrm{D}_b + 3\,a + b\right] \tag{1.1.3}$$

```
> redct(Int(u^(2*m)/sqrt(1-u^2),u=0..1),[m::shift]);
```

$$\left[2\,\mathrm{D}_m\,m - 2\,m + 2\,\mathrm{D}_m - 1\right] \tag{1.1.4}$$

```
> redct(Int(1/(x^4+2*a*x^2+1)^(m+1),x=0..infinity),[a::diff,
  m::shift]);
```

$$\left[-2\,a\,D_a + 4\,D_m\,m - 4\,m + 4\,D_m - 3,\, 4\,a^2\,D_a^2 + 8\,a\,m\,D_a + 12\,a\,D_a - 4\,D_a^2 + 4\,m + 3\right] \quad (1.1.5)$$

```
> redct(Int(LegendreP(2*n,u)/sqrt(1-u^2),u=-1..1),[n::shift]);
```

$$\left[4\,n^2\,D_n - 4\,n^2 + 8\,n\,D_n - 4\,n + 4\,D_n - 1\right] \quad (1.1.6)$$

```
> redct(Int(sin(m*x)*sin(n*x),x=0..2*Pi),[n::diff,m::diff]);
```

$$[1] \quad (1.1.7)$$

```
> redct(Int(exp(-u*(a+1))*log(u),u=0..infinity),[a::diff]);
```

$$\left[D_a\,a + D_a + 1\right] \quad (1.1.8)$$

```
> redct(subs(s=4,Int((x^2/(x^4+2*a*x^2+1))^r*(x^2+1)/x^2/
  (x^s+1),x=0..infinity)),[r::shift,a::diff]);
```

$$\left[2\,r\,D_r + D_a,\, 4\,a^3\,D_a^3 + 12\,a^2\,r\,D_a^2 + \textit{[...11 terms...]} - 3\,r - 4\,D_a\right] \quad (1.1.9)$$

```
> redct(Int(arcsinh(x)*exp(-z*x)/sqrt(x^2+1),x=0..infinity),
  [z::diff]);
```

$$\left[z\,D_z^2 + z + D_z\right] \quad (1.1.10)$$

```
> redct(Int(arcsinh(x)^2*exp(-z*x)*z/2,x=0..infinity),
  [z::diff]);
```

$$\left[z\,D_z^2 + z + D_z\right] \quad (1.1.11)$$

```
> redct(Int(x*arcsinh(x)*exp(-z*x)/sqrt(x^2+1),x=0..infinity),
  [z::diff]);
```

$$\left[z^2\,D_z^2 + z^2 + D_z\,z - 1\right] \quad (1.1.12)$$

```
> redct(Int(arcsinh(x)/(1+x^2)^(n+1),x=0..infinity),[n::shift]
  );
```

$$\left[2\,n\,D_n - 2\,n + 2\,D_n - 1\right] \quad (1.1.13)$$

```
> redct(Int(x^n*BesselJ(n,x)*LommelS2(-1,0,x),x=0..infinity),
  [n::shift]);
```

$$\left[8\,n^3 - 8\,n^2\,D_n + 2\,n\,D_n^2 + 12\,n^2 - 18\,n\,D_n + 4\,D_n^2 + 6\,n - 11\,D_n + 1\right] \quad (1.1.14)$$

```
> redct(Int((1+x*t+t^2)^(-2),t),[x::diff]);
```

$$\left[x^2\,D_x + 3\,x - 4\,D_x\right] \quad (1.1.15)$$

```
> redct(Int(1/(1+x^2)^n,x),[n::shift]);
```

$$\left[2\,n\,D_n - 2\,n + 1\right] \quad (1.1.16)$$

```
> redct(Int(x^(mu-1)*exp(-g*x-b*x^2)*sin(a*x),x=0..infinity),
  [mu::shift,g::diff,b::diff,a::diff]);
```

$$\left[D_a\,a + 2\,b\,D_b + D_g\,g + \mu,\, -D_a\,a - 2\,b\,D_b + D_\mu\,g - \mu,\, D_a^2 - D_b,\, 4\,a\,b\,D_a\,D_b + 4\,b^2\,D_b^2 + a^2\,D_b + 2\,a\,\mu\,D_a + 4\,b\,\mu\,D_b + g^2\,D_b + 2\,D_a\,a + 6\,b\,D_b + \mu^2 + \mu\right] \quad (1.1.17)$$

```
> redct(Int(arctan(p*x)/(1+p^2*x),x=0..1),[p::diff]);
```

$$\left[p^4\,D_p^2 + 6\,p^3\,D_p + p^2\,D_p^2 + 6\,p^2 + 4\,p\,D_p + 2\right] \quad (1.1.18)$$

```
> redct(Int(BesselJ(m,a*x)*BesselJ(n,b*x),x=0..infinity),
  [n::shift,m::shift,a::diff,b::diff]);
```

$$\left[a\,D_a + b\,D_b + 1,\, a^4\,D_a^2 - a^2\,b^2\,D_a^2 + 3\,a^3\,D_a - a^2\,n^2 - a\,b^2\,D_a + b^2\,m^2 + a^2,\right. \quad (1.1.19)$$

$$a^3 D_a D_m - a b^2 D_a D_m + a^2 n D_m - a b D_n m + a b D_n n - b^2 m D_m + a^2 D_m - b^2 D_m, a^3 D_a D_n - a b^2 D_a D_n - a^2 n D_n + b a D_m m - b a D_m n + b^2 m D_n, a^2 m^2 D_m^2 - a^2 n^2 D_m^2 + [\ldots 9\ terms\ldots] + 2 b^2 m + a^2, a b m D_m D_n + a b n D_m D_n - a^3 D_a + a b^2 D_a + a b D_n D_m - a^2 n - b^2 m - a^2, b^2 m^2 D_n^2 - b^2 n^2 D_n^2 + [\ldots 11\ terms\ldots] - 2 a^2 + b^2\Big]$$

```
> redct(Int(1/sqrt(x^2+a^2)*log((sqrt(x^2+a^2)+x)/(sqrt(x^2+
  a^2)-x))*BesselJ(0,b*x),x=0..infinity),[a::diff,b::diff]);
```

$$\left[-D_a a + b D_b, -a^2 b^2 D_a + a^2 D_a^3 - a b^2 + 3 a D_a^2 + D_a\right] \quad (1.1.20)$$

```
> redct(Int((1-x^2)^(nu-1/2)*GegenbauerC(m,nu,x)*GegenbauerC
  (n,nu,x),x=-1..1),[n::shift,m::shift,nu::shift]);
```

$$[1] \quad (1.1.21)$$

```
> redct(Int((x*(2*a-x))^(nu-1/2)*GegenbauerC(n,nu,x/a-1)*exp(-
  b*x),x=0..2*a),[n::shift,nu::shift,a::diff,b::diff]);
```

$$\Big[-D_a a + b D_b + 2 \nu, b a D_n n + a b n + 2 a b \nu + b a D_n + D_a a n + 2 D_a a \nu - n^2 - 4 n \nu - 4 \nu^2, -a b n^2 - 4 a b n \nu + [\ldots 14\ terms\ldots] + 4 n \nu + 4 \nu^2, 2 a^2 b D_a + a^2 D_a^2 - 2 a b \nu - 2 D_a a \nu + a b + D_a a - n^2 - 2 n \nu\Big] \quad (1.1.22)$$

```
> redct(Int(ChebyshevT(n,1-x^2*y)/sqrt(1-x^2),x=-1..1),
  [y::diff,n::shift]);
```

$$\Big[2 n y^2 D_y + 2 n^2 y + 2 n^2 D_n - 4 n y D_y + y^2 D_y - 2 n^2 + n y + 2 n D_n - 2 D_y y - 2 n, 4 n^2 y D_n + 2 n^2 D_n^2 + [\ldots 7\ terms\ldots] + 3 n - 2 D_n\Big] \quad (1.1.23)$$

```
> redct(Int(x^(r-1)*(1-x)^(s-1)*hypergeom([a,b],[c],x),x=0.
  .1),[r::shift,s::shift,a::shift,b::shift,c::shift]);
```

$$\Big[-D_a a + b D_b + a - b, -D_a a^2 c - D_a a c b + [\ldots 17\ terms\ldots] + r c^2 + a c, D_a a^2 + D_a a b + [\ldots 18\ terms\ldots] - s r - a, -D_a a^2 - D_a a b + [\ldots 15\ terms\ldots] - s^2 + a, a^2 D_a^2 + a b D_a^2 + [\ldots 30\ terms\ldots] - 3 D_a + 1\Big] \quad (1.1.24)$$

```
> redct(Int((1-t^2)^(n-1/2)*cos(z*t)*(z/2)^n/GAMMA(n+1/2)
  /GAMMA(1/2),t=-1..1),[z::diff]);
```

