Refined Product Formulas for Tamari Intervals

This Maple session complements the work by Alin Bostan, Frédéric Chyzak, and Vincent Pilaud (March 2023). It contains all calculations described in the arXiv prepublication (v1).

The present version, dated April 3, 2023, also contains and evolution of Remark 20 in the article (v1) with a conceptually simpler computational proof of Theorem 1.

```
> restart;
```

For some of the calculations, we will need a package by Frédéric Chyzak, Mgfun, which can be found at https://mathexp.eu/chyzak/.

```
> libname := "/home/chyzak/Mgfun.mla", libname:
```

[4.1 Theorem 1 by creative telescoping]

Goal

We are interested in the generating series A(t,z) in Q[z][[t]], whose coefficient with respect to t^n*z^k is:

```
> CF_A := 2 * binomial(n+1, k+2) * binomial(3*n, k) / (n * (n+1));
```

$$CF\_A := \frac{2 \binom{n+1}{k+2} \binom{3n}{k}}{n\,(n+1)} \qquad \textbf{(1.1.1)}$$

That is, the series is:

```
> A(t,z) = Sum(Sum(CF_A * t^n*z^k, k=0..n-1), n=0..infinity);
```

$$A(t,z) = \sum_{n=0}^{\infty} \sum_{k=0}^{n-1} \frac{2 \binom{n+1}{k+2} \binom{3n}{k} t^n z^k}{n\,(n+1)} \qquad \textbf{(1.1.2)}$$

We want to prove that this series is an algebraic series, solution of the polynomial:

```
> X^4*t^3*z^6 + X^3*t^3*z^6 + 6*X^3*t^3*z^5 - 3*X^3*t^3*z^4 +
  6*X^2*t^3*z^5 + 3*X^3*t^2*z^4 + 9*X^2*t^3*z^4 - 12*X^2*t^3*
  z^3 + 2*X^2*t^2*z^4 + 12*X*t^3*z^4 + 3*X^2*t^3*z^2 - 6*X^2*
  t^2*z^3 - 4*X*t^3*z^3 + 21*X^2*t^2*z^2 - 9*X*t^3*z^2 - 10*X*
  t^2*z^3 + 8*t^3*z^3 + 6*X*t^3*z + 3*X^2*t*z^2 + 26*X*t^2*z^2
  - 12*t^3*z^2 - X*t^3 + 6*X*t^2*z + 6*t^3*z + X*t*z^2 - t^2*
  z^2 + 3*X*t^2 - t^3 - 12*X*t*z + 10*t^2*z - 3*X*t + 2*t^2 +
  X - t:
  P := collect(%, X, factor);
```

$$P := X^4 t^3 z^6 + t^2 z^4 \left(t z^2 + 6 t z - 3 t + 3\right) X^3 + t z^2 \left(6 t^2 z^3 + 9 t^2 z^2 - 12 t^2 z + 2 t z^2 + 3 t^2 - 6 t z + 21 t + 3\right) X^2 + \left(12 t^3 z^4 - 4 t^3 z^3\right. \qquad \textbf{(1.1.3)}$$

$$K\ 9t^3z^2K\ 10t^2z^3+6t^3z+26t^2z^2K\ t^3+6t^2z+tz^2+3t^2K\ 12tz$$
$$K\ 3t+1)\,X+t\,(8t^2z^3K\ 12t^2z^2+6t^2zK\ tz^2K\ t^2+10tz+2tK\ 1)$$

Let us first check that there is no mistake in the posed problem, that is, that P(t,z,A(t,z)) = 0.

```
> N := 20:
  ser_A := add(add(CF_A*z^k*t^n, k=0..n-1), n=0..N):
  series(ser_A, t);
```

$$t+(2z+1)\,t^2+(6z^2+6z+1)\,t^3+(22z^3+33z^2+12z+1)\,t^4+(91z^4+182z^3+105z^2+20z+1)\,t^5+\mathrm{O}(t^6) \qquad \textbf{(1.1.4)}$$

```
> series(expand(subs(X = ser_A + O(t^(N+1)), P)), t, infinity)
  ;
```

$$\mathrm{O}(t^{21}) \qquad \textbf{(1.1.5)}$$

First step: compute a recurrence relation on the coefficients a[n](z) of the solution A(t,z) of P(t,z,X).

The gfun package is now classic. We use it to transform the polynomial equation into a differential equation, then into a recurrence equation on the coefficients with respect to t of any series solution.

```
> with(gfun);
```

[*Laplace, Parameters, algebraicsubs, algeqtodiffeq, algeqtoseries, algfuntoalgeq, borel, cauchyproduct, diffeq*diffeq, diffeq+diffeq, diffeqtohomdiffeq, diffeqtorec, guesseqn, guessgf, hadamardproduct, holexprtodiffeq, invborel, listtoalgeq, listtodiffeq, listtohypergeom, listtolist, listtoratpoly, listtorec, listtoseries, poltodiffeq, poltorec, ratpolytocoeff, rec*rec, rec+rec, rectodiffeq, rectohomrec, rectoproc, seriestoalgeq, seriestodiffeq, seriestohypergeom, seriestolist, seriestoratpoly, seriestorec, seriestoseries*] **(1.2.1)**

```
> deq_A := algeqtodiffeq(P, X(t));
```

$$deq\_A := K\ 24t^3z^7+276t^3z^6K\ 1152t^3z^5+2436t^3z^4K\ 2904t^3z^3 \qquad \textbf{(1.2.2)}$$
$$+228t^2z^4+1980t^3z^2K\ 1488t^2z^3K\ 720t^3z+3816t^2z^2+108t^3$$
$$K\ 3312t^2z+756t^2K\ 720tz+1080t+(12t^3z^7+6t^3z^6K\ 288t^3z^5$$
$$+906t^3z^4K\ 1284t^3z^3K\ 114t^2z^4+954t^3z^2+456t^2z^3K\ 360t^3z$$
$$+36t^2z^2+54t^3K\ 216t^2zK\ 342tz^2K\ 162t^2+540tz+162t$$
$$+36zK\ 54)\,X(t)+(360t^5z^9K\ 3420t^5z^8+14400t^5z^7K\ 240t^4z^8$$
$$K\ 35280t^5z^6+2628t^4z^7+55440t^5z^5K\ 13746t^4z^6+24t^3z^7$$
$$K\ 57960t^5z^4+36168t^4z^5K\ 276t^3z^6+40320t^5z^3K\ 48726t^4z^4$$
$$+3456t^3z^5K\ 18000t^5z^2+29604t^4z^3K\ 16338t^3z^4+4680t^5z$$

$$K\ 414\,t^4 z^2 + 33744\,t^3 z^3 K\ 228\,t^2 z^4 K\ 540\,t^5 K\ 7920\,t^4 z K\ 9648\,t^3 z^2$$
$$+ 1488\,t^2 z^3 + 2646\,t^4 K\ 6264\,t^3 z K\ 9522\,t^2 z^2 K\ 4698\,t^3 + 8820\,t^2 z$$
$$+ 3618\,t^2 + 684\,t\,z K\ 1026\,t) \left(\frac{d}{dt} X(t)\right) + (648\,t^6 z^9 K\ 6156\,t^6 z^8$$
$$+ 25920\,t^6 z^7 K\ 354\,t^5 z^8 K\ 63504\,t^6 z^6 + 3810\,t^5 z^7 + 99792\,t^6 z^5$$
$$K\ 21663\,t^5 z^6 + 36\,t^4 z^7 K\ 104328\,t^6 z^4 + 62112\,t^5 z^5 K\ 414\,t^4 z^6$$
$$+ 72576\,t^6 z^3 K\ 91497\,t^5 z^4 + 5184\,t^4 z^5 K\ 32400\,t^6 z^2 + 64602\,t^5 z^3$$
$$K\ 24165\,t^4 z^4 + 8424\,t^6 z K\ 11097\,t^5 z^2 + 49248\,t^4 z^3 K\ 348\,t^3 z^4$$
$$K\ 972\,t^6 K\ 10044\,t^5 z K\ 14580\,t^4 z^2 + 2304\,t^3 z^3 + 4131\,t^5$$
$$K\ 8748\,t^4 z K\ 11907\,t^3 z^2 K\ 6561\,t^4 + 9558\,t^3 z + 4617\,t^3 + 810\,t^2 z$$
$$K\ 1215\,t^2) \left(\frac{d^2}{dt^2} X(t)\right) + (162\,t^7 z^9 K\ 1539\,t^7 z^8 + 6480\,t^7 z^7$$
$$K\ 78\,t^6 z^8 K\ 15876\,t^7 z^6 + 834\,t^6 z^7 + 24948\,t^7 z^5 K\ 4986\,t^6 z^6$$
$$+ 8\,t^5 z^7 K\ 26082\,t^7 z^4 + 14946\,t^6 z^5 K\ 92\,t^5 z^6 + 18144\,t^7 z^3$$
$$K\ 22902\,t^6 z^4 + 1152\,t^5 z^5 K\ 8100\,t^7 z^2 + 17046\,t^6 z^3 K\ 5370\,t^5 z^4$$
$$+ 2106\,t^7 z K\ 3726\,t^6 z^2 + 10944\,t^5 z^3 K\ 78\,t^4 z^4 K\ 243\,t^7 K\ 2106\,t^6 z$$
$$K\ 3240\,t^5 z^2 + 522\,t^4 z^3 + 972\,t^6 K\ 1944\,t^5 z K\ 2430\,t^4 z^2 K\ 1458\,t^5$$
$$+ 1782\,t^4 z + 972\,t^4 + 162\,t^3 z K\ 243\,t^3) \left(\frac{d^3}{dt^3} X(t)\right)$$

```
> collect(deq_A, {diff, X}, factor);
```

$$(12\,t^3 z^7 + 6\,t^3 z^6 K\ 288\,t^3 z^5 + 906\,t^3 z^4 K\ 1284\,t^3 z^3 K\ 114\,t^2 z^4 \qquad \textbf{(1.2.3)}$$
$$+ 954\,t^3 z^2 + 456\,t^2 z^3 K\ 360\,t^3 z + 36\,t^2 z^2 + 54\,t^3 K\ 216\,t^2 z$$
$$K\ 342\,t\,z^2 K\ 162\,t^2 + 540\,t\,z + 162\,t + 36\,z K\ 54)\, X(t) + 6\,t\,(60\,t^4 z^9$$
$$K\ 570\,t^4 z^8 + 2400\,t^4 z^7 K\ 40\,t^3 z^8 K\ 5880\,t^4 z^6 + 438\,t^3 z^7 + 9240\,t^4 z^5$$
$$K\ 2291\,t^3 z^6 + 4\,t^2 z^7 K\ 9660\,t^4 z^4 + 6028\,t^3 z^5 K\ 46\,t^2 z^6 + 6720\,t^4 z^3$$
$$K\ 8121\,t^3 z^4 + 576\,t^2 z^5 K\ 3000\,t^4 z^2 + 4934\,t^3 z^3 K\ 2723\,t^2 z^4$$
$$+ 780\,t^4 z K\ 69\,t^3 z^2 + 5624\,t^2 z^3 K\ 38\,t\,z^4 K\ 90\,t^4 K\ 1320\,t^3 z$$
$$K\ 1608\,t^2 z^2 + 248\,t\,z^3 + 441\,t^3 K\ 1044\,t^2 z K\ 1587\,t\,z^2 K\ 783\,t^2$$
$$+ 1470\,t\,z + 603\,t + 114\,z K\ 171) \left(\frac{d}{dt} X(t)\right) + 3\,t^2\,(216\,t^4 z^9$$
$$K\ 2052\,t^4 z^8 + 8640\,t^4 z^7 K\ 118\,t^3 z^8 K\ 21168\,t^4 z^6 + 1270\,t^3 z^7$$
$$+ 33264\,t^4 z^5 K\ 7221\,t^3 z^6 + 12\,t^2 z^7 K\ 34776\,t^4 z^4 + 20704\,t^3 z^5$$
$$K\ 138\,t^2 z^6 + 24192\,t^4 z^3 K\ 30499\,t^3 z^4 + 1728\,t^2 z^5 K\ 10800\,t^4 z^2$$
$$+ 21534\,t^3 z^3 K\ 8055\,t^2 z^4 + 2808\,t^4 z K\ 3699\,t^3 z^2 + 16416\,t^2 z^3$$
$$K\ 116\,t\,z^4 K\ 324\,t^4 K\ 3348\,t^3 z K\ 4860\,t^2 z^2 + 768\,t\,z^3 + 1377\,t^3$$
$$K\ 2916\,t^2 z K\ 3969\,t\,z^2 K\ 2187\,t^2 + 3186\,t\,z + 1539\,t + 270\,z K\ 405)$$