$$\left[z^2 D_z^2 - n^2 + z^2 + z D_z\right] \quad (1.1.25)$$

```
> redct(Int(t^(-n-1)*exp(t-z^2/4/t),t=c-I*infinity..c+I*
  infinity),[z::diff,n::shift]);
```

$$\left[D_n z + 2 D_z, z^2 D_n^2 - 4 n D_n - 4 D_n + 4\right] \quad (1.1.26)$$

```
> redct(Int(sin(z*t)*(1-t^2)^(n+1/2),t=0..1),[n::shift,
  z::diff]);
```

$$\left[2 D_z n + D_n z + 3 D_z, z^2 D_n^2 - 4 n^2 D_n + 4 n^2 - 18 n D_n + 16 n - 20 D_n + 15\right] \quad (1.1.27)$$

```
> redct(Int(exp(-z*t)*(1-t^2)^(n+1/2),t=0..1),[n::shift,
  z::diff]);
```

$$\left[2 D_z n - D_n z + 3 D_z, z^2 D_n^2 + 4 n^2 D_n - 4 n^2 + 18 n D_n - 16 n + 20 D_n - 15\right] \quad (1.1.28)$$

```
> redct(Int(BesselJ(n,b*t)*exp(-p^2*t^2)*t^(n+1),t=0..
  infinity),[n::shift,b::diff,p::diff]);
```

(1.1.29)

$$\left[2p^2 b D_b - 2np^2 + b^2, 2p^2 D_n - b, 2p^3 D_p + 4np^2 - b^2 + 4p^2\right] \tag{1.1.29}$$

```
> redct(Int(t^n*BesselY(n,a*t)/(t^2+k^2),t=0..infinity),
  [n::shift, a::diff, k::diff]);
```

$$\left[-D_a a + D_k k - n + 1, D_a a + a D_n - n, -a^3 k^2 D_a + a^3 D_a^3 - a^2 k^2 n + a^2 n D_a^2 - a^2 k^2 + 2 a^2 D_a^2 - a n^2 D_a + a n D_a - n^3 + n^2\right] \tag{1.1.30}$$

```
> redct(Int(exp(-p^2*t^2)*BesselJ(0,a*t)*BesselY(0,a*t),t=0..
  infinity),[a::diff,p::diff]);
```

$$\left[D_a a + D_p p + 1, a^2 D_a^3 p^2 + 2a^3 D_a^2 + 3ap^2 D_a^2 + 6 D_a a^2 + p^2 D_a + 2a\right] \tag{1.1.31}$$

```
> redct(Int(z^(n+1)*BesselI(n,z),z),[n::shift]);
```

$$[1] \tag{1.1.32}$$

```
> redct(Int(x^(a-1)*(1-x)^(b-1),x=0..1),[a::shift,b::shift]);
```

$$\left[D_a a + D_a b - a, D_b a + b D_b - b\right] \tag{1.1.33}$$

```
> redct(Int(exp(-a*t)*StruveH(0,t),t=0..infinity),[a::diff]);
```

$$\left[D_a a^2 + a + D_a\right] \tag{1.1.34}$$

```
> redct(Int(AiryAi(2^(2/3)*(u^2+x)),u=0..infinity),[x::diff]);
```

$$\left[D_x^3 - 4 D_x x - 2\right] \tag{1.1.35}$$

Total time for this section:

```
> time()-st;
```

$$3.475 \tag{1.1.36}$$

Longer ones

```
> st:=time():
> redct(Int(2*BesselJ(m+n,2*z*t)*ChebyshevT(m-n,t)/sqrt(1-
  t^2),t=0..1),[m::shift,n::shift,z::diff]);
```

$$\left[z D_m + z D_n + z D_z - m - n, z D_m^2 - 2 m D_m + z - 2 D_m, z D_n^2 - 2 n D_n + z - 2 D_n\right] \tag{1.2.1}$$

time taken:

```
> time()-st;
```

$$12.861 \tag{1.2.2}$$

Problematic ones

```
> Int(GegenbauerC(l,lambda,x)*GegenbauerC(m,lambda,x)*
  GegenbauerC(n,lambda,x)*(1-x^2)^(lambda-1/2),x=-1..1),
  [n::shift,m::shift,l::shift];
```

$$\int_{-1}^{1} \mathrm{GegenbauerC}(l,\lambda,x)\,\mathrm{GegenbauerC}(m,\lambda,x)\,\mathrm{GegenbauerC}(n,\lambda,x)\,\left(-x^2+1\right)^{\lambda - \frac{1}{2}}\,dx, [n::shift, m::shift, l::shift] \tag{1.3.1}$$

```
> timelimit(3600,redct(%));
Error, (in gcd/degrees) time expired
```

and

```
> Int(x*BesselJ(1,a*x)*BesselI(1,a*x)*BesselY(0,x)*BesselK(0,
  x),x=0..infinity);
```

$$\int_0^\infty x \operatorname{BesselJ}(1, x\, a) \operatorname{BesselI}(1, x\, a) \operatorname{BesselY}(0, x) \operatorname{BesselK}(0, x)\, dx \qquad \textbf{(1.3.2)}$$

```
> timelimit(3600,redct(%,[a::diff]));
Error, (in factor/lift) time expired
```

take way too long with this version of the code.

```
> st:=time():
```

Currently, dedicated code is necessary for:

```
> Int((c+I*u*(-c^2+1)^(1/2))^n/(-u^2+1)^(1/2),u=0..Pi);
```

$$\int_0^{\pi} \frac{\left(c + \mathrm{I}\, u \sqrt{-c^2+1}\right)^n}{\sqrt{-u^2+1}}\, \mathrm{d}u \qquad \textbf{(1.3.3)}$$

we can reencode the input as a solution of a linear system and integrate:

```
> lfs := LFSol({(-c^2*n*u^2+c^2*n+n*u^2-u^2-n)*_f(c, n, u)+
  (c^3*u^2-c^3-c*u^2+c)*(diff(_f(c, n, u), c))+(-u^3+u)*(diff
  (_f(c, n, u), u)), (-c^2*n*u^4+c^2*u^4+n*u^4-c^2*u^2-u^4+
  c^2*n-n*u^2)*_f(c, n, u)+(c*n*u^2-c*n)*_f(c, n+1, u)+(c^2*
  u^5-2*c^2*u^3-u^5+c^2*u+u^3)*(diff(_f(c, n, u), u)), (c^2*
  n^2*u^4-3*c^2*n*u^4-2*c^2*n^2*u^2+2*c^2*u^4-n^2*u^4+4*c^2*n*
  u^2+3*n*u^4+c^2*n^2-3*c^2*u^2+2*n^2*u^2-2*u^4-c^2*n-4*n*u^2+
  c^2-n^2+3*u^2+n)*_f(c, n, u)+(-2*c^2*n*u^5+4*c^2*u^5+4*c^2*
  n*u^3+2*n*u^5-8*c^2*u^3-4*u^5-2*c^2*n*u-4*n*u^3+4*c^2*u+6*
  u^3+2*n*u-2*u)*(diff(_f(c, n, u), u))+(c^2*u^6-3*c^2*u^4-
  u^6+3*c^2*u^2+2*u^4-c^2-u^2)*(diff(_f(c, n, u), u, u))}):
> redct(Int(lfs, u = 0..Pi), [n::shift, c::diff]);
```

$$\left[c^2 D_c + c\, n - D_n\, n + c - D_c - D_n,\ -2\, c\, D_n\, n + D_n^2\, n - 3\, c\, D_n + 2\, D_n^2 + n + 1\right] \qquad \textbf{(1.3.4)}$$

```
> Int(exp(x)*x^(-a/2)*exp(-t)*t^(n+a/2)*BesselJ(a,2*sqrt(t*x))
  /n!,t=0..infinity);
```

$$\int_0^{\infty} \frac{e^x\, x^{-\frac{a}{2}}\, e^{-t}\, t^{n+\frac{a}{2}}\, \mathrm{BesselJ}\left(a, 2\sqrt{t\, x}\right)}{n!}\, \mathrm{d}t \qquad \textbf{(1.3.5)}$$

is not recognized as D-finite by Mgfun. Till this is fixed, one can simply change a into 2a :

```
> redct(subs(a=2*a,Int(exp(x)*x^(-a/2)*exp(-t)*t^(n+a/2)*
  BesselJ(a,2*sqrt(t*x))/n!,t=0..infinity)),[n::shift,
  a::shift,x::diff]);
```

$$\left[2\, a\, n\, D_n + n\, x\, D_n + \textit{[...8 terms...]} + D_n - 1,\ 2\, a\, D_x + x\, D_a + x\, D_x + n - x + D_x,\ 2\, x^2\, D_a^2\, a + x^3\, D_a^2 + \textit{[...27 terms...]} - 6\, D_a + 6\right] \qquad \textbf{(1.3.6)}$$

```
> time()-st;
```

$$0.931 \qquad \textbf{(1.3.7)}$$

Bivariate rational integrands

```
> st:=time():
> for d to 16 do
    bigredct(Int(subs([x=y,y=x/y],1/(1-x-y-x*y*(1-x^d)))/y,y),
  [x::diff]);
  od;
```

$$\int \frac{1}{\left(1 - y - \frac{x}{y} - x\,(1-y)\right) y}\, \mathrm{d}y$$

$$5\, x^2\, D_x - 6\, x\, D_x + 5\, x + D_x - 3$$

```
time = 0.026 sec.
```

$$\int \frac{1}{\left(1 - y - \frac{x}{y} - x\left(-y^2 + 1\right)\right) y} \, dy$$

$$253 x^6 D_x^2 + 123 x^5 D_x^2 + [\ldots 14 \text{ terms} \ldots] + 6 D_x + 8$$

```
time = 0.041 sec.
```

$$\int \frac{1}{\left(1 - y - \frac{x}{y} - x\left(-y^3 + 1\right)\right) y} \, dy$$

$$833435 x^{11} D_x^3 - 1461243 x^{10} D_x^3 + [\ldots 37 \text{ terms} \ldots] + 48 D_x + 12$$

```
time = 0.107 sec.
```

$$\int \frac{1}{\left(1 - y - \frac{x}{y} - x\left(-y^4 + 1\right)\right) y} \, dy$$

$$546380568209 x^{17} D_x^4 + 174179973156 x^{16} D_x^4 + [\ldots 72 \text{ terms} \ldots] + 544320 x + 4680 D_x$$

```
time = 0.082 sec.
```

$$\int \frac{1}{\left(1 - y - \frac{x}{y} - x\left(-y^5 + 1\right)\right) y} \, dy$$

$$271[\ldots 13 \text{ digits} \ldots]447 x^{24} D_x^5 - 118[\ldots 13 \text{ digits} \ldots]240 x^{23} D_x^5 + [\ldots 125 \text{ terms} \ldots] + 166320 D_x - 18480$$

```
time = 0.139 sec.
```

$$\int \frac{1}{\left(1 - y - \frac{x}{y} - x\left(-y^6 + 1\right)\right) y} \, dy$$

$$937[\ldots 24 \text{ digits} \ldots]868 x^{32} D_x^6 - 274[\ldots 25 \text{ digits} \ldots]858 x^{31} D_x^6 + [\ldots 196 \text{ terms} \ldots] - 8589672000 D_x + 1493856000$$

```
time = 0.268 sec.
```

$$\int \frac{1}{\left(1 - y - \frac{x}{y} - x\left(-y^7 + 1\right)\right) y} \, dy$$

$$200[\ldots 37 \text{ digits} \ldots]475 x^{41} D_x^7 - 687[\ldots 37 \text{ digits} \ldots]465 x^{40} D_x^7 + [\ldots 289 \text{ terms} \ldots] - 15375651984000 D_x + 3294782568000$$

```
time = 0.407 sec.
```

$$\int \frac{1}{\left(1 - y - \frac{x}{y} - x\left(-y^8 + 1\right)\right) y} \, dy$$

$$271[\ldots 53 \text{ digits} \ldots]417 x^{51} D_x^8 - 514[\ldots 53 \text{ digits} \ldots]824 x^{50} D_x^8 + [\ldots 407 \text{ terms} \ldots] - 829[\ldots 13 \text{ digits} \ldots]000 D_x + 201[\ldots 13 \text{ digits} \ldots]000$$

```
time = 0.685 sec.
```

$$\int \frac{1}{\left(1-y-\frac{x}{y}-x\left(-y^{9}+1\right)\right)y}\,dy$$

$252[...69\text{ digits}...]313\,x^{62}\,D_x^{9} - 848[...69\text{ digits}...]878\,x^{61}\,D_x^{9} + [...553\text{ terms}...]$
$- 578[...17\text{ digits}...]200\,D_x + 152[...17\text{ digits}...]000$

```
time = 1.283 sec.
```

$$\int \frac{1}{\left(1-y-\frac{x}{y}-x\left(-y^{10}+1\right)\right)y}\,dy$$

$249[...92\text{ digits}...]936\,x^{74}\,D_x^{10} + 885[...92\text{ digits}...]300\,x^{73}\,D_x^{10} + [...730\text{ terms}...]$
$- 335[...27\text{ digits}...]800\,D_x + 936[...26\text{ digits}...]200$

```
time = 1.729 sec.
```

$$\int \frac{1}{\left(1-y-\frac{x}{y}-x\left(-y^{11}+1\right)\right)y}\,dy$$

$453[...113\text{ digits}...]891\,x^{87}\,D_x^{11} - 264[...114\text{ digits}...]639\,x^{86}\,D_x^{11} + [...941\text{ terms}...]$
$+ 162[...32\text{ digits}...]000\,D_x - 472[...31\text{ digits}...]000$

```
time = 2.949 sec.
```

$$\int \frac{1}{\left(1-y-\frac{x}{y}-x\left(-y^{12}+1\right)\right)y}\,dy$$

$315[...144\text{ digits}...]500\,x^{101}\,D_x^{12} - 124[...145\text{ digits}...]800\,x^{100}\,D_x^{12} + [...1189\text{ terms}...]$
$+ 112[...45\text{ digits}...]000\,D_x - 339[...44\text{ digits}...]000$

```
time = 4.273 sec.
```

$$\int \frac{1}{\left(1-y-\frac{x}{y}-x\left(-y^{13}+1\right)\right)y}\,dy$$

$183[...168\text{ digits}...]942\,x^{116}\,D_x^{13} - 102[...169\text{ digits}...]584\,x^{115}\,D_x^{13} + [...1477\text{ terms}...]$
$+ 964[...48\text{ digits}...]000\,D_x - 299[...48\text{ digits}...]000$

```
time = 7.086 sec.
```

$$\int \frac{1}{\left(1-y-\frac{x}{y}-x\left(-y^{14}+1\right)\right)y}\,dy$$

$431[...206\text{ digits}...]009\,x^{132}\,D_x^{14} - 113[...207\text{ digits}...]184\,x^{131}\,D_x^{14} + [...1808\text{ terms}...]$
$+ 774[...65\text{ digits}...]000\,D_x - 245[...65\text{ digits}...]000$

```
time = 11.455 sec.
```

$$\int \frac{1}{\left(1-y-\frac{x}{y}-x\left(-y^{15}+1\right)\right) y}\,dy$$

$629[...239\text{ digits}...]985\, x^{149}\, D_x^{15} - 556[...240\text{ digits}...]975\, x^{148}\, D_x^{15} + [...2185\text{ terms}...]$
$- 142[...75\text{ digits}...]000\, D_x + 460[...74\text{ digits}...]000$

```
time = 21.197 sec.
```

$$\int \frac{1}{\left(1-y-\frac{x}{y}-x\left(-y^{16}+1\right)\right) y}\,dy$$

$166[...283\text{ digits}...]825\, x^{167}\, D_x^{16} - 101[...284\text{ digits}...]280\, x^{166}\, D_x^{16} + [...2611\text{ terms}...]$
$- 202[...92\text{ digits}...]000\, D_x + 665[...91\text{ digits}...]000$

```
time = 27.174 sec.
```

Total time for this section:

```
> time()-st;
```

$$79.060 \qquad (2.1)$$

Bivariate hyperexponential integrands

```
> st:=time():
> for lambda to 2 do
    p:=randpoly([x,y],degree=lambda,dense);
    q:=randpoly([x,y],degree=lambda,dense);
    for mu to 2 do
       a:=randpoly([x,y],degree=mu,dense);
       b:=randpoly([x,y],degree=mu,dense);
       for nu to 2 do
          u:=randpoly([x,y],degree=nu,dense);
          v:=randpoly([x,y],degree=nu,dense);
          for m from 5 to 5 do
             f:=p/q^m*sqrt(a/b)*exp(u/v);
             bigredct(Int(f,y),[x::diff])
          od
       od
     od
  od:
```

$$\int \frac{(12x-19y-34)\sqrt{\frac{89x+65y+64}{-33x-76y+96}}\, e^{\frac{-8x+79y-64}{-75x-81y+98}}}{(68x+69y+56)^5}\,dy$$