$$\left(\frac{d^2}{dt^2}\,X(t)\right) + t^3\,(27\,t^2 z^4 \mathrm{K}\ 108\,t^2 z^3 + 162\,t^2 z^2 \mathrm{K}\ 4\,t\,z^3 \mathrm{K}\ 108\,t^2 z$$
$$+ 18\,t\,z^2 + 27\,t^2 \mathrm{K}\ 216\,t\,z \mathrm{K}\ 54\,t + 27)\,(6\,t^2 z^5 \mathrm{K}\ 33\,t^2 z^4 + 72\,t^2 z^3$$
$$\mathrm{K}\ 2\,t\,z^4 \mathrm{K}\ 78\,t^2 z^2 + 14\,t\,z^3 + 42\,t^2 z \mathrm{K}\ 36\,t\,z^2 \mathrm{K}\ 9\,t^2 + 6\,t\,z + 18\,t + 6\,z$$
$$\mathrm{K}\ 9)\,\left(\frac{d^3}{dt^3}\,X(t)\right) \mathrm{K}\ 12\,t\,(2\,t^2 z^7 \mathrm{K}\ 23\,t^2 z^6 + 96\,t^2 z^5 \mathrm{K}\ 203\,t^2 z^4$$
$$+ 242\,t^2 z^3 \mathrm{K}\ 19\,t\,z^4 \mathrm{K}\ 165\,t^2 z^2 + 124\,t\,z^3 + 60\,t^2 z \mathrm{K}\ 318\,t\,z^2 \mathrm{K}\ 9\,t^2$$
$$+ 276\,t\,z \mathrm{K}\ 63\,t + 60\,z \mathrm{K}\ 90)$$

```
> nops(expand(deq_A));
```

135 **(1.2.4)**

```
> sys_A := diffeqtorec(deq_A, X(t), a(n));
```

(1.2.5)

$$sys\_A := \{((162\,z^9 \mathrm{K}\ 1539\,z^8 + 6480\,z^7 \mathrm{K}\ 15876\,z^6 + 24948\,z^5$$
$$\mathrm{K}\ 26082\,z^4 + 18144\,z^3 \mathrm{K}\ 8100\,z^2 + 2106\,z \mathrm{K}\ 243)\,n^3 + (162\,z^9$$
$$\mathrm{K}\ 1539\,z^8 + 6480\,z^7 \mathrm{K}\ 15876\,z^6 + 24948\,z^5 \mathrm{K}\ 26082\,z^4 + 18144\,z^3$$
$$\mathrm{K}\ 8100\,z^2 + 2106\,z \mathrm{K}\ 243)\,n^2 + (36\,z^9 \mathrm{K}\ 342\,z^8 + 1440\,z^7 \mathrm{K}\ 3528\,z^6$$
$$+ 5544\,z^5 \mathrm{K}\ 5796\,z^4 + 4032\,z^3 \mathrm{K}\ 1800\,z^2 + 468\,z \mathrm{K}\ 54)\,n)\,a(n)$$
$$+ ((\mathrm{K}\,78\,z^8 + 834\,z^7 \mathrm{K}\ 4986\,z^6 + 14946\,z^5 \mathrm{K}\ 22902\,z^4 + 17046\,z^3$$
$$\mathrm{K}\ 3726\,z^2 \mathrm{K}\ 2106\,z + 972)\,n^3 + (\mathrm{K}\,354\,z^8 + 3810\,z^7 \mathrm{K}\ 21663\,z^6$$
$$+ 62112\,z^5 \mathrm{K}\ 91497\,z^4 + 64602\,z^3 \mathrm{K}\ 11097\,z^2 \mathrm{K}\ 10044\,z + 4131)$$
$$n^2 + (\mathrm{K}\,516\,z^8 + 5604\,z^7 \mathrm{K}\ 30423\,z^6 + 83334\,z^5 \mathrm{K}\ 117321\,z^4$$
$$+ 77160\,z^3 \mathrm{K}\ 7785\,z^2 \mathrm{K}\ 15858\,z + 5805)\,n \mathrm{K}\ 240\,z^8 + 2640\,z^7$$
$$\mathrm{K}\ 13740\,z^6 + 35880\,z^5 \mathrm{K}\ 47820\,z^4 + 28320\,z^3 + 540\,z^2 \mathrm{K}\ 8280\,z$$
$$+ 2700)\,a(n+1) + ((8\,z^7 \mathrm{K}\ 92\,z^6 + 1152\,z^5 \mathrm{K}\ 5370\,z^4 + 10944\,z^3$$
$$\mathrm{K}\ 3240\,z^2 \mathrm{K}\ 1944\,z \mathrm{K}\ 1458)\,n^3 + (60\,z^7 \mathrm{K}\ 690\,z^6 + 8640\,z^5$$
$$\mathrm{K}\ 40275\,z^4 + 82080\,z^3 \mathrm{K}\ 24300\,z^2 \mathrm{K}\ 14580\,z \mathrm{K}\ 10935)\,n^2$$
$$+ (148\,z^7 \mathrm{K}\ 1702\,z^6 + 21312\,z^5 \mathrm{K}\ 99573\,z^4 + 203376\,z^3 \mathrm{K}\ 59868\,z^2$$
$$\mathrm{K}\ 36396\,z \mathrm{K}\ 27297)\,n + 120\,z^7 \mathrm{K}\ 1380\,z^6 + 17280\,z^5 \mathrm{K}\ 81120\,z^4$$
$$+ 166440\,z^3 \mathrm{K}\ 48420\,z^2 \mathrm{K}\ 30240\,z \mathrm{K}\ 22680)\,a(n+2) + ((\mathrm{K}\,78\,z^4$$
$$+ 522\,z^3 \mathrm{K}\ 2430\,z^2 + 1782\,z + 972)\,n^3 + (\mathrm{K}\,816\,z^4 + 5436\,z^3$$
$$\mathrm{K}\ 26487\,z^2 + 20250\,z + 10449)\,n^2 + (\mathrm{K}\,2826\,z^4 + 18750\,z^3$$
$$\mathrm{K}\ 95787\,z^2 + 76212\,z + 37395)\,n \mathrm{K}\ 3240\,z^4 + 21420\,z^3$$
$$\mathrm{K}\ 114930\,z^2 + 95040\,z + 44550)\,a(n+3) + ((162\,z \mathrm{K}\ 243)\,n^3$$
$$+ (2268\,z \mathrm{K}\ 3402)\,n^2 + (10566\,z \mathrm{K}\ 15849)\,n + 16380\,z \mathrm{K}\ 24570)$$
$$a(n+4),\ a(0) = 0,\ a(1) = 1,\ a(2) = 2\,z + 1,\ a(3) = 6\,z^2 + 6\,z + 1\}$$

```
> rec_A := collect(op(remove(type, sys_A, equation)), a,
  factor);
```

$$rec\_A := 9\,n\,(2\,z \mathrm{K}\ 3)\,(z \mathrm{K}\ 1)^8\,(3\,n + 2)\,(3\,n + 1)\,a(n) \mathrm{K}\ 3\,(z$$ **(1.2.6)**

$- 1)^4 (26 n^3 z^4 - 174 n^3 z^3 + 118 n^2 z^4 + 810 n^3 z^2 - 798 n^2 z^3$
$+ 172 n z^4 - 594 n^3 z + 3321 n^2 z^2 - 1180 n z^3 + 80 z^4 - 324 n^3$
$- 2160 n^2 z + 4389 n z^2 - 560 z^3 - 1377 n^2 - 2454 z n + 1860 z^2$
$- 1935 n - 840 z - 900) a(n+1) + (z - 1) (2 n + 5) (4 n^2 z^6$
$- 42 n^2 z^5 + 20 n z^6 + 534 n^2 z^4 - 210 n z^5 + 24 z^6 - 2151 n^2 z^3$
$+ 2670 n z^4 - 252 z^5 + 3321 n^2 z^2 - 10755 n z^3 + 3204 z^4$
$+ 1701 n^2 z + 16605 n z^2 - 13020 z^3 + 729 n^2 + 8505 z n$
$+ 20268 z^2 + 3645 n + 10584 z + 4536) a(n+2) + (- 78 n^3 z^4$
$+ 522 n^3 z^3 - 816 n^2 z^4 - 2430 n^3 z^2 + 5436 n^2 z^3 - 2826 n z^4$
$+ 1782 n^3 z - 26487 n^2 z^2 + 18750 n z^3 - 3240 z^4 + 972 n^3$
$+ 20250 n^2 z - 95787 n z^2 + 21420 z^3 + 10449 n^2 + 76212 z n$
$- 114930 z^2 + 37395 n + 95040 z + 44550) a(n+3) + 9 (n$
$+ 5) (3 n + 14) (3 n + 13) (2 z - 3) a(n+4)$

To fix ideas, let us numerically verify that this recurrence relation is indeed satisfied for a few values of n (from 10 to 15):

```
> seq(normal(eval(subs(a = unapply(coeff(ser_A, t, n), n),
  subs(n=i, rec_A)))), i=10..15);
```

$$0, 0, 0, 0, 0, 0 \qquad (1.2.7)$$

Second step: compute a recurrence relation on the polynomials aa[n](z) = Sum(CF_A * z^k, k=0..n-1).