$423[...112\text{ digits}...]376\, x^{25}\, D_x^{3} + 365[...114\text{ digits}...]668\, x^{24}\, D_x^{3} + [...94\text{ terms}...]$
$- 186[...121\text{ digits}...]224\, D_x - 632[...121\text{ digits}...]416$

```
time = 0.559 sec.
```

$$\int \frac{(12x-19y-34)\sqrt{\frac{89x+65y+64}{-33x-76y+96}}\, e^{\frac{-10x^2-13xy-37y^2+83x+49y-43}{-4x^2+86xy+84y^2-58x-25y-91}}}{(68x+69y+56)^5}\,dy$$

$263[...272\ digits...]000\, x^{89}\, D_x^5 + 453[...272\ digits...]000\, x^{88}\, D_x^5 + [...521\ terms...]$
$- 624[...283\ digits...]840\, D_x - 277[...284\ digits...]160$

```
time = 1.77 sec.
```

$$\int \frac{1}{(68x+69y+56)^5}\left((12x-19y-34)\sqrt{\frac{-93x^2-40xy+83y^2+74x-54y+11}{60x^2-64xy+18y^2-78x-78y+75}}\; e^{\frac{-29x-7y+13}{-29x+25y+74}}\right) dy$$

$457[...253\ digits...]000\, x^{89}\, D_x^5 + 476[...255\ digits...]000\, x^{88}\, D_x^5 + [...521\ terms...]$
$- 522[...282\ digits...]000\, D_x - 100[...283\ digits...]000$

```
time = 1.17 sec.
```

$$\int \frac{1}{(68x+69y+56)^5}\left((12x-19y-34)\sqrt{\frac{-93x^2-40xy+83y^2+74x-54y+11}{60x^2-64xy+18y^2-78x-78y+75}}\; e^{\frac{90x^2-33xy+20y^2+98x-77y-93}{-20x^2-32xy-78y^2-98x-67y+51}}\right) dy$$

$208[...543\ digits...]000\, x^{221}\, D_x^7 - 179[...545\ digits...]000\, x^{220}\, D_x^7 + [...1744\ terms...]$
$- 113[...544\ digits...]000\, D_x - 667[...544\ digits...]000$

```
time = 11.077 sec.
```

$$\int \left((-48x^2+29xy+64y^2+94x-31y-25)\sqrt{\frac{55x-53y+72}{50x-6y+40}}\; e^{\frac{8x+70y-20}{33x+47y-73}}\right) \Big/ (36x^2-71xy+25y^2-56x-17y+70)^5\, dy$$

$102[...202\ digits...]000\, x^{59}\, D_x^4 + 152[...203\ digits...]000\, x^{58}\, D_x^4 + [...286\ terms...]$
$+ 691[...200\ digits...]000\, D_x + 103[...202\ digits...]000$

```
time = 0.689 sec.
```

$$\int \left((-48x^2+29xy+64y^2+94x-31y\right.$$

$$-25\Big)\sqrt{\frac{55x-53y+72}{50x-6y+40}}\,e^{\frac{-37x^2+70xy+81y^2+95x-65y-58}{-50x^2-74xy-48y^2+63x+96y-82}}\Big)\Big/(36x^2-71xy+25y^2-56x-17y+70)^5\,dy$$

$$977[...406\text{ digits...}]000\,x^{156}\,D_x^6+210[...408\text{ digits...}]000\,x^{155}\,D_x^6+[...1074\text{ terms...}]-183[...404\text{ digits...}]000\,D_x+245[...404\text{ digits...}]200$$

```
time = 4.581 sec.
```

$$\int\!\!\int\Big(\big(-48x^2+29xy+64y^2+94x-31y-25\big)\sqrt{\frac{58x^2-27xy+82y^2-23x+68y-46}{-17x^2+47xy+59y^2-44x-10y+44}}\,e^{\frac{98x+46y-97}{13x-57y-32}}\Big)\Big/(36x^2-71xy+25y^2-56x-17y+70)^5\,dy$$

$$699[...453\text{ digits...}]000\,x^{156}\,D_x^6-584[...454\text{ digits...}]000\,x^{155}\,D_x^6+[...1074\text{ terms...}]-874[...480\text{ digits...}]808\,D_x+169[...481\text{ digits...}]760$$

```
time = 6.481 sec.
```

$$\int\!\!\int\Big(\big(-48x^2+29xy+64y^2+94x-31y-25\big)\sqrt{\frac{58x^2-27xy+82y^2-23x+68y-46}{-17x^2+47xy+59y^2-44x-10y+44}}\,e^{\frac{43x^2+96xy-37y^2+46x-41y-96}{66x^2+54xy-40y^2-71x+61y+51}}\Big)\Big/(36x^2-71xy+25y^2-56x-17y+70)^5\,dy$$

[Length of output exceeds limit of 1000000]

```
time = 63.546 sec.
```

Total time for this section:

```
> time()-st;
```

90.072 **(3.1)**

Bivariate mixed terms

```
> st:=time():
> Int((1+x/(n^2+1))*((x+1)^2/(x-4)/(x-3)^2/(x^2-5)^3)^n*sqrt(x^2-5)*exp((x^3+1)/x/(x-3)/(x-4)^2),x);
```

(4.1)

$$\int\left(1+\frac{x}{n^2+1}\right)\left(\frac{(x+1)^2}{(x-4)\,(x-3)^2\,(x^2-5)^3}\right)^n\sqrt{x^2-5}\,\mathrm{e}^{\frac{x^3+1}{x\,(x-3)\,(x-4)^2}}\,\mathrm{d}x \qquad \textbf{(4.1)}$$

```
> bigredct(%,[n::shift]);
```

$$\int\left(1+\frac{x}{n^2+1}\right)\left(\frac{(x+1)^2}{(x-4)\,(x-3)^2\,(x^2-5)^3}\right)^n\sqrt{x^2-5}\,\mathrm{e}^{\frac{x^3+1}{x\,(x-3)\,(x-4)^2}}\,\mathrm{d}x$$

$$950[...175\text{ digits...}]000\,n^{89}\,D_n^8+103[...171\text{ digits...}]000\,n^{90}\,D_n^6+\textit{[...900 terms...]}+101[...207\text{ digits...}]000\,n-988[...210\text{ digits...}]000\,D_n$$

```
time = 1.511 sec.
> redct(Int(((z^2-1)/2/(z-x))^n*(1-z)^alpha*(1+z)^beta/(z-x),z),
  [n::shift]);
```

$$\left[-\alpha^3\,x\,D_n-3\,\alpha^2\,\beta\,x\,D_n+\textit{[...58 terms...]}+20\,n+8\right] \qquad \textbf{(4.2)}$$

```
> for i from 50 by 50 to 300 do
      bigredct(Int((1+x)^(i*n)/x^(n+1),x),[n::shift]);
  od;
```

$$\int\frac{(x+1)^{50\,n}}{x^{n+1}}\,\mathrm{d}x$$

$$660[...77\text{ digits...}]449\,n^{49}\,D_n-888[...79\text{ digits...}]000\,n^{49}+\textit{[...96 terms...]}+608[...57\text{ digits...}]000\,D_n-304[...59\text{ digits...}]000$$

```
time = 0.189 sec.
```

$$\int\frac{(x+1)^{100\,n}}{x^{n+1}}\,\mathrm{d}x$$

$$369[...192\text{ digits...}]899\,n^{99}\,D_n-100[...195\text{ digits...}]000\,n^{99}+\textit{[...196 terms...]}+933[...150\text{ digits...}]000\,D_n-933[...152\text{ digits...}]000$$

```
time = 1.832 sec.
```

$$\int\frac{(x+1)^{150\,n}}{x^{n+1}}\,\mathrm{d}x$$

$$637[...318\text{ digits...}]349\,n^{149}\,D_n-259[...321\text{ digits...}]000\,n^{149}+\textit{[...296 terms...]}+380[...255\text{ digits...}]000\,D_n-571[...257\text{ digits...}]000$$

```
time = 3.566 sec.
```

$$\int\frac{(x+1)^{200\,n}}{x^{n+1}}\,\mathrm{d}x$$

$$296[...452\text{ digits...}]799\,n^{199}\,D_n-160[...455\text{ digits...}]000\,n^{199}+\textit{[...396 terms...]}+394[...367\text{ digits...}]000\,D_n-788[...369\text{ digits...}]000$$

```
time = 12.827 sec.
```

$$\int\frac{(x+1)^{250\,n}}{x^{n+1}}\,\mathrm{d}x$$

$$450[...591\text{ digits...}]249\,n^{249}\,D_n-305[...594\text{ digits...}]000\,n^{249}+\textit{[...496 terms...]}$$

$+ 129[...485 \text{ digits...}]000\, D_n - 323[...487 \text{ digits...}]000$

time = 19.699 sec.

$$\int \frac{(x+1)^{300\,n}}{x^{n+1}}\, dx$$

$168[...735 \text{ digits...}]699\, n^{299}\, D_n - 136[...738 \text{ digits...}]000\, n^{299} + [...596 \text{ terms...}]$

$+ 102[...607 \text{ digits...}]000\, D_n - 306[...609 \text{ digits...}]000$

time = 60.283 sec.

Another family:

```
> for i to 8 do
      p:=x*randpoly(x,degree=i,dense)^2/randpoly(x,degree=i,
  dense);
      f:=Int(p^(-n-1)*x*normal(diff(p,x)),x);
      bigredct(f,[n::shift])
  od:
```

$$\int \frac{\left(\frac{x\,(98\,x+71)^2}{28\,x+45}\right)^{-n-1} x\,(98\,x+71)\left(5488\,x^2+13230\,x+3195\right)}{(28\,x+45)^2}\, dx$$

$210399412994\, n^3\, D_n^2 + 14629662551\, n^3\, D_n + [...7 \text{ terms...}] + 3700480\, n$

$+ 4687558176\, D_n$

time = 0.062 sec.

$$\int \frac{1}{\left(60\,x^2+82\,x+87\right)^2}\left(\left(\frac{x\left(12\,x^2+68\,x-45\right)^2}{60\,x^2+82\,x+87}\right)^{-n-1} x\left(12\,x^2+68\,x - 45\right)\left(2160\,x^4+8016\,x^3+19072\,x^2+17748\,x-3915\right)\right) dx$$

$364[...42 \text{ digits...}]000\, n^{10}\, D_n^4 - 196[...42 \text{ digits...}]576\, n^{10}\, D_n^3 + [...50 \text{ terms...}]$

$+ 265[...35 \text{ digits...}]750\, n + 243[...38 \text{ digits...}]000\, D_n$

time = 0.111 sec.

$$\int \frac{1}{\left(94\,x^3+13\,x^2+2\,x-31\right)^2}\left(\left(\frac{x\left(-86\,x^3-31\,x^2-89\,x-51\right)^2}{94\,x^3+13\,x^2+2\,x-31}\right)^{-n-1} x\left(86\,x^3 + 31\,x^2+89\,x+51\right)\left(32336\,x^6+11418\,x^5+2241\,x^4-26845\,x^3-5112\,x^2 - 8277\,x-1581\right)\right) dx$$

$644[...147 \text{ digits...}]816\, n^{22}\, D_n^6 - 108[...147 \text{ digits...}]544\, n^{22}\, D_n^5 + [...156 \text{ terms...}]$

$- 187[...142 \text{ digits...}]200\, n - 775[...144 \text{ digits...}]000\, D_n$

time = 0.194 sec.

$$\int$$

$-1 \Big/ \left(29\,x^4 + 74\,x^3 + 84\,x^2 - 23\,x\right.$

$$- 53)^2 \left(\left(\frac{x\,(50\,x^4 + 57\,x^3 - 30\,x^2 + 45\,x - 14)^2}{-29\,x^4 - 74\,x^3 - 84\,x^2 + 23\,x + 53} \right)^{-n-1} x\,(50\,x^4 + 57\,x^3 - 30\,x^2 + 45\,x - 14)\,(7250\,x^8 + 27159\,x^7 + 45402\,x^6 + 8995\,x^5 - 38058\,x^4 - 12535\,x^3 + 7056\,x^2 - 7155\,x + 742) \right) dx$$

257[...282 digits...]192 $n^{39}\,D_n^8$ − 184[...283 digits...]344 $n^{39}\,D_n^7$ + *[...355 terms...]* + 366[...283 digits...]000 n − 192[...287 digits...]000 D_n

```
time = 0.388 sec.
```

$$\int \left(\left(\frac{x\,(65\,x^5 - 60\,x^3 + 74\,x^2 - 49\,x - 80)^2}{58\,x^5 - 76\,x^4 + 21\,x^3 + 95\,x^2 - 27\,x + 25} \right)^{-n-1} x\,(65\,x^5 - 60\,x^3 + 74\,x^2 - 49\,x - 80)\,(22620\,x^{10} - 34580\,x^9 + [\ldots 7\ terms\ldots] - 3675\,x - 2000) \right) \Big/ (58\,x^5 - 76\,x^4 + 21\,x^3 + 95\,x^2 - 27\,x + 25)^2\, dx$$

222[...433 digits...]000 $n^{61}\,D_n^{10}$ + 777[...430 digits...]000 $n^{61}\,D_n^9$ + *[...677 terms...]* + 145[...441 digits...]000 n − 184[...445 digits...]000 D_n

```
time = 1.967 sec.
```

$$\int -\left(\left(\frac{x\,(-40\,x^6 - 48\,x^5 - 33\,x^4 + 98\,x^3 - 36\,x^2 + 19\,x - 10)^2}{-37\,x^6 + 92\,x^5 + 92\,x^4 - 69\,x^3 - 50\,x^2 - 54\,x + 68} \right)^{-n-1} x\,(40\,x^6 + 48\,x^5 + 33\,x^4 - 98\,x^3 + 36\,x^2 - 19\,x + 10)\,(10360\,x^{12} - 20560\,x^{11} + [\ldots 9\ terms\ldots] + 3876\,x - 680) \right) \Big/ (37\,x^6 - 92\,x^5 - 92\,x^4 + 69\,x^3 + 50\,x^2 + 54\,x - 68)^2\, dx$$

269[...658 digits...]000 $n^{88}\,D_n^{12}$ − 145[...661 digits...]760 $n^{88}\,D_n^{11}$ + *[...1152 terms...]* + 544[...693 digits...]000 n − 466[...700 digits...]000 D_n

```
time = 2.029 sec.
```

$$\int \left(\left(\frac{x\,(-54\,x^7 + 84\,x^6 - 88\,x^5 - 19\,x^4 - 19\,x^3 + 27\,x^2 + 27\,x - 12)^2}{9\,x^7 - 18\,x^6 + 71\,x^5 + 88\,x^4 - 78\,x^3 - 20\,x^2 - 20\,x + 83} \right)^{-n-1} x\,(54\,x^7 - 84\,x^6 + 88\,x^5 + 19\,x^4 + 19\,x^3 - 27\,x^2 - 27\,x + 12)\,(3888\,x^{14} - 13284\,x^{13} + [\ldots 11\ terms\ldots] - 6723\,x + 996) \right) \Big/ (9\,x^7 - 18\,x^6 + 71\,x^5 + 88\,x^4 - 78\,x^3 - 20\,x^2 - 20\,x + 83)^2\, dx$$

[Length of output exceeds limit of 1000000]

```
time = 6.817 sec.
```

$$\int$$

$$-\left(\left(\left(x\left(51x^8+15x^7+77x^6+55x^5-58x^4+85x^3+18x^2+35x-72\right)^2\right)\Big/\left(-32x^8+23x^7+66x^6+75x^5+95x^4+93x^3-51x^2-37x-70\right)\right)^{-n-1} x\left(51x^8+15x^7+77x^6+55x^5-58x^4+85x^3+18x^2+35x-72\right)\left(14688x^{16}-8370x^{15}+[\ldots 13\ terms\ldots]+7350x-5040\right)\right)\Big/\left(32x^8-23x^7-66x^6-75x^5-95x^4-93x^3+51x^2+37x+70\right)^2\,dx$$

[Length of output exceeds limit of 1000000]

time = 22.814 sec.