We use Zeilberger's algorithm for the computation of sums. On our instance, the output is a bit large. It is a pair [T(n,z,sn), C(n,k,z)] consisting of a skew polynomial T in Q(n,z)<sn; sn*n = (n+1)*sn>, the "telescoper", and of an expression C(n,k,z) in Q(n,k,z)*CF_A*z^k, the "certificate", satisfying a specific recurrence relation (to be provided below).

```
> zb_A := SumTools:-Hypergeometric:-Zeilberger(CF_A * z^k, n,
  k, sn):
  zb_A := applyop(collect, 1, zb_A, sn, factor):
  zb_A := applyop(factor, 2, zb_A);
```

$$zb\_A := \Big[3 (3 n + 7) (n + 3) (3 n + 8) (n^2 z^2 - 6 n^2 z + 2 n z^2 - 27 n^2 \qquad (1.3.1)$$

$- 12 z n - 54 n - 30) sn^2 - (2 n + 3) (2 n^4 z^5 - 21 n^4 z^4 + 12 n^3 z^5$

$+ 108 n^4 z^3 - 126 n^3 z^4 + 22 n^2 z^5 - 378 n^4 z^2 + 648 n^3 z^3 - 231 n^2 z^4$

$$+ 12\, n\, z^5 \mathrm{K}\ 3078\, n^4\, z \mathrm{K}\ 2268\, n^3\, z^2 + 1188\, n^2\, z^3 \mathrm{K}\ 126\, n\, z^4 \mathrm{K}\ 729\, n^4$$

$$\mathrm{K}\ 18468\, n^3\, z \mathrm{K}\ 4188\, n^2\, z^2 + 648\, n\, z^3 \mathrm{K}\ 4374\, n^3 \mathrm{K}\ 39078\, n^2\, z$$

$$\mathrm{K}\ 2358\, n\, z^2 \mathrm{K}\ 10449\, n^2 \mathrm{K}\ 34128\, z\, n \mathrm{K}\ 11664\, n \mathrm{K}\ 10080\, z \mathrm{K}\ 5040)$$

$$sn + 3\, n\, (z \mathrm{K}\ 1)^4\, (3\, n + 2)\, (3\, n + 1)\, (n^2\, z^2 \mathrm{K}\ 6\, n^2\, z + 4\, n\, z^2 \mathrm{K}\ 27\, n^2$$

$$\mathrm{K}\ 24\, z\, n + 3\, z^2 \mathrm{K}\ 108\, n \mathrm{K}\ 18\, z \mathrm{K}\ 111),\ \left(6\, z^k \binom{3\, n}{k} \binom{n+1}{k+2}\, k\, (k\right.$$

$$+ 2)\, (3\, n + 2)\, (3\, n + 1)\, (k^6\, n^2\, z^5 \mathrm{K}\ 16\, k^5\, n^3\, z^5 + 105\, k^4\, n^4\, z^5$$

$$\mathrm{K}\ 360\, k^3\, n^5\, z^5 + 675\, k^2\, n^6\, z^5 \mathrm{K}\ 648\, k\, n^7\, z^5 + 243\, n^8\, z^5 \mathrm{K}\ 9\, k^6\, n^2\, z^4$$

$$+ 4\, k^6\, n\, z^5 + 148\, k^5\, n^3\, z^4 \mathrm{K}\ 85\, k^5\, n^2\, z^5 \mathrm{K}\ 996\, k^4\, n^4\, z^4 + 695\, k^4\, n^3\, z^5$$

$$+ 3492\, k^3\, n^5\, z^4 \mathrm{K}\ 2850\, k^3\, n^4\, z^5 \mathrm{K}\ 6673\, k^2\, n^6\, z^4 + 6210\, k^2\, n^5\, z^5$$

$$+ 6504\, k\, n^7\, z^4 \mathrm{K}\ 6777\, k\, n^6\, z^5 \mathrm{K}\ 2466\, n^8\, z^4 + 2835\, n^7\, z^5 \mathrm{K}\ 6\, k^6\, n^2\, z^3$$

$$\mathrm{K}\ 36\, k^6\, n\, z^4 + 3\, k^6\, z^5 + 64\, k^5\, n^3\, z^3 + 784\, k^5\, n^2\, z^4 \mathrm{K}\ 132\, k^5\, n\, z^5$$

$$\mathrm{K}\ 206\, k^4\, n^4\, z^3 \mathrm{K}\ 6572\, k^4\, n^3\, z^4 + 1590\, k^4\, n^2\, z^5 \mathrm{K}\ 8\, k^3\, n^5\, z^3$$

$$+ 27582\, k^3\, n^4\, z^4 \mathrm{K}\ 8510\, k^3\, n^3\, z^5 + 1245\, k^2\, n^6\, z^3 \mathrm{K}\ 61324\, k^2\, n^5\, z^4$$

$$+ 22725\, k^2\, n^4\, z^5 \mathrm{K}\ 2196\, k\, n^7\, z^3 + 68018\, k\, n^6\, z^4 \mathrm{K}\ 29214\, k\, n^5\, z^5$$

$$+ 1107\, n^8\, z^3 \mathrm{K}\ 28788\, n^7\, z^4 + 13986\, n^6\, z^5 + 62\, k^6\, n^2\, z^2 \mathrm{K}\ 24\, k^6\, n\, z^3$$

$$\mathrm{K}\ 27\, k^6\, z^4 \mathrm{K}\ 1048\, k^5\, n^3\, z^2 + 358\, k^5\, n^2\, z^3 + 1212\, k^5\, n\, z^4 \mathrm{K}\ 63\, k^5\, z^5$$

$$+ 7112\, k^4\, n^4\, z^2 \mathrm{K}\ 1542\, k^4\, n^3\, z^3 \mathrm{K}\ 14972\, k^4\, n^2\, z^4 + 1525\, k^4\, n\, z^5$$

$$\mathrm{K}\ 24720\, k^3\, n^5\, z^2 + 524\, k^3\, n^4\, z^3 + 82130\, k^3\, n^3\, z^4 \mathrm{K}\ 12125\, k^3\, n^2\, z^5$$

$$+ 46053\, k^2 n^6 z^2 + 10785\, k^2 n^5 z^3 \mathrm{K}\; 224156\, k^2 n^4 z^4 + 42625\, k^2 n^3 z^5$$

$$\mathrm{K}\; 42768\, k\, n^7 z^2 \mathrm{K}\; 22935\, k\, n^6 z^3 + 293282\, k\, n^5 z^4 \mathrm{K}\; 67560\, k\, n^4 z^5$$

$$+ 14580\, n^8 z^2 + 13104\, n^7 z^3 \mathrm{K}\; 142173\, n^6 z^4 + 38214\, n^5 z^5$$

$$\mathrm{K}\; 75\, k^6 n^2 z + 248\, k^6 n\, z^2 \mathrm{K}\; 48\, k^6 z^3 + 1392\, k^5 n^3 z \mathrm{K}\; 5536\, k^5 n^2 z^2$$

$$+ 1080\, k^5 n\, z^3 + 576\, k^5 z^4 \mathrm{K}\; 10659\, k^4 n^4 z + 46884\, k^4 n^3 z^2$$

$$\mathrm{K}\; 7222\, k^4 n^2 z^3 \mathrm{K}\; 14292\, k^4 n\, z^4 + 525\, k^4 z^5 + 41904\, k^3 n^5 z$$

$$\mathrm{K}\; 195460\, k^3 n^4 z^2 + 14484\, k^3 n^3 z^3 + 116660\, k^3 n^2 z^4 \mathrm{K}\; 8310\, k^3 n\, z^5$$

$$\mathrm{K}\; 86508\, k^2 n^6 z + 423945\, k^2 n^5 z^2 + 16691\, k^2 n^4 z^3 \mathrm{K}\; 420030\, k^2 n^3 z^4$$

$$+ 43424\, k^2 n^2 z^5 + 84564\, k\, n^7 z \mathrm{K}\; 447444\, k\, n^6 z^2 \mathrm{K}\; 80131\, k\, n^5 z^3$$

$$+ 678748\, k\, n^4 z^4 \mathrm{K}\; 90786\, k\, n^3 z^5 \mathrm{K}\; 26244\, n^8 z + 169128\, n^7 z^2$$

$$+ 58926\, n^6 z^3 \mathrm{K}\; 389130\, n^5 z^4 + 63387\, n^4 z^5 + 27\, k^6 n^2 \mathrm{K}\; 300\, k^6 n\, z$$

$$+ 276\, k^6 z^2 \mathrm{K}\; 540\, k^5 n^3 + 7287\, k^5 n^2 z \mathrm{K}\; 10080\, k^5 n\, z^2 + 936\, k^5 z^3$$

$$+ 4644\, k^4 n^4 \mathrm{K}\; 69273\, k^4 n^3 z + 117624\, k^4 n^2 z^2 \mathrm{K}\; 12732\, k^4 n\, z^3$$

$$\mathrm{K}\; 4896\, k^4 z^4 \mathrm{K}\; 22356\, k^3 n^5 + 327318\, k^3 n^4 z \mathrm{K}\; 622600\, k^3 n^3 z^2$$

$$+ 50834\, k^3 n^2 z^3 + 79698\, k^3 n\, z^4 \mathrm{K}\; 2205\, k^3 z^5 + 64152\, k^2 n^6$$

$$\mathrm{K}\; 791775\, k^2 n^5 z + 1631197\, k^2 n^4 z^2 \mathrm{K}\; 40203\, k^2 n^3 z^3$$

$$\mathrm{K}\; 427618\, k^2 n^2 z^4 + 22861\, k^2 n\, z^5 \mathrm{K}\; 104976\, k\, n^7 + 885006\, k\, n^6 z$$

$$\mathrm{K}\; 2008236\, k\, n^5 z^2 \mathrm{K}\; 109619\, k\, n^4 z^3 + 913468\, k\, n^3 z^4 \mathrm{K}\; 71043\, k\, n^2 z^5$$

$$+ 78732\, n^8 \mathrm{K}\; 302535\, n^7 z + 859311\, n^6 z^2 + 131844\, n^5 z^3$$

$$\mathrm{K}\; 647172\, n^4 z^4 + 65475\, n^3 z^5 + 108\, k^6 n \mathrm{K}\; 315\, k^6 z \mathrm{K}\; 2808\, k^5 n^2$$

$$+ 12732\, k^5\, n\, z \,\mathrm{K}\, 6012\, k^5\, z^2 + 29808\, k^4\, n^3 \,\mathrm{K}\, 167628\, k^4\, n^2\, z$$

$$+ 129704\, k^4\, n\, z^2 \,\mathrm{K}\, 6996\, k^4\, z^3 \,\mathrm{K}\, 171450\, k^3\, n^4 + 1013538\, k^3\, n^3\, z$$

$$\mathrm{K}\, 982920\, k^3\, n^2\, z^2 + 61872\, k^3\, n\, z^3 + 21078\, k^3\, z^4 + 576639\, k^2\, n^5$$

$$\mathrm{K}\, 2994270\, k^2\, n^4\, z + 3321975\, k^2\, n^3\, z^2 \,\mathrm{K}\, 136502\, k^2\, n^2\, z^3$$

$$\mathrm{K}\, 225090\, k^2\, n\, z^4 + 4872\, k^2\, z^5 \,\mathrm{K}\, 1093500\, k\, n^6 + 3945483\, k\, n^5\, z$$

$$\mathrm{K}\, 4974200\, k\, n^4\, z^2 \,\mathrm{K}\, 13357\, k\, n^3\, z^3 + 716688\, k\, n^2\, z^4 \,\mathrm{K}\, 30024\, k\, n\, z^5$$

$$+ 944784\, n^7 \,\mathrm{K}\, 1526040\, n^6\, z + 2484891\, n^5\, z^2 + 154383\, n^4\, z^3$$

$$\mathrm{K}\, 671100\, n^3\, z^4 + 41184\, n^2\, z^5 + 111\, k^6 \,\mathrm{K}\, 4812\, k^5\, n + 7227\, k^5\, z$$

$$+ 70608\, k^4\, n^2 \,\mathrm{K}\, 176733\, k^4\, n\, z + 52152\, k^4\, z^2 \,\mathrm{K}\, 517698\, k^3\, n^3$$

$$+ 1543083\, k^3\, n^2\, z \,\mathrm{K}\, 759900\, k^3\, n\, z^2 + 24840\, k^3\, z^3 + 2128005\, k^2\, n^4$$

$$\mathrm{K}\, 5957676\, k^2\, n^3\, z + 3743200\, k^2\, n^2\, z^2 \,\mathrm{K}\, 130884\, k^2\, n\, z^3$$

$$\mathrm{K}\, 47997\, k^2\, z^4 \,\mathrm{K}\, 4815288\, k\, n^5 + 9681987\, k\, n^4\, z \,\mathrm{K}\, 7294564\, k\, n^3\, z^2$$

$$+ 111078\, k\, n^2\, z^3 + 304146\, k\, n\, z^4 \,\mathrm{K}\, 5292\, k\, z^5 + 4895964\, n^6$$

$$\mathrm{K}\, 4400892\, n^5\, z + 4453731\, n^4\, z^2 + 83076\, n^3\, z^3 \,\mathrm{K}\, 424485\, n^2\, z^4$$

$$+ 14436\, n\, z^5 \,\mathrm{K}\, 2664\, k^5 + 72528\, k^4\, n \,\mathrm{K}\, 67869\, k^4\, z \,\mathrm{K}\, 765708\, k^3\, n^2$$

$$+ 1147854\, k^3\, n\, z \,\mathrm{K}\, 228240\, k^3\, z^2 + 4116501\, k^2\, n^3 \,\mathrm{K}\, 6549708\, k^2\, n^2\, z$$

$$+ 2198132\, k^2\, n\, z^2 \,\mathrm{K}\, 41556\, k^2\, z^3 \,\mathrm{K}\, 11607570\, k\, n^4$$

$$+ 14090571\, k\, n^3\, z \,\mathrm{K}\, 6301640\, k\, n^2\, z^2 + 101352\, k\, n\, z^3 + 53946\, k\, z^4$$

$$+ 14306625\, n^5 \,\mathrm{K}\, 7949322\, n^4\, z + 5052393\, n^3\, z^2 + 2304\, n^2\, z^3$$

$$\mathrm{K}\, 149958\, n\, z^4 + 2160\, z^5 + 27084\, k^4 \,\mathrm{K}\, 552582\, k^3\, n + 332145\, k^3\, z$$

$+ 4393572\, k^2 n^2 \mathrm{K}\ 3760251\, k^2 n z + 523212\, k^2 z^2 \mathrm{K}\ 16532586\, k n^3$

$+ 12146199\, k n^2 z \mathrm{K}\ 2959764\, k n z^2 + 27144\, k z^3 + 25788699\, n^4$

$\mathrm{K}\ 9237861\, n^3 z + 3539610\, n^2 z^2 \mathrm{K}\ 15624\, n z^3 \mathrm{K}\ 22680\, z^4$

$\mathrm{K}\ 155178\, k^3 + 2449704\, k^2 n \mathrm{K}\ 878976\, k^2 z \mathrm{K}\ 13912644\, k n^2$

$+ 5742162\, k n z \mathrm{K}\ 582228\, k z^2 + 29384127\, n^3 \mathrm{K}\ 6772266\, n^2 z$

$+ 1401660\, n z^2 \mathrm{K}\ 4320\, z^3 + 557085\, k^2 \mathrm{K}\ 6409002\, k n$

$+ 1149948\, k z + 20691837\, n^2 \mathrm{K}\ 2874744\, z n + 240840\, z^2$

$\mathrm{K}\ 1248318\, k + 8246376\, n \mathrm{K}\ 542160\, z + 1426680)) \Big/ ((n+1)\, ($

$\mathrm{K}\ 3 n \mathrm{K}\ 6 + k)\, (\mathrm{K}\ 3 n \mathrm{K}\ 5 + k)\, (\mathrm{K}\ 3 n + k \mathrm{K}\ 4)\, (\mathrm{K}\ n + k \mathrm{K}\ 1)\, (\mathrm{K}\ n$
$+ k)\, (\mathrm{K}\ 3 n + k \mathrm{K}\ 1)\, (\mathrm{K}\ 3 n + k \mathrm{K}\ 2)\, (\mathrm{K}\ 3 n + k \mathrm{K}\ 3))]$

```
> reczb_A := add(coeff(zb_A[1], sn, i) * aa[n+i,k], i=0..2);
```

$reczb\_A := 3 n (z \mathrm{K}\ 1)^4 (3 n + 2)(3 n + 1)(n^2 z^2 \mathrm{K}\ 6 n^2 z + 4 n z^2 \mathrm{K}\ 27 n^2$ **(1.3.2)**
$\mathrm{K}\ 24 z n + 3 z^2 \mathrm{K}\ 108 n \mathrm{K}\ 18 z \mathrm{K}\ 111)\, aa_{n,k} \mathrm{K}\ (2 n + 3)(2 n^4 z^5$
$\mathrm{K}\ 21 n^4 z^4 + 12 n^3 z^5 + 108 n^4 z^3 \mathrm{K}\ 126 n^3 z^4 + 22 n^2 z^5 \mathrm{K}\ 378 n^4 z^2$
$+ 648 n^3 z^3 \mathrm{K}\ 231 n^2 z^4 + 12 n z^5 \mathrm{K}\ 3078 n^4 z \mathrm{K}\ 2268 n^3 z^2$
$+ 1188 n^2 z^3 \mathrm{K}\ 126 n z^4 \mathrm{K}\ 729 n^4 \mathrm{K}\ 18468 n^3 z \mathrm{K}\ 4188 n^2 z^2$
$+ 648 n z^3 \mathrm{K}\ 4374 n^3 \mathrm{K}\ 39078 n^2 z \mathrm{K}\ 2358 n z^2 \mathrm{K}\ 10449 n^2$
$\mathrm{K}\ 34128 z n \mathrm{K}\ 11664 n \mathrm{K}\ 10080 z \mathrm{K}\ 5040)\, aa_{n+1,k} + 3 (3 n + 7)(n$
$+ 3)(3 n + 8)(n^2 z^2 \mathrm{K}\ 6 n^2 z + 2 n z^2 \mathrm{K}\ 27 n^2 \mathrm{K}\ 12 z n \mathrm{K}\ 54 n$
$\mathrm{K}\ 30)\, aa_{n+2,k}$

This provides the eta[i](n) given in the article:

```
> for i from 2 to 0 by -1 do eta[i](n) = coeff(zb_A[1], sn, i)
  od;
```

$\eta_2(n) = 3 (3 n + 7)(n + 3)(3 n + 8)(n^2 z^2 \mathrm{K}\ 6 n^2 z + 2 n z^2 \mathrm{K}\ 27 n^2$
$\mathrm{K}\ 12 z n \mathrm{K}\ 54 n \mathrm{K}\ 30)$

$\eta_1(n) = \mathrm{K}\ (2 n + 3)(2 n^4 z^5 \mathrm{K}\ 21 n^4 z^4 + 12 n^3 z^5 + 108 n^4 z^3 \mathrm{K}\ 126 n^3 z^4$
$+ 22 n^2 z^5 \mathrm{K}\ 378 n^4 z^2 + 648 n^3 z^3 \mathrm{K}\ 231 n^2 z^4 + 12 n z^5 \mathrm{K}\ 3078 n^4 z$
$\mathrm{K}\ 2268 n^3 z^2 + 1188 n^2 z^3 \mathrm{K}\ 126 n z^4 \mathrm{K}\ 729 n^4 \mathrm{K}\ 18468 n^3 z$
$\mathrm{K}\ 4188 n^2 z^2 + 648 n z^3 \mathrm{K}\ 4374 n^3 \mathrm{K}\ 39078 n^2 z \mathrm{K}\ 2358 n z^2$
$\mathrm{K}\ 10449 n^2 \mathrm{K}\ 34128 z n \mathrm{K}\ 11664 n \mathrm{K}\ 10080 z \mathrm{K}\ 5040)$

(1.3.3)

$$\eta_0(n) = 3\,n\,(z-1)^4\,(3\,n+2)\,(3\,n+1)\,(n^2\,z^2-6\,n^2\,z+4\,n\,z^2-27\,n^2 - 24\,z\,n+3\,z^2-108\,n-18\,z-111) \qquad \textbf{(1.3.3)}$$

The algorithm justifies the following recurrence relation involving a forward finite operator Delta[k]:

```
> reczb_A = Delta[k](factor(zb_A[2]/CF_A/z^k) * aa[n,k]);
```

$$\begin{aligned}
&3\,n\,(z-1)^4\,(3\,n+2)\,(3\,n+1)\,(n^2\,z^2-6\,n^2\,z+4\,n\,z^2-27\,n^2-24\,z\,n \qquad \textbf{(1.3.4)}\\
&\quad +3\,z^2-108\,n-18\,z-111)\,aa_{n,k}-(2\,n+3)\,(2\,n^4\,z^5-21\,n^4\,z^4\\
&\quad +12\,n^3\,z^5+108\,n^4\,z^3-126\,n^3\,z^4+22\,n^2\,z^5-378\,n^4\,z^2+648\,n^3\,z^3\\
&\quad -231\,n^2\,z^4+12\,n\,z^5-3078\,n^4\,z-2268\,n^3\,z^2+1188\,n^2\,z^3\\
&\quad -126\,n\,z^4-729\,n^4-18468\,n^3\,z-4188\,n^2\,z^2+648\,n\,z^3-4374\,n^3\\
&\quad -39078\,n^2\,z-2358\,n\,z^2-10449\,n^2-34128\,z\,n-11664\,n\\
&\quad -10080\,z-5040)\,aa_{n+1,k}+3\,(3\,n+7)\,(n+3)\,(3\,n+8)\,(n^2\,z^2\\
&\quad -6\,n^2\,z+2\,n\,z^2-27\,n^2-12\,z\,n-54\,n-30)\,aa_{n+2,k}\\
&\quad =\Delta_k\big((3\,k\,(k+2)\,(3\,n+2)\,(3\,n+1)\,(k^6\,n^2\,z^5-16\,k^5\,n^3\,z^5\\
&\quad +105\,k^4\,n^4\,z^5-360\,k^3\,n^5\,z^5+675\,k^2\,n^6\,z^5-648\,k\,n^7\,z^5+243\,n^8\,z^5\\
&\quad -9\,k^6\,n^2\,z^4+4\,k^6\,n\,z^5+148\,k^5\,n^3\,z^4-85\,k^5\,n^2\,z^5-996\,k^4\,n^4\,z^4\\
&\quad +695\,k^4\,n^3\,z^5+3492\,k^3\,n^5\,z^4-2850\,k^3\,n^4\,z^5-6673\,k^2\,n^6\,z^4\\
&\quad +6210\,k^2\,n^5\,z^5+6504\,k\,n^7\,z^4-6777\,k\,n^6\,z^5-2466\,n^8\,z^4\\
&\quad +2835\,n^7\,z^5-6\,k^6\,n^2\,z^3-36\,k^6\,n\,z^4+3\,k^6\,z^5+64\,k^5\,n^3\,z^3\\
&\quad +784\,k^5\,n^2\,z^4-132\,k^5\,n\,z^5-206\,k^4\,n^4\,z^3-6572\,k^4\,n^3\,z^4\\
&\quad +1590\,k^4\,n^2\,z^5-8\,k^3\,n^5\,z^3+27582\,k^3\,n^4\,z^4-8510\,k^3\,n^3\,z^5\\
&\quad +1245\,k^2\,n^6\,z^3-61324\,k^2\,n^5\,z^4+22725\,k^2\,n^4\,z^5-2196\,k\,n^7\,z^3\\
&\quad +68018\,k\,n^6\,z^4-29214\,k\,n^5\,z^5+1107\,n^8\,z^3-28788\,n^7\,z^4\\
&\quad +13986\,n^6\,z^5+62\,k^6\,n^2\,z^2-24\,k^6\,n\,z^3-27\,k^6\,z^4-1048\,k^5\,n^3\,z^2\\
&\quad +358\,k^5\,n^2\,z^3+1212\,k^5\,n\,z^4-63\,k^5\,z^5+7112\,k^4\,n^4\,z^2\\
&\quad -1542\,k^4\,n^3\,z^3-14972\,k^4\,n^2\,z^4+1525\,k^4\,n\,z^5-24720\,k^3\,n^5\,z^2\\
&\quad +524\,k^3\,n^4\,z^3+82130\,k^3\,n^3\,z^4-12125\,k^3\,n^2\,z^5+46053\,k^2\,n^6\,z^2\\
&\quad +10785\,k^2\,n^5\,z^3-224156\,k^2\,n^4\,z^4+42625\,k^2\,n^3\,z^5-42768\,k\,n^7\,z^2\\
&\quad -22935\,k\,n^6\,z^3+293282\,k\,n^5\,z^4-67560\,k\,n^4\,z^5+14580\,n^8\,z^2\\
&\quad +13104\,n^7\,z^3-142173\,n^6\,z^4+38214\,n^5\,z^5-75\,k^6\,n^2\,z\\
&\quad +248\,k^6\,n\,z^2-48\,k^6\,z^3+1392\,k^5\,n^3\,z-5536\,k^5\,n^2\,z^2+1080\,k^5\,n\,z^3\\
&\quad +576\,k^5\,z^4-10659\,k^4\,n^4\,z+46884\,k^4\,n^3\,z^2-7222\,k^4\,n^2\,z^3\\
&\quad -14292\,k^4\,n\,z^4+525\,k^4\,z^5+41904\,k^3\,n^5\,z-195460\,k^3\,n^4\,z^2\\
&\quad +14484\,k^3\,n^3\,z^3+116660\,k^3\,n^2\,z^4-8310\,k^3\,n\,z^5-86508\,k^2\,n^6\,z
\end{aligned}$$