Another family, for which Mgfun needs some help:

```
> for i from 7 to 15 do
    p:= randpoly(x,degree=i,dense);
    f:=exp(Int(1/p,x))/p^n;
    print(Int(f,x));
    bigredct(Int(LFSol_of_multi_hyper(f,{x::diff,n::shift}),x),
  [n::shift])
  od:
```

$$\int \frac{e^{\int \frac{1}{-67x^7+8x^6+11x^5-3x^4+51x^3+52x^2-19x+11}dx}}{\left(-67x^7+8x^6+11x^5-3x^4+51x^3+52x^2-19x+11\right)^n}\,dx$$

$$\int LFSol\left(\left\{\_f(n+1,x)+\frac{\_f(n,x)}{67x^7-8x^6-11x^5+3x^4-51x^3-52x^2+19x-11}, \frac{\partial}{\partial x}\_f(n,x)+\left(\left(469nx^6-48nx^5-55nx^4+12nx^3-153nx^2-104nx+19n+1\right)\_f(n,x)\right)\Big/\left(67x^7-8x^6-11x^5+3x^4-51x^3-52x^2+19x-11\right)\right\}\right)dx$$

$$564[\ldots 114\ digits\ldots]000\,n^{36}D_n^6-737[\ldots 113\ digits\ldots]000\,n^{36}D_n^5+[\ldots 255\ terms\ldots]-206[\ldots 117\ digits\ldots]000\,D_n-421[\ldots 116\ digits\ldots]000$$

time = 0.277 sec.

$$\int \frac{e^{\int \frac{1}{-95x^8+15x^7+19x^6+20x^5+61x^4-92x^3+68x^2-23x+25}dx}}{\left(-95x^8+15x^7+19x^6+20x^5+61x^4-92x^3+68x^2-23x+25\right)^n}\,dx$$

$$\int LFSol\left(\left\{\_f(n+1,x)\right.\right.$$

$$+\frac{\_f(n,x)}{95x^8-15x^7-19x^6-20x^5-61x^4+92x^3-68x^2+23x-25}, \frac{\partial}{\partial x}\_f(n,x)+\left(\left(760\,n\,x^7-105\,n\,x^6-114\,n\,x^5-100\,n\,x^4-244\,n\,x^3+276\,n\,x^2-136\,n\,x+23\,n+1\right)\_f(n,x)\right)\Big/\left(95x^8-15x^7-19x^6-20x^5-61x^4+92x^3-68x^2+23x-25\right)\}\Big)\,dx$$

111[...158 digits...]040 $n^{49}\,D_n^7$ − 122[...157 digits...]880 $n^{49}\,D_n^6$ + [...396 terms...] + 226[...168 digits...]000 D_n + 337[...167 digits...]000

time = 0.549 sec.

$$\int \frac{e^{\int \frac{1}{-9x^9+83x^8+35x^7+77x^6+89x^5+44x^4-32x^3-75x^2+53x+16}\,dx}}{\left(-9x^9+83x^8+35x^7+77x^6+89x^5+44x^4-32x^3-75x^2+53x+16\right)^n}\,dx$$

$$\int LFSol\Big(\Big\{\_f(n+1,x)+\frac{\_f(n,x)}{9x^9-83x^8-35x^7-77x^6-89x^5-44x^4+32x^3+75x^2-53x-16}, \frac{\partial}{\partial x}\_f(n,x)+\left(\left(81\,n\,x^8-664\,n\,x^7-245\,n\,x^6-462\,n\,x^5-445\,n\,x^4-176\,n\,x^3+96\,n\,x^2+150\,n\,x-53\,n+1\right)\_f(n,x)\right)\Big/\left(9x^9-83x^8-35x^7-77x^6-89x^5-44x^4+32x^3+75x^2-53x-16\right)\Big\}\Big)\,dx$$

205[...225 digits...]032 $n^{64}\,D_n^8$ − 125[...224 digits...]104 $n^{64}\,D_n^7$ + [...581 terms...] + 378[...232 digits...]000 D_n + 435[...231 digits...]000

time = 1.626 sec.

$$\int \frac{e^{\int \frac{1}{58x^{10}+60x^9+[...7\ terms...]-12x-57}\,dx}}{\left(58x^{10}+60x^9+[...7\ terms...]-12x-57\right)^n}\,dx$$

$$\int LFSol\Big(\Big\{\_f(n+1,x)-\frac{\_f(n,x)}{58x^{10}+60x^9+[...7\ terms...]-12x-57}, \frac{\partial}{\partial x}\_f(n,x)+\frac{\left(580\,n\,x^9+540\,n\,x^8+[...7\ terms...]-12\,n-1\right)\_f(n,x)}{58x^{10}+60x^9+[...7\ terms...]-12x-57}\Big\}\Big)\,dx$$

103[...258 digits...]000 $n^{81}\,D_n^9$ + 114[...257 digits...]000 $n^{81}\,D_n^8$ + [...816 terms...] + 767[...281 digits...]000 D_n − 220[...281 digits...]000

time = 1.82 sec.

$$\int \frac{e^{\int \frac{1}{18x^{11}-14x^{10}+[...8\ terms...]+40x+24}\,dx}}{\left(18x^{11}-14x^{10}+[...8\ terms...]+40x+24\right)^n}\,dx$$

$$\int LFSol\left(\left\{\_f(n+1,x) - \frac{\_f(n,x)}{18x^{11} - 14x^{10} + [...8\ terms...] + 40x + 24}, \frac{\partial}{\partial x}\_f(n,x) + \frac{(198nx^{10} - 140nx^{9} + [...8\ terms...] + 40n - 1)\_f(n,x)}{18x^{11} - 14x^{10} + [...8\ terms...] + 40x + 24}\right\}\right) dx$$

$$305[...315\ digits...]712\, n^{100}\, D_n^{10} - 581[...314\ digits...]408\, n^{100}\, D_n^{9} + [...1107\ terms...] + 888[...348\ digits...]000\, D_n + 227[...349\ digits...]000$$

time = 3.34 sec.

$$\int \frac{e^{\int \frac{1}{9x^{12} - 89x^{11} + [...9\ terms...] + 41x + 30} dx}}{(9x^{12} - 89x^{11} + [...9\ terms...] + 41x + 30)^n} dx$$

$$\int LFSol\left(\left\{\_f(n+1,x) - \frac{\_f(n,x)}{9x^{12} - 89x^{11} + [...9\ terms...] + 41x + 30}, \frac{\partial}{\partial x}\_f(n,x) + \frac{(108nx^{11} - 979nx^{10} + [...9\ terms...] + 41n - 1)\_f(n,x)}{9x^{12} - 89x^{11} + [...9\ terms...] + 41x + 30}\right\}\right) dx$$

$$547[...455\ digits...]888\, n^{121}\, D_n^{11} - 312[...454\ digits...]096\, n^{121}\, D_n^{10} + [...1460\ terms...] + 110[...497\ digits...]000\, D_n + 115[...496\ digits...]000$$

time = 8.021 sec.

$$\int \frac{e^{\int \frac{1}{-x^{13} + 24x^{12} + [...10\ terms...] - 54x - 76} dx}}{(-x^{13} + 24x^{12} + [...10\ terms...] - 54x - 76)^n} dx$$

$$\int LFSol\left(\left\{\_f(n+1,x) + \frac{\_f(n,x)}{x^{13} - 24x^{12} + [...10\ terms...] + 54x + 76}, \frac{\partial}{\partial x}\_f(n,x) + \frac{(13nx^{12} - 288nx^{11} + [...10\ terms...] + 54n + 1)\_f(n,x)}{x^{13} - 24x^{12} + [...10\ terms...] + 54x + 76}\right\}\right) dx$$

$$193[...502\ digits...]064\, n^{144}\, D_n^{12} + 570[...500\ digits...]992\, n^{144}\, D_n^{11} + [...1881\ terms...] + 841[...550\ digits...]000\, D_n + 409[...549\ digits...]000$$

time = 13.01 sec.

$$\int \frac{e^{\int \frac{1}{-3x^{14} + 62x^{13} + [...11\ terms...] - 6x + 8} dx}}{(-3x^{14} + 62x^{13} + [...11\ terms...] - 6x + 8)^n} dx$$

$$\int LFSol\left(\left\{\_f(n+1,x) + \frac{\_f(n,x)}{3x^{14} - 62x^{13} + [...11\ terms...] + 6x - 8}, \frac{\partial}{\partial x}\_f(n,x) + \frac{(42nx^{13} - 806nx^{12} + [...11\ terms...] + 6n + 1)\_f(n,x)}{3x^{14} - 62x^{13} + [...11\ terms...] + 6x - 8}\right\}\right) dx$$

[Length of output exceeds limit of 1000000]

time = 28.539 sec.

$$\int \frac{e^{\int \frac{1}{77\,x^{15} + 99\,x^{14} + [...12\ terms...] + 98\,x - 20}\,dx}}{\left(77\,x^{15} + 99\,x^{14} + [...12\ terms...] + 98\,x - 20\right)^{n}}\,dx$$

$$\int LFSol\left(\left\{\_f(n+1,x) - \frac{\_f(n,x)}{77\,x^{15} + 99\,x^{14} + [...12\ terms...] + 98\,x - 20}, \frac{\partial}{\partial x}\,\_f(n,x) + \frac{\left(1155\,n\,x^{14} + 1386\,n\,x^{13} + [...12\ terms...] + 98\,n - 1\right)\,\_f(n,x)}{77\,x^{15} + 99\,x^{14} + [...12\ terms...] + 98\,x - 20}\right\}\right)\,dx$$