$$+ 423945\, k^2 n^5 z^2 + 16691\, k^2 n^4 z^3 \mathrm{K}\; 420030\, k^2 n^3 z^4$$
$$+ 43424\, k^2 n^2 z^5 + 84564\, k\, n^7 z \mathrm{K}\; 447444\, k\, n^6 z^2 \mathrm{K}\; 80131\, k\, n^5 z^3$$
$$+ 678748\, k\, n^4 z^4 \mathrm{K}\; 90786\, k\, n^3 z^5 \mathrm{K}\; 26244\, n^8 z + 169128\, n^7 z^2$$
$$+ 58926\, n^6 z^3 \mathrm{K}\; 389130\, n^5 z^4 + 63387\, n^4 z^5 + 27\, k^6 n^2 \mathrm{K}\; 300\, k^6 n\, z$$
$$+ 276\, k^6 z^2 \mathrm{K}\; 540\, k^5 n^3 + 7287\, k^5 n^2 z \mathrm{K}\; 10080\, k^5 n\, z^2 + 936\, k^5 z^3$$
$$+ 4644\, k^4 n^4 \mathrm{K}\; 69273\, k^4 n^3 z + 117624\, k^4 n^2 z^2 \mathrm{K}\; 12732\, k^4 n\, z^3$$
$$\mathrm{K}\; 4896\, k^4 z^4 \mathrm{K}\; 22356\, k^3 n^5 + 327318\, k^3 n^4 z \mathrm{K}\; 622600\, k^3 n^3 z^2$$
$$+ 50834\, k^3 n^2 z^3 + 79698\, k^3 n\, z^4 \mathrm{K}\; 2205\, k^3 z^5 + 64152\, k^2 n^6$$
$$\mathrm{K}\; 791775\, k^2 n^5 z + 1631197\, k^2 n^4 z^2 \mathrm{K}\; 40203\, k^2 n^3 z^3$$
$$\mathrm{K}\; 427618\, k^2 n^2 z^4 + 22861\, k^2 n\, z^5 \mathrm{K}\; 104976\, k\, n^7 + 885006\, k\, n^6 z$$
$$\mathrm{K}\; 2008236\, k\, n^5 z^2 \mathrm{K}\; 109619\, k\, n^4 z^3 + 913468\, k\, n^3 z^4 \mathrm{K}\; 71043\, k\, n^2 z^5$$
$$+ 78732\, n^8 \mathrm{K}\; 302535\, n^7 z + 859311\, n^6 z^2 + 131844\, n^5 z^3$$
$$\mathrm{K}\; 647172\, n^4 z^4 + 65475\, n^3 z^5 + 108\, k^6 n \mathrm{K}\; 315\, k^6 z \mathrm{K}\; 2808\, k^5 n^2$$
$$+ 12732\, k^5 n\, z \mathrm{K}\; 6012\, k^5 z^2 + 29808\, k^4 n^3 \mathrm{K}\; 167628\, k^4 n^2 z$$
$$+ 129704\, k^4 n\, z^2 \mathrm{K}\; 6996\, k^4 z^3 \mathrm{K}\; 171450\, k^3 n^4 + 1013538\, k^3 n^3 z$$
$$\mathrm{K}\; 982920\, k^3 n^2 z^2 + 61872\, k^3 n\, z^3 + 21078\, k^3 z^4 + 576639\, k^2 n^5$$
$$\mathrm{K}\; 2994270\, k^2 n^4 z + 3321975\, k^2 n^3 z^2 \mathrm{K}\; 136502\, k^2 n^2 z^3$$
$$\mathrm{K}\; 225090\, k^2 n\, z^4 + 4872\, k^2 z^5 \mathrm{K}\; 1093500\, k\, n^6 + 3945483\, k\, n^5 z$$
$$\mathrm{K}\; 4974200\, k\, n^4 z^2 \mathrm{K}\; 13357\, k\, n^3 z^3 + 716688\, k\, n^2 z^4 \mathrm{K}\; 30024\, k\, n\, z^5$$
$$+ 944784\, n^7 \mathrm{K}\; 1526040\, n^6 z + 2484891\, n^5 z^2 + 154383\, n^4 z^3$$
$$\mathrm{K}\; 671100\, n^3 z^4 + 41184\, n^2 z^5 + 111\, k^6 \mathrm{K}\; 4812\, k^5 n + 7227\, k^5 z$$
$$+ 70608\, k^4 n^2 \mathrm{K}\; 176733\, k^4 n\, z + 52152\, k^4 z^2 \mathrm{K}\; 517698\, k^3 n^3$$
$$+ 1543083\, k^3 n^2 z \mathrm{K}\; 759900\, k^3 n\, z^2 + 24840\, k^3 z^3 + 2128005\, k^2 n^4$$
$$\mathrm{K}\; 5957676\, k^2 n^3 z + 3743200\, k^2 n^2 z^2 \mathrm{K}\; 130884\, k^2 n\, z^3$$
$$\mathrm{K}\; 47997\, k^2 z^4 \mathrm{K}\; 4815288\, k\, n^5 + 9681987\, k\, n^4 z \mathrm{K}\; 7294564\, k\, n^3 z^2$$
$$+ 111078\, k\, n^2 z^3 + 304146\, k\, n\, z^4 \mathrm{K}\; 5292\, k\, z^5 + 4895964\, n^6$$
$$\mathrm{K}\; 4400892\, n^5 z + 4453731\, n^4 z^2 + 83076\, n^3 z^3 \mathrm{K}\; 424485\, n^2 z^4$$
$$+ 14436\, n\, z^5 \mathrm{K}\; 2664\, k^5 + 72528\, k^4 n \mathrm{K}\; 67869\, k^4 z \mathrm{K}\; 765708\, k^3 n^2$$
$$+ 1147854\, k^3 n\, z \mathrm{K}\; 228240\, k^3 z^2 + 4116501\, k^2 n^3 \mathrm{K}\; 6549708\, k^2 n^2 z$$
$$+ 2198132\, k^2 n\, z^2 \mathrm{K}\; 41556\, k^2 z^3 \mathrm{K}\; 11607570\, k\, n^4$$
$$+ 14090571\, k\, n^3 z \mathrm{K}\; 6301640\, k\, n^2 z^2 + 101352\, k\, n\, z^3 + 53946\, k\, z^4$$
$$+ 14306625\, n^5 \mathrm{K}\; 7949322\, n^4 z + 5052393\, n^3 z^2 + 2304\, n^2 z^3$$

$$
\begin{aligned}
&\mathrm{K}\ 149958\, n\, z^{4} + 2160\, z^{5} + 27084\, k^{4} \mathrm{K}\ 552582\, k^{3}\, n + 332145\, k^{3}\, z \\
&+ 4393572\, k^{2}\, n^{2} \mathrm{K}\ 3760251\, k^{2}\, n\, z + 523212\, k^{2}\, z^{2} \mathrm{K}\ 16532586\, k\, n^{3} \\
&+ 12146199\, k\, n^{2}\, z \mathrm{K}\ 2959764\, k\, n\, z^{2} + 27144\, k\, z^{3} + 25788699\, n^{4} \\
&\mathrm{K}\ 9237861\, n^{3}\, z + 3539610\, n^{2}\, z^{2} \mathrm{K}\ 15624\, n\, z^{3} \mathrm{K}\ 22680\, z^{4} \\
&\mathrm{K}\ 155178\, k^{3} + 2449704\, k^{2}\, n \mathrm{K}\ 878976\, k^{2}\, z \mathrm{K}\ 13912644\, k\, n^{2} \\
&+ 5742162\, k\, n\, z \mathrm{K}\ 582228\, k\, z^{2} + 29384127\, n^{3} \mathrm{K}\ 6772266\, n^{2}\, z \\
&+ 1401660\, n\, z^{2} \mathrm{K}\ 4320\, z^{3} + 557085\, k^{2} \mathrm{K}\ 6409002\, k\, n \\
&+ 1149948\, k\, z + 20691837\, n^{2} \mathrm{K}\ 2874744\, z\, n + 240840\, z^{2} \\
&\mathrm{K}\ 1248318\, k + 8246376\, n \mathrm{K}\ 542160\, z + 1426680)\ n\, aa_{n,k}\big) \big/ ((\mathrm{K}\ 3\, n \\
&\mathrm{K}\ 6 + k)\ (\mathrm{K}\ 3\, n \mathrm{K}\ 5 + k)\ (\mathrm{K}\ 3\, n + k \mathrm{K}\ 4)\ (\mathrm{K}\ n + k \mathrm{K}\ 1)\ (\mathrm{K}\ n + k)\ (\\
&\mathrm{K}\ 3\, n + k \mathrm{K}\ 1)\ (\mathrm{K}\ 3\, n + k \mathrm{K}\ 2)\ (\mathrm{K}\ 3\, n + k \mathrm{K}\ 3)))
\end{aligned}
$$

The set of poles to be avoided in the summation argument in the article is the following.

```
> Z = {solve(denom(zb_A[2]), k)};
```

$$Z = \{n, n + 1, 3\, n + 1, 3\, n + 2, 3\, n + 3, 3\, n + 4, 3\, n + 5, 3\, n + 6\} \qquad \textbf{(1.3.5)}$$