[Length of output exceeds limit of 1000000]

```
time = 52.758 sec.
```

Total time for this section:

```
> time()-st;
```

$$244.832 \qquad (4.3)$$

Gegenbauer examples

```
> m:='m':mu:='mu':nu:='nu':a:='a':lambda:='lambda':
> infolevel[redct]:=2: # display intermediate information
> F:=Int(x^ell*GegenbauerC(m,mu,x)*GegenbauerC(n,nu,x)*(-x^2+1)^
  (nu-1/2),x = -1 .. 1);
```

$$F := \int_{-1}^{1} x^{\ell}\,\mathrm{GegenbauerC}(m,\mu,x)\,\mathrm{GegenbauerC}(n,\nu,x)\,\left(-x^2+1\right)^{\nu - \frac{1}{2}}\,dx \qquad (5.1)$$

With ell=0

```
> st:=time():
  redct(subs(ell=0,F),[m::shift, n::shift,mu::shift,nu::shift]);
  time()-st;
redct: dimension 4
scalar_red_ct: dealing with monomial 1 at time: 10465.206
exceptionalbasis: entering exceptionalbasis at time 10465.206
exceptionalbasis: dimension of the exceptional set 2
exceptionalbasis: degrees of its elements [2 3]
exceptionalbasis: exceptional basis computed in  .441 sec.
scalar_red_ct: reduction done at time: 10465.648
scalar_red_ct: dealing with monomial D[m] at time: 10465.648
scalar_red_ct: reduction done at time: 10465.769
scalar_red_ct: dealing with monomial D[mu] at time: 10465.769
scalar_red_ct: reduction done at time: 10465.863
scalar_red_ct: dimension of the matrix: 2
scalar_red_ct: col. degrees of the matrix entries: 0 0
scalar_red_ct: relation found
scalar_red_ct: degree of the solution: 0
scalar_red_ct: dealing with monomial D[n] at time: 10465.865
scalar_red_ct: reduction done at time: 10465.967
scalar_red_ct: dimension of the matrix: 2
scalar_red_ct: col. degrees of the matrix entries: 0 0
scalar_red_ct: relation found
scalar_red_ct: degree of the solution: 0
scalar_red_ct: dealing with monomial D[nu] at time: 10465.974
scalar_red_ct: reduction done at time: 10466.081
scalar_red_ct: dimension of the matrix: 2
scalar_red_ct: col. degrees of the matrix entries: 0 0
scalar_red_ct: relation found
scalar_red_ct: degree of the solution: 0
scalar_red_ct: dealing with monomial D[m]^2 at time:
10466.083
```

```
scalar_red_ct: reduction done at time: 10466.222
scalar_red_ct: dimension of the matrix: 2
scalar_red_ct: col. degrees of the matrix entries: 0 0
scalar_red_ct: relation found
scalar_red_ct: degree of the solution: 0
scalar_red_ct: dim quotient: 2
scalar_red_ct: degree coeffs: 3
```

$$\left[4\,\mu^2\,D_\mu - 4\,\mu\,D_\mu\,\nu - m^2 - 4\,m\,\mu + 2\,\nu\,m - 4\,\mu^2 + 4\,\mu\,\nu + n^2 + 2\,n\,\nu, \text{-}m\,n\,D_m + m\,n\,D_n + \textit{[...11 terms...]} - 2\,\nu\,D_n - D_n, m^2\,\nu\,D_\nu + 2\,m\,\mu\,\nu\,D_\nu + \textit{[...20 terms...]} + n + 2\,\nu, m^2\,D_m^2 + 2\,m\,\nu\,D_m^2 + \textit{[...10 terms...]} + 2\,n\,\nu + 4\,D_m^2\right]$$

$$1.516 \qquad \textbf{(5.2)}$$

General ell:

```
> st:=time():
  redct(F,[ell::shift,m::shift, n::shift,mu::shift,nu::shift]);
  time()-st;
redct: dimension 4
scalar_red_ct: dealing with monomial 1 at time: 10467.230
exceptionalbasis: entering exceptionalbasis at time 10467.230
exceptionalbasis: dimension of the exceptional set 2
exceptionalbasis: degrees of its elements [6 7]
exceptionalbasis: exceptional basis computed in  1.758 sec.
scalar_red_ct: reduction done at time: 10468.988
scalar_red_ct: dealing with monomial D[ell] at time:
10468.988
scalar_red_ct: reduction done at time: 10468.988
scalar_red_ct: dealing with monomial D[m] at time: 10468.988
scalar_red_ct: reduction done at time: 10470.974
scalar_red_ct: dealing with monomial D[mu] at time: 10470.982
scalar_red_ct: reduction done at time: 10474.921
scalar_red_ct: dealing with monomial D[n] at time: 10474.926
scalar_red_ct: reduction done at time: 10476.741
scalar_red_ct: dealing with monomial D[nu] at time: 10476.752
scalar_red_ct: reduction done at time: 10478.543
scalar_red_ct: dealing with monomial D[ell]^2 at time:
10478.549
scalar_red_ct: reduction done at time: 10478.549
scalar_red_ct: dimension of the matrix: 6
scalar_red_ct: col. degrees of the matrix entries: 0 0 6 5 6
4
scalar_red_ct: relation found
scalar_red_ct: degree of the solution: 2
scalar_red_ct: dealing with monomial D[ell]*D[m] at time:
10478.613
scalar_red_ct: reduction done at time: 10479.337
scalar_red_ct: dimension of the matrix: 6
scalar_red_ct: col. degrees of the matrix entries: 0 0 6 5 6
4
scalar_red_ct: relation found
scalar_red_ct: degree of the solution: 5
scalar_red_ct: dealing with monomial D[ell]*D[mu] at time:
10479.411
scalar_red_ct: reduction done at time: 10481.418
scalar_red_ct: dimension of the matrix: 6
scalar_red_ct: col. degrees of the matrix entries: 0 0 6 5 6
4
scalar_red_ct: relation found
scalar_red_ct: degree of the solution: 5
scalar_red_ct: dealing with monomial D[ell]*D[n] at time:
10481.492
scalar_red_ct: reduction done at time: 10482.174
scalar_red_ct: dimension of the matrix: 6
```

```
scalar_red_ct: col. degrees of the matrix entries: 0 0 6 5 6
4
scalar_red_ct: relation found
scalar_red_ct: degree of the solution: 4
scalar_red_ct: dealing with monomial D[ell]*D[nu] at time:
10482.239
scalar_red_ct: reduction done at time: 10484.884
scalar_red_ct: dimension of the matrix: 6
scalar_red_ct: col. degrees of the matrix entries: 0 0 6 5 6
4
scalar_red_ct: relation found
scalar_red_ct: degree of the solution: 6
scalar_red_ct: dealing with monomial D[m]^2 at time:
10484.976
scalar_red_ct: reduction done at time: 10494.768
scalar_red_ct: dimension of the matrix: 6
scalar_red_ct: col. degrees of the matrix entries: 0 0 6 5 6
4
scalar_red_ct: relation found
scalar_red_ct: degree of the solution: 6
scalar_red_ct: dealing with monomial D[m]*D[mu] at time:
10494.925
scalar_red_ct: reduction done at time: 10509.631
scalar_red_ct: dimension of the matrix: 6
scalar_red_ct: col. degrees of the matrix entries: 0 0 6 5 6
4
scalar_red_ct: relation found
scalar_red_ct: degree of the solution: 8
scalar_red_ct: dealing with monomial D[m]*D[n] at time:
10509.850
scalar_red_ct: reduction done at time: 10522.605
scalar_red_ct: dimension of the matrix: 6
scalar_red_ct: col. degrees of the matrix entries: 0 0 6 5 6
4
scalar_red_ct: relation found
scalar_red_ct: degree of the solution: 6
scalar_red_ct: dealing with monomial D[m]*D[nu] at time:
10522.732
scalar_red_ct: reduction done at time: 10533.158
scalar_red_ct: dimension of the matrix: 6
scalar_red_ct: col. degrees of the matrix entries: 0 0 6 5 6
4
scalar_red_ct: relation found
scalar_red_ct: degree of the solution: 8
scalar_red_ct: dealing with monomial D[mu]^2 at time:
10533.437
scalar_red_ct: reduction done at time: 10559.472
scalar_red_ct: dimension of the matrix: 6
scalar_red_ct: col. degrees of the matrix entries: 0 0 6 5 6
4
scalar_red_ct: relation found
scalar_red_ct: degree of the solution: 7
scalar_red_ct: dealing with monomial D[mu]*D[n] at time:
10559.640
scalar_red_ct: reduction done at time: 10576.089
scalar_red_ct: dimension of the matrix: 6
scalar_red_ct: col. degrees of the matrix entries: 0 0 6 5 6
4
scalar_red_ct: relation found
scalar_red_ct: degree of the solution: 8
scalar_red_ct: dealing with monomial D[mu]*D[nu] at time:
10576.342
scalar_red_ct: reduction done at time: 10592.542
scalar_red_ct: dimension of the matrix: 6
scalar_red_ct: col. degrees of the matrix entries: 0 0 6 5 6
4
```

```
scalar_red_ct: relation found
scalar_red_ct: degree of the solution: 5
scalar_red_ct: dealing with monomial D[n]^2 at time:
10592.639
scalar_red_ct: reduction done at time: 10605.031
scalar_red_ct: dimension of the matrix: 6
scalar_red_ct: col. degrees of the matrix entries: 0 0 6 5 6
4
scalar_red_ct: relation found
scalar_red_ct: degree of the solution: 5
scalar_red_ct: dealing with monomial D[n]*D[nu] at time:
10605.137
scalar_red_ct: reduction done at time: 10615.155
scalar_red_ct: dimension of the matrix: 6
scalar_red_ct: col. degrees of the matrix entries: 0 0 6 5 6
4
scalar_red_ct: relation found
scalar_red_ct: degree of the solution: 9
scalar_red_ct: dealing with monomial D[nu]^2 at time:
10615.331
scalar_red_ct: reduction done at time: 10631.434
scalar_red_ct: dimension of the matrix: 6
scalar_red_ct: col. degrees of the matrix entries: 0 0 6 5 6
4
scalar_red_ct: relation found
scalar_red_ct: degree of the solution: 7
scalar_red_ct: dim quotient: 6
scalar_red_ct: degree coeffs: 4
```