We apply sound creative telescoping. Here, this justifies the homogeneous recurrence relation in the article:

```
> rhs = collect(expand(subs(k=n-1, zb_A[2]) - subs(k=-1, zb_A
  [2]) + add(coeff(zb_A[1], sn, i) * add(subs(n = n+i, k=n+j,
  CF_A * z^k), j=-1..i), i=0..2)), binomial, factor);
```

$$
\begin{aligned}
rhs = {} & \frac{1}{(3\, n + 4)\ (3\, n + 5)\ (n + 2)\ (n + 1)\ z} \Big(2\ (3\, n + 1)\ (27\, n^{5}\, z^{5} \qquad \textbf{(1.3.6)} \\
&\mathrm{K}\ 274\, n^{5}\, z^{4} + 243\, n^{4}\, z^{5} + 123\, n^{5}\, z^{3} \mathrm{K}\ 2460\, n^{4}\, z^{4} + 843\, n^{3}\, z^{5} \\
&+ 1620\, n^{5}\, z^{2} + 1044\, n^{4}\, z^{3} \mathrm{K}\ 8510\, n^{3}\, z^{4} + 1413\, n^{2}\, z^{5} \mathrm{K}\ 2916\, n^{5}\, z \\
&+ 14904\, n^{4}\, z^{2} + 2559\, n^{3}\, z^{3} \mathrm{K}\ 14220\, n^{2}\, z^{4} + 1146\, n\, z^{5} + 8748\, n^{5} \\
&\mathrm{K}\ 27459\, n^{4}\, z + 56244\, n^{3}\, z^{2} + 1134\, n^{2}\, z^{3} \mathrm{K}\ 11496\, n\, z^{4} + 360\, z^{5} \\
&+ 69984\, n^{4} \mathrm{K}\ 104814\, n^{3}\, z + 107424\, n^{2}\, z^{2} \mathrm{K}\ 2844\, n\, z^{3} \mathrm{K}\ 3600\, z^{4} \\
&+ 220644\, n^{3} \mathrm{K}\ 200259\, n^{2}\, z + 102624\, n\, z^{2} \mathrm{K}\ 2520\, z^{3} + 343764\, n^{2} \\
&\mathrm{K}\ 189768\, z\, n + 38880\, z^{2} + 265356\, n \mathrm{K}\ 70920\, z + 81360)\ \binom{3\, n}{\mathrm{K}\ 1} \Big) \\
&+ \Bigg(\frac{1}{n + 1} \Big(6\, z^{n}\ (z \mathrm{K}\ 1)^{4}\ (3\, n + 2)\ (3\, n + 1)\ (n^{2}\, z^{2} \mathrm{K}\ 6\, n^{2}\, z
\end{aligned}
$$

$$+ 4\,n\,z^2 \mathrm{K}\; 27\,n^2 \mathrm{K}\; 24\,z\,n + 3\,z^2 \mathrm{K}\; 108\,n \mathrm{K}\; 18\,z \mathrm{K}\; 111)\, \binom{n+1}{\mathrm{K}\,1} \Big)$$

$$\mathrm{K}\; \frac{1}{(n+2)\,(2\,n+1)\,(n+1)^2} \Big(3\,(3\,n+2)\,(2\,n+3)\,(3\,n+1)\,(2\,n^4\,z^5$$

$$\mathrm{K}\; 21\,n^4\,z^4 + 12\,n^3\,z^5 + 108\,n^4\,z^3 \mathrm{K}\; 126\,n^3\,z^4 + 22\,n^2\,z^5 \mathrm{K}\; 378\,n^4\,z^2$$

$$+ 648\,n^3\,z^3 \mathrm{K}\; 231\,n^2\,z^4 + 12\,n\,z^5 \mathrm{K}\; 3078\,n^4\,z \mathrm{K}\; 2268\,n^3\,z^2$$

$$+ 1188\,n^2\,z^3 \mathrm{K}\; 126\,n\,z^4 \mathrm{K}\; 729\,n^4 \mathrm{K}\; 18468\,n^3\,z \mathrm{K}\; 4188\,n^2\,z^2$$

$$+ 648\,n\,z^3 \mathrm{K}\; 4374\,n^3 \mathrm{K}\; 39078\,n^2\,z \mathrm{K}\; 2358\,n\,z^2 \mathrm{K}\; 10449\,n^2$$

$$\mathrm{K}\; 34128\,z\,n \mathrm{K}\; 11664\,n \mathrm{K}\; 10080\,z \mathrm{K}\; 5040)\, z^n\, z\, \binom{n+2}{\mathrm{K}\,1} \Big)$$

$$n \mathrm{K}\; 30)\, z^n\, z^2\, \binom{n+3}{\mathrm{K}\,1} \Big) \Big/ \left(2\,(2\,n+1)\,(n+1)\,(2\,n+3)\,(n+2)^2\right) \Bigg) \binom{3\,n}{n}$$

```
> subs(seq(binomial(e,-1) = 0, e=[n+1,n+2,n+3,3*n]), %);
```

$$rhs = 0 \qquad \textbf{(1.3.7)}$$

We can also validate numerically that the recurrence is satisfied for a few values of n (in the range 10 to 15):

```
> seq(normal(eval(subs(aa = unapply(coeff(ser_A, t, n), n),
  subs(seq(aa[n+j,k] = aa(n+j,k), j=0..2), n=i, reczb_A)))),
  i=10..15);
```

$$0, 0, 0, 0, 0, 0 \qquad \textbf{(1.3.8)}$$

So we have justified the following recurrence relation:

```
> rec2_A := subs(seq(aa[n+i,k] = aa(n+i), i=0..2), reczb_A);
```

$$rec2\_A := 3\,n\,(z \mathrm{K}\; 1)^4\,(3\,n+2)\,(3\,n+1)\,(n^2\,z^2 \mathrm{K}\; 6\,n^2\,z + 4\,n\,z^2 \mathrm{K}\; 27\,n^2 \qquad \textbf{(1.3.9)}$$

$$\mathrm{K}\; 24\,z\,n + 3\,z^2 \mathrm{K}\; 108\,n \mathrm{K}\; 18\,z \mathrm{K}\; 111)\, aa(n) \mathrm{K}\; (2\,n+3)\,(2\,n^4\,z^5$$

$$\mathrm{K}\; 21\,n^4\,z^4 + 12\,n^3\,z^5 + 108\,n^4\,z^3 \mathrm{K}\; 126\,n^3\,z^4 + 22\,n^2\,z^5 \mathrm{K}\; 378\,n^4\,z^2$$

$$+ 648\,n^3\,z^3 \mathrm{K}\; 231\,n^2\,z^4 + 12\,n\,z^5 \mathrm{K}\; 3078\,n^4\,z \mathrm{K}\; 2268\,n^3\,z^2$$

$$+ 1188\,n^2\,z^3 \mathrm{K}\; 126\,n\,z^4 \mathrm{K}\; 729\,n^4 \mathrm{K}\; 18468\,n^3\,z \mathrm{K}\; 4188\,n^2\,z^2$$

$$+ 648\,n\,z^3 \mathrm{K}\; 4374\,n^3 \mathrm{K}\; 39078\,n^2\,z \mathrm{K}\; 2358\,n\,z^2 \mathrm{K}\; 10449\,n^2$$

$$\mathrm{K}\; 34128\,z\,n \mathrm{K}\; 11664\,n \mathrm{K}\; 10080\,z \mathrm{K}\; 5040)\, aa(n+1) + 3\,(3\,n$$

$$+ 7)\,(n+3)\,(3\,n+8)\,(n^2\,z^2 \mathrm{K}\; 6\,n^2\,z + 2\,n\,z^2 \mathrm{K}\; 27\,n^2 \mathrm{K}\; 12\,z\,n \mathrm{K}\; 54\,n$$

$\mathrm{K}\ 30)\ aa(n+2)$

Third step: compute a common recurrence relation for a[n](z) and aa[n](z), and conclude.

We proceed to prove the equality between the solution a[n](z) of a fourth-order recurrence equation and the solution aa[n](z) of a second-order recurrence equation.

```
> indets(rec_A);
```

$$\{n, z, a(n), a(n+1), a(n+2), a(n+3), a(n+4)\} \quad \textbf{(1.4.1)}$$

```
> indets(rec2_A);
```

$$\{n, z, aa(n), aa(n+1), aa(n+2)\} \quad \textbf{(1.4.2)}$$

To this end, we first derive a common recurrence relation, which is a relation satisfied by any linear combination with constant coefficients of the two sequences. It will in particular be valid for a(n) = a(n) + 0 and for aa(n) = 0 + aa(n). The syntax of `rec+rec` uses the same name for all input and output equations.

```
> reclclm_A := collect(`rec+rec`(rec_A, subs(aa=a, rec2_A), a
  (n)), a, factor);
```

$$reclclm\_A := 9\,n\,(2\,z\mathrm{K}\ 3)\,(z\mathrm{K}\ 1)^8\,(3\,n+2)\,(3\,n+1)\,a(n)\mathrm{K}\ 3\,(z \mathrm{K}\ 1)^4\,(26\,n^3 z^4\mathrm{K}\ 174\,n^3 z^3+118\,n^2 z^4+810\,n^3 z^2\mathrm{K}\ 798\,n^2 z^3 +172\,n\,z^4\mathrm{K}\ 594\,n^3 z+3321\,n^2 z^2\mathrm{K}\ 1180\,n\,z^3+80\,z^4\mathrm{K}\ 324\,n^3 \mathrm{K}\ 2160\,n^2 z+4389\,n\,z^2\mathrm{K}\ 560\,z^3\mathrm{K}\ 1377\,n^2\mathrm{K}\ 2454\,z\,n+1860\,z^2 \mathrm{K}\ 1935\,n\mathrm{K}\ 840\,z\mathrm{K}\ 900)\,a(n+1)+(z\mathrm{K}\ 1)\,(2\,n+5)\,(4\,n^2 z^6 \mathrm{K}\ 42\,n^2 z^5+20\,n\,z^6+534\,n^2 z^4\mathrm{K}\ 210\,n\,z^5+24\,z^6\mathrm{K}\ 2151\,n^2 z^3 +2670\,n\,z^4\mathrm{K}\ 252\,z^5+3321\,n^2 z^2\mathrm{K}\ 10755\,n\,z^3+3204\,z^4 +1701\,n^2 z+16605\,n\,z^2\mathrm{K}\ 13020\,z^3+729\,n^2+8505\,z\,n +20268\,z^2+3645\,n+10584\,z+4536)\,a(n+2)+(\mathrm{K}\ 78\,n^3 z^4 +522\,n^3 z^3\mathrm{K}\ 816\,n^2 z^4\mathrm{K}\ 2430\,n^3 z^2+5436\,n^2 z^3\mathrm{K}\ 2826\,n\,z^4 +1782\,n^3 z\mathrm{K}\ 26487\,n^2 z^2+18750\,n\,z^3\mathrm{K}\ 3240\,z^4+972\,n^3 +20250\,n^2 z\mathrm{K}\ 95787\,n\,z^2+21420\,z^3+10449\,n^2+76212\,z\,n \mathrm{K}\ 114930\,z^2+37395\,n+95040\,z+44550)\,a(n+3)+9\,(n +5)\,(3\,n+14)\,(3\,n+13)\,(2\,z\mathrm{K}\ 3)\,a(n+4) \quad \textbf{(1.4.3)}$$

It turns out that this is the already known fourth-order equation.

```
> rec_A - reclclm_A;
```

$$0 \quad \textbf{(1.4.4)}$$

The recurrence can be unrolled over all natural integers, as its leading coefficient has no nonnegative integer root.

```
> solve(coeff(reclclm_A, a(n+4)), n);
```

(1.4.5)

$$-5, -\frac{14}{3}, -\frac{13}{3} \tag{1.4.5}$$

So the solution to the recurrence is identified by its first 4 terms, and to end the proof, it is sufficient to compare enough initial conditions.

```
> for i from 0 to 3 do
      coeff(ser_A, t, i) = eval(add(subs(n=i, k=j, CF_A*z^k), j=0..i-1))
  od;
```

$$0 = 0$$

$$1 = 1$$

$$2z + 1 = 2z + 1$$

$$6z^2 + 6z + 1 = 6z^2 + 6z + 1 \tag{1.4.6}$$

[4.2 Theorem 2 by creative telescoping]

This section is very similar to the previous one about Theorem 1.

Goal

We are interested in the generating series B(t,z) in Q[z][[t]], whose coefficient with respect to t^n*z^k is:

```
> CF_B := 2 * binomial(n-1, k) * binomial(4*n+1-k, n+1) / ((3*n+1) * (3*n+2));
```

$$CF\_B := \frac{2 \binom{n-1}{k} \binom{4n+1-k}{n+1}}{(3n+2)(3n+1)} \tag{2.1.1}$$

That is, the series is:

```
> B(t,z) = Sum(Sum(CF_B * t^n*z^k, k=0..n-1), n=0..infinity);
```

$$B(t, z) = \sum_{n=0}^{\infty} \sum_{k=0}^{n-1} \frac{2 \binom{n-1}{k} \binom{4n+1-k}{n+1} t^n z^k}{(3n+2)(3n+1)} \tag{2.1.2}$$

We want to prove that this series is an algebraic series, solution of the polynomial:

```
> P_mod := collect(subs(z=z+1, P), X, factor);
```

$$P\_mod := X^4 t^3 (z+1)^6 + t^2 (z+1)^4 (t z^2 + 8 t z + 4 t + 3) X^3 + t (z+1)^2 (6 t^2 z^3 + 27 t^2 z^2 + 24 t^2 z + 2 t z^2 + 6 t^2 - 2 t z + 17 t + 3) X^2 + (12 t^3 z^4 + 44 t^3 z^3 + 51 t^3 z^2 - 10 t^2 z^3 + 24 t^3 z - 4 t^2 z^2 + 4 t^3 + 28 t^2 z + t z^2 + 25 t^2 - 10 t z - 14 t + 1) X + t (8 t^2 z^3 + 12 t^2 z^2 + 6 t^2 z - t z^2 + t^2 + 8 t z + 11 t - 1) \tag{2.1.3}$$

Let us first check that there is no mistake in the posed problem.

```
> N := 20:
  ser_B := add(add(CF_B*z^k*t^n, k=0..n-1), n=0..N);
```

(2.1.4)

$$
\begin{aligned}
ser\_B := {} & 9737153323590\, t^{20} z^{19} + 277508869722315\, t^{20} z^{18} \\
& + 3717328115350080\, t^{20} z^{17} + 1633292229030\, t^{19} z^{18} \\
& + 31118542177741200\, t^{20} z^{16} + 44098890183810\, t^{19} z^{17} \\
& + 182562114109415040\, t^{20} z^{15} + 557689623422085\, t^{19} z^{16} \\
& + 275750636070\, t^{18} z^{17} + 797717063825922240\, t^{20} z^{14} \\
& + 4390699257418320\, t^{19} z^{15} + 7031641219785\, t^{18} z^{16} \\
& + 2693002286391056640\, t^{20} z^{13} + 24123318594536700\, t^{19} z^{14} \\
& + 83658500666160\, t^{18} z^{15} + 46892282815\, t^{17} z^{16} \\
& + 7189354318133267280\, t^{20} z^{12} + 98247697548658560\, t^{19} z^{13} \\
& + 616981442412930\, t^{18} z^{14} + 1125414787560\, t^{17} z^{15} \\
& + 15405759253142715600\, t^{20} z^{11} + 307478905291172160\, t^{19} z^{12} \\
& + 3160148851383300\, t^{18} z^{13} + 12546854050500\, t^{17} z^{14} \\
& + 8038677054\, t^{16} z^{15} + 26737551059343246408\, t^{20} z^{10} \\
& + 756283518604373760\, t^{19} z^{11} + 11933323995937890\, t^{18} z^{12} \\
& + 86287136628000\, t^{17} z^{13} + 180870233715\, t^{16} z^{14} \\
& + 37747130907308112576\, t^{20} z^{9} + 1482396152158041120\, t^{19} z^{10} \\
& + 34412376174332520\, t^{18} z^{11} + 409863898983000\, t^{17} z^{12} \\
& + 1881050430636\, t^{16} z^{13} + 1390594458\, t^{15} z^{14} \\
& + 43356407353324178256\, t^{20} z^{8} + 2333401350619138800\, t^{19} z^{9} \\
& + 77427846392248170\, t^{18} z^{10} + 1426326368460840\, t^{17} z^{11} \\
& + 12000405062113\, t^{16} z^{12} + 29202483618\, t^{15} z^{13} \\
& + 40356907473534455232\, t^{20} z^{7} + 2957229058641806520\, t^{19} z^{8} \\
& + 137649504697330080\, t^{18} z^{9} + 3762950459882460\, t^{17} z^{10} \\
& + 52542314055738\, t^{16} z^{11} + 281848213101\, t^{15} z^{12} \\
& + 243035536\, t^{14} z^{13} + 30181447896874058400\, t^{20} z^{6} \\
& + 3010996859708021184\, t^{19} z^{7} + 194504734898401200\, t^{18} z^{8} \\
& + 7679490734454000\, t^{17} z^{9} + 167305789493271\, t^{16} z^{10} \\
& + 1657930665300\, t^{15} z^{11} + 4739192952\, t^{14} z^{12} \\
& + 17873688624642299520\, t^{20} z^{5} + 2445204966135435504\, t^{19} z^{6} \\
& + 218507446864586880\, t^{18} z^{7} + 12255931433765250\, t^{17} z^{8} \\
& + 400389923573640\, t^{16} z^{9} + 6643565023095\, t^{15} z^{10} \\
& + 42194104992\, t^{14} z^{11} + 42975796\, t^{13} z^{12} \\
& + 8192107286294387280\, t^{20} z^{4} + 1562616191376491328\, t^{19} z^{5} \\
& + 194090895036915240\, t^{18} z^{6} + 15350863614009000\, t^{17} z^{7} \\
& + 733571538547419\, t^{16} z^{8} + 19192521177830\, t^{15} z^{9} \\
& + 227232836259\, t^{14} z^{10} + 773564328\, t^{13} z^{11}
\end{aligned}
\qquad \textbf{(2.1.4)}
$$

$$+ 2802563018995448280\, t^{20} z^{3} + 768672385245848160\, t^{19} z^{4}$$
$$+ 134675314923573840\, t^{18} z^{5} + 15043846341728820\, t^{17} z^{6}$$
$$+ 1037735347213422\, t^{16} z^{7} + 41237984692905\, t^{15} z^{8}$$
$$+ 826301222760\, t^{14} z^{9} + 6308550468\, t^{13} z^{10} + 7702632\, t^{12} z^{11}$$
$$+ 6736383727200012600\, t^{20} z^{2} + 2808975630034210560\, t^{19} z^{3}$$
$$+ 714815133055891920\, t^{18} z^{4} + 114166739036044800\, t^{17} z^{5}$$
$$+ 11338219534368870\, t^{16} z^{6} + 669729676215600\, t^{15} z^{7}$$
$$+ 21435225837480\, t^{14} z^{8} + 308418022880\, t^{13} z^{9} + 1270934280\, t^{12} z^{10}$$
$$+ 10148977366779680\, t^{20} z + 7182039963155652\, t^{19} z^{2}$$
$$+ 28031966002191840\, t^{18} z^{3} + 6578757125304000\, t^{17} z^{4}$$
$$+ 949246286598324\, t^{16} z^{5} + 82642924789425\, t^{15} z^{6}$$
$$+ 4082900159520\, t^{14} z^{7} + 100733305860\, t^{13} z^{8} + 941432800\, t^{12} z^{9}$$
$$+ 1402440\, t^{11} z^{10} + 7211115497448720\, t^{20}$$
$$+ 11467122630248520\, t^{19} z + 7654883023675464\, t^{18} z^{2}$$
$$+ 2783320322244000\, t^{17} z^{3} + 598182060769605\, t^{16} z^{4}$$
$$+ 77133396470130\, t^{15} z^{5} + 5784108559320\, t^{14} z^{6}$$
$$+ 231686603478\, t^{13} z^{7} + 4135579800\, t^{12} z^{8} + 21036600\, t^{11} z^{9}$$
$$+ 860593023907540\, t^{19} + 1299885796473192\, t^{18} z$$
$$+ 815520969054000\, t^{17} z^{2} + 274720650131226\, t^{16} z^{3}$$
$$+ 53617117058505\, t^{15} z^{4} + 6096763076040\, t^{14} z^{5}$$
$$+ 384974204769\, t^{13} z^{6} + 11978920800\, t^{12} z^{7} + 140103756\, t^{11} z^{8}$$
$$+ 260130\, t^{10} z^{9} + 103367824774012\, t^{18} + 147881135721792\, t^{17} z$$
$$+ 86826426212043\, t^{16} z^{2} + 26924612895180\, t^{15} z^{3}$$
$$+ 4724100044300\, t^{14} z^{4} + 465851138544\, t^{13} z^{5}$$
$$+ 24037701072\, t^{12} z^{6} + 546045408\, t^{11} z^{7} + 3511755\, t^{10} z^{8}$$
$$+ 12504654858828\, t^{17} + 16890247044288\, t^{16} z$$
$$+ 9235768376835\, t^{15} z^{2} + 2616424639920\, t^{14} z^{3}$$
$$+ 407619746226\, t^{13} z^{4} + 34118027328\, t^{12} z^{5} + 1380281448\, t^{11} z^{6}$$
$$+ 20765160\, t^{10} z^{7} + 49335\, t^{9} z^{8} + 1524813969276\, t^{16}$$
$$+ 1937573785350\, t^{15} z + 981159239970\, t^{14} z^{2}$$
$$+ 251617127300\, t^{13} z^{3} + 34270339950\, t^{12} z^{4} + 2366196768\, t^{11} z^{5}$$
$$+ 70659225\, t^{10} z^{6} + 592020\, t^{9} z^{7} + 187606350645\, t^{15}$$
$$+ 223353322920\, t^{14} z + 104047082370\, t^{13} z^{2} + 23885388450\, t^{12} z^{3}$$
$$+ 2787760560\, t^{11} z^{4} + 152623926\, t^{10} z^{5} + 3058770\, t^{9} z^{6}$$
$$+ 9614\, t^{8} z^{7} + 23317105140\, t^{14} + 25887312360\, t^{13} z$$
$$+ 11006012325\, t^{12} z^{2} + 2230208448\, t^{11} z^{3} + 217195587\, t^{10} z^{4}$$

$+ 8898240\, t^9 z^5 + 100947\, t^8 z^6 + 2931682810\, t^{13}$
$+ 3018791952\, t^{12} z + 1160068104\, t^{11} z^2 + 203788452\, t^{10} z^3$
$+ 15958800\, t^9 z^4 + 446292\, t^8 z^5 + 1938\, t^7 z^6 + 373537388\, t^{12}$
$+ 354465254\, t^{11} z + 121649229\, t^{10} z^2 + 18086640\, t^9 z^3$
$+ 1078539\, t^8 z^4 + 17442\, t^7 z^5 + 48336171\, t^{11} + 41948010\, t^{10} z$
$+ 12660648\, t^9 z^2 + 1540770\, t^8 z^3 + 64125\, t^7 z^4 + 408\, t^6 z^5$
$+ 6369883\, t^{10} + 5008608\, t^9 z + 1302651\, t^8 z^2 + 123500\, t^7 z^3$
$+ 3060\, t^6 z^4 + 857956\, t^9 + 604128\, t^8 z + 131625\, t^7 z^2 + 8976\, t^6 z^3$
$+ 91\, t^5 z^4 + 118668\, t^8 + 73710\, t^7 z + 12903\, t^6 z^2 + 546\, t^5 z^3$
$+ 16965\, t^7 + 9108\, t^6 z + 1197\, t^5 z^2 + 22\, t^4 z^3 + 2530\, t^6 + 1140\, t^5 z$
$+ 99\, t^4 z^2 + 399\, t^5 + 144\, t^4 z + 6\, t^3 z^2 + 68\, t^4 + 18\, t^3 z + 13\, t^3 + 2\, t^2 z$
$+ 3\, t^2 + t$

```
> series(expand(subs(X = ser_B + O(t^(N+1)), P_mod)), t,
  infinity);
```

$$O(t^{21}) \qquad (2.1.5)$$

First step: compute a recurrence relation on the coefficients b[n](z) of the solution B(t,z) of P(t,z+1, X).