$$\Big[\ell^2 D_\ell^2 - 2\,\ell\,\mu\, D_\ell^2 + \textit{[...28 terms...]} + 6\,\nu - 2, \ell\, m\, D_\ell\, D_m + m^2\, D_\ell\, D_m + \textit{[...24 terms...]} - n - 2\,\nu, 4\,\mu^2\, D_\ell\, D_\mu - 4\,\mu\,\nu\, D_\ell\, D_\mu + \textit{[...21 terms...]} - D_m + D_n, n\, D_\ell\, D_n + 2\,\nu\, D_\nu + D_\ell\, D_n - n - 2\,\nu, 2\,\ell^2\,\nu\, D_\ell\, D_\nu - 4\,\ell\,\mu\,\nu\, D_\ell\, D_\nu + \textit{[...58 terms...]} + 2\,\nu\, D_n + 2\, D_n, \ell\, m^2\, D_m^2 + m^3\, D_m^2 + \textit{[...55 terms...]} - 2\,n - 4\,\nu, 4\,\mu^2\, D_m\, D_\mu - 4\,\mu\,\nu\, D_m\, D_\mu + \textit{[...27 terms...]} - D_m + D_n, \ell\, m\, n\, D_m\, D_n + 2\,\ell\, m\,\nu\, D_\nu + \textit{[...49 terms...]} - n - 2\,\nu, 2\,\ell\,\nu\, D_m\, D_\nu + 2\, m\,\nu\, D_m\, D_\nu + \textit{[...22 terms...]} - 4\,\nu\, D_m + D_n, 8\,\mu^3\, D_\mu^2 - 8\,\mu^2\,\nu\, D_\mu^2 + \textit{[...66 terms...]} - 2\,n - 4\,\nu, 4\,\mu^2\, n\, D_\mu\, D_n - 4\,\mu\, n\,\nu\, D_\mu\, D_n + \textit{[...40 terms...]} + n\, D_n - 2\,\nu\, D_m, 4\,\mu\,\nu\, D_\mu\, D_\nu + 2\,\ell\,\nu\, D_\nu + \textit{[...8 terms...]} - n - 2\,\nu, n^2\, D_n^2 + 4\, n\,\nu\, D_\nu + \textit{[...7 terms...]} - n - 2\,\nu, 2\,\ell^2\, n\,\nu\, D_n\, D_\nu - 4\,\ell\,\mu\, n\,\nu\, D_n\, D_\nu + \textit{[...139 terms...]} - 22\,\nu\, D_n - 5\, D_n, 4\,\ell^2\,\nu^2\, D_\nu^2 - 8\,\ell\,\mu\,\nu^2\, D_\nu^2 + \textit{[...117 terms...]} + 6\,n + 12\,\nu\Big]$$

165.381 **(5.3)**

Another integral with Gegenbauer polynomials:

```
> F:=Int((x+a)^(g+lambda-1)*(a-x)^(beta-1)*GegenbauerC(m,g,x/a)*
  GegenbauerC(n,lambda,x/a),x = -a .. a);
```

$$F := \int_{-a}^{a} (x+a)^{g+\lambda-1}\,(a-x)^{\beta-1}\,\mathrm{GegenbauerC}\left(m, g, \frac{x}{a}\right)\,\mathrm{GegenbauerC}\left(n, \lambda,\right. \qquad \textbf{(5.4)}$$

$\left.\frac{x}{a}\right) \mathrm{d}x$

```
> st:=time():
  redct(F,[m::shift, n::shift,a::diff,beta::shift]);
  time()-st;
redct: dimension 4
scalar_red_ct: dealing with monomial 1 at time: 11882.785
exceptionalbasis: entering exceptionalbasis at time 11882.785
exceptionalbasis: dimension of the exceptional set 2
exceptionalbasis: degrees of its elements [3 4]
exceptionalbasis: exceptional basis computed in  8.093 sec.
scalar_red_ct: reduction done at time: 11890.878
scalar_red_ct: dealing with monomial D[a] at time: 11890.878
scalar_red_ct: reduction done at time: 11890.878
scalar_red_ct: dimension of the matrix: 1
scalar_red_ct: col. degrees of the matrix entries: 0
scalar_red_ct: relation found
scalar_red_ct: degree of the solution: 0
scalar_red_ct: dealing with monomial D[beta] at time:
11890.880
scalar_red_ct: reduction done at time: 11890.880
scalar_red_ct: dealing with monomial D[m] at time: 11890.880
scalar_red_ct: reduction done at time: 11894.052
scalar_red_ct: dealing with monomial D[n] at time: 11894.054
scalar_red_ct: reduction done at time: 11897.329
scalar_red_ct: dimension of the matrix: 3
scalar_red_ct: col. degrees of the matrix entries: 0 1 6
scalar_red_ct: relation found
scalar_red_ct: degree of the solution: 5
scalar_red_ct: dealing with monomial D[beta]^2 at time:
11897.351
scalar_red_ct: reduction done at time: 11897.352
scalar_red_ct: dimension of the matrix: 3
scalar_red_ct: col. degrees of the matrix entries: 0 1 6
scalar_red_ct: relation found
scalar_red_ct: degree of the solution: 6
scalar_red_ct: dealing with monomial D[beta]*D[m] at time:
11897.363
scalar_red_ct: reduction done at time: 11898.531
scalar_red_ct: dimension of the matrix: 3
scalar_red_ct: col. degrees of the matrix entries: 0 1 6
scalar_red_ct: relation found
scalar_red_ct: degree of the solution: 7
scalar_red_ct: dealing with monomial D[m]^2 at time:
11898.576
scalar_red_ct: reduction done at time: 11934.147
scalar_red_ct: dimension of the matrix: 3
scalar_red_ct: col. degrees of the matrix entries: 0 1 6
scalar_red_ct: relation found
scalar_red_ct: degree of the solution: 8
scalar_red_ct: dim quotient: 3
scalar_red_ct: degree coeffs: 6
```

$$\Big[a\,D_a - \beta - g - \lambda + 1,\ a\,m\,D_m + a\,n\,D_n + [\ldots 10\ terms\ldots] + m\,D_\beta + n\,D_\beta,$$
$$-2\,a^2\,\beta^2\,g\,m\,D_m + 2\,a^2\,\beta^2\,\lambda\,m\,D_m + [\ldots 258\ terms\ldots] - 2\,a\,D_\beta + D_\beta^2,$$
$$-2\,a\,\beta^2\,m\,D_m - 4\,a\,\beta\,g\,m\,D_m + [\ldots 77\ terms\ldots] - a\,D_m + D_\beta\,D_m,\ a\,\beta^2\,m^2\,D_m^2$$
$$+ 2\,a\,\beta\,g\,m^2\,D_m^2 + [\ldots 169\ terms\ldots] - 2\,a\,g - a\,m\Big]$$

$$53.075 \tag{5.5}$$