The gfun package is now classic. We use it to transform the polynomial equation into a differential equation, then into a recurrence equation on the coefficients with respect to t of any series solution.

```
> with(gfun);
```

[*Laplace, Parameters, algebraicsubs, algeqtodiffeq, algeqtoseries, algfuntoalgeq, borel, cauchyproduct, diffeq*diffeq, diffeq+diffeq, diffeqtohomdiffeq, diffeqtorec, guesseqn, guessgf, hadamardproduct, holexprtodiffeq, invborel, listtoalgeq, listtodiffeq, listtohypergeom, listtolist, listtoratpoly, listtorec, listtoseries, poltodiffeq, poltorec, ratpolytocoeff, rec*rec, rec+rec, rectodiffeq, rectohomrec, rectoproc, seriestoalgeq, seriestodiffeq, seriestohypergeom, seriestolist, seriestoratpoly, seriestorec, seriestoseries*] **(2.2.1)**

```
> deq_B := algeqtodiffeq(P_mod, X(t));
```

$deq\_B := K\, 24\, t^3 z^7 + 108\, t^3 z^6 K\; 24\, t^3 z^4 + 228\, t^2 z^4 K\; 576\, t^2 z^3 + 720\, t^2 z^2$ **(2.2.2)**
$+ 768\, t^2 z K\; 720\, t z + 360\, t + (12\, t^3 z^7 + 90\, t^3 z^6 K\; 24\, t^3 z^4 K\; 114\, t^2 z^4$
$+ 720\, t^2 z^2 + 768\, t^2 z K\; 342\, t z^2 K\; 144\, t z + 360\, t + 36\, z K\; 18)\, X(t)$
$+ (360\, t^5 z^9 K\; 180\, t^5 z^8 K\; 240\, t^4 z^8 + 708\, t^4 z^7 K\; 2070\, t^4 z^6 + 24\, t^3 z^7$

$$K\ 4560\,t^4 z^5 K\ 108\,t^3 z^6 + 1104\,t^4 z^4 + 2304\,t^3 z^5 K\ 2358\,t^3 z^4$$
$$K\ 1728\,t^3 z^3 K\ 228\,t^2 z^4 + 24480\,t^3 z^2 + 576\,t^2 z^3 + 26112\,t^3 z$$
$$K\ 6426\,t^2 z^2 K\ 6672\,t^2 z + 4176\,t^2 + 684\,t\,z K\ 342\,t) \left(\frac{d}{dt} X(t)\right)$$
$$+ (648\,t^6 z^9 K\ 324\,t^6 z^8 K\ 354\,t^5 z^8 + 978\,t^5 z^7 K\ 4905\,t^5 z^6 + 36\,t^4 z^7$$
$$K\ 7680\,t^5 z^5 K\ 162\,t^4 z^6 + 2688\,t^5 z^4 + 3456\,t^4 z^5 K\ 3195\,t^4 z^4$$
$$K\ 2592\,t^4 z^3 K\ 348\,t^3 z^4 + 34560\,t^4 z^2 + 912\,t^3 z^3 + 36864\,t^4 z$$
$$K\ 7083\,t^3 z^2 K\ 8736\,t^3 z + 4224\,t^3 + 810\,t^2 z K\ 405\,t^2) \left(\frac{d^2}{dt^2} X(t)\right)$$
$$+ (162\,t^7 z^9 K\ 81\,t^7 z^8 K\ 78\,t^6 z^8 + 210\,t^6 z^7 K\ 1332\,t^6 z^6 + 8\,t^5 z^7$$
$$K\ 1824\,t^6 z^5 K\ 36\,t^5 z^6 + 768\,t^6 z^4 + 768\,t^5 z^5 K\ 710\,t^5 z^4 K\ 576\,t^5 z^3$$
$$K\ 78\,t^4 z^4 + 7680\,t^5 z^2 + 210\,t^4 z^3 + 8192\,t^5 z K\ 1332\,t^4 z^2 K\ 1824\,t^4 z$$
$$+ 768\,t^4 + 162\,t^3 z K\ 81\,t^3) \left(\frac{d^3}{dt^3} X(t)\right)$$

```
> collect(deq_B, {diff, f}, factor);
```

$$6\,t\,(60\,t^4 z^9 K\ 30\,t^4 z^8 K\ 40\,t^3 z^8 + 118\,t^3 z^7 K\ 345\,t^3 z^6 + 4\,t^2 z^7 K\ 760\,t^3 z^5 \quad \textbf{(2.2.3)}$$
$$K\ 18\,t^2 z^6 + 184\,t^3 z^4 + 384\,t^2 z^5 K\ 393\,t^2 z^4 K\ 288\,t^2 z^3 K\ 38\,t\,z^4$$
$$+ 4080\,t^2 z^2 + 96\,t\,z^3 + 4352\,t^2 z K\ 1071\,t\,z^2 K\ 1112\,t\,z + 696\,t$$
$$+ 114\,z K\ 57) \left(\frac{d}{dt} X(t)\right) + 3\,t^2\,(216\,t^4 z^9 K\ 108\,t^4 z^8 K\ 118\,t^3 z^8$$
$$+ 326\,t^3 z^7 K\ 1635\,t^3 z^6 + 12\,t^2 z^7 K\ 2560\,t^3 z^5 K\ 54\,t^2 z^6 + 896\,t^3 z^4$$
$$+ 1152\,t^2 z^5 K\ 1065\,t^2 z^4 K\ 864\,t^2 z^3 K\ 116\,t\,z^4 + 11520\,t^2 z^2$$
$$+ 304\,t\,z^3 + 12288\,t^2 z K\ 2361\,t\,z^2 K\ 2912\,t\,z + 1408\,t + 270\,z$$
$$K\ 135) \left(\frac{d^2}{dt^2} X(t)\right) + t^3\,(6\,t^2 z^5 K\ 3\,t^2 z^4 K\ 2\,t\,z^4 + 6\,t\,z^3 K\ 6\,t\,z^2$$
$$K\ 32\,t\,z + 6\,z K\ 3)\,(27\,t^2 z^4 K\ 4\,t\,z^3 + 6\,t\,z^2 K\ 192\,t\,z K\ 256\,t$$
$$+ 27) \left(\frac{d^3}{dt^3} X(t)\right) + 12\,X(t)\,t^3 z^7 + 90\,X(t)\,t^3 z^6 K\ 24\,t^3 z^7 + 108\,t^3 z^6$$
$$K\ 24\,X(t)\,t^3 z^4 K\ 114\,X(t)\,t^2 z^4 K\ 24\,t^3 z^4 + 228\,t^2 z^4 + 720\,X(t)\,t^2 z^2$$
$$K\ 576\,t^2 z^3 + 768\,X(t)\,t^2 z K\ 342\,X(t)\,t\,z^2 + 720\,t^2 z^2 K\ 144\,X(t)\,t\,z$$
$$+ 768\,t^2 z + 360\,X(t)\,t + 36\,X(t)\,z K\ 720\,t\,z K\ 18\,X(t) + 360\,t$$

```
> nops(expand(deq_B));
```

$$83 \quad \textbf{(2.2.4)}$$

```
> sys_B := diffeqtorec(deq_B, X(t), b(n));
```

$$sys\_B := \{((162\,z^9 K\ 81\,z^8)\,n^3 + (162\,z^9 K\ 81\,z^8)\,n^2 + (36\,z^9 \quad \textbf{(2.2.5)}$$
$$K\ 18\,z^8)\,n)\,b(n) + ((K\,78\,z^8 + 210\,z^7 K\ 1332\,z^6 K\ 1824\,z^5$$
$$+ 768\,z^4)\,n^3 + (K\,354\,z^8 + 978\,z^7 K\ 4905\,z^6 K\ 7680\,z^5 + 2688\,z^4)\,n^2$$

$$+ (K\ 516 z^8 + 1476 z^7 K\ 5643 z^6 K\ 10416 z^5 + 3024 z^4)\ n K\ 240 z^8$$
$$+ 720 z^7 K\ 1980 z^6 K\ 4560 z^5 + 1080 z^4)\ b(n+1) + ((8 z^7 K\ 36 z^6$$
$$+ 768 z^5 K\ 710 z^4 K\ 576 z^3 + 7680 z^2 + 8192 z)\ n^3 + (60 z^7 K\ 270 z^6$$
$$+ 5760 z^5 K\ 5325 z^4 K\ 4320 z^3 + 57600 z^2 + 61440 z)\ n^2 + (148 z^7$$
$$K\ 666 z^6 + 14208 z^5 K\ 13363 z^4 K\ 10656 z^3 + 143520 z^2$$
$$+ 153088 z)\ n + 120 z^7 K\ 540 z^6 + 11520 z^5 K\ 11220 z^4 K\ 8640 z^3$$
$$+ 118800 z^2 + 126720 z)\ b(n+2) + ((K\ 78 z^4 + 210 z^3 K\ 1332 z^2$$
$$K\ 1824 z + 768)\ n^3 + (K\ 816 z^4 + 2172 z^3 K\ 15075 z^2 K\ 19680 z$$
$$+ 8832)\ n^2 + (K\ 2826 z^4 + 7446 z^3 K\ 56493 z^2 K\ 70416 z$$
$$+ 33744)\ n K\ 3240 z^4 + 8460 z^3 K\ 70110 z^2 K\ 83520 z + 42840)$$
$$b(n+3) + ((162 z K\ 81)\ n^3 + (2268 z K\ 1134)\ n^2 + (10566 z$$
$$K\ 5283)\ n + 16380 z K\ 8190)\ b(n+4),\ b(0) = 0,\ b(1) = 1,\ b(2)$$
$$= 2 z + 3,\ b(3) = 6 z^2 + 18 z + 13\}$$

```
> rec_B := collect(op(remove(type, sys_B, equation)), b,
  factor);
```

$$rec\_B := 9 n z^8 (3 n + 2) (3 n + 1) (2 z K\ 1)\ b(n) K\ 3 z^4 (26 n^3 z^4 \tag{2.2.6}$$
$$K\ 70 n^3 z^3 + 118 n^2 z^4 + 444 n^3 z^2 K\ 326 n^2 z^3 + 172 n z^4 + 608 n^3 z$$
$$+ 1635 n^2 z^2 K\ 492 n z^3 + 80 z^4 K\ 256 n^3 + 2560 n^2 z + 1881 n z^2$$
$$K\ 240 z^3 K\ 896 n^2 + 3472 z n + 660 z^2 K\ 1008 n + 1520 z K\ 360)\ b(n$$
$$+ 1) + z (2 n + 5) (4 n^2 z^6 K\ 18 n^2 z^5 + 20 n z^6 + 384 n^2 z^4 K\ 90 n z^5$$
$$+ 24 z^6 K\ 355 n^2 z^3 + 1920 n z^4 K\ 108 z^5 K\ 288 n^2 z^2 K\ 1775 n z^3$$
$$+ 2304 z^4 + 3840 n^2 z K\ 1440 n z^2 K\ 2244 z^3 + 4096 n^2 + 19200 z n$$
$$K\ 1728 z^2 + 20480 n + 23760 z + 25344)\ b(n+2) + (K\ 78 n^3 z^4$$
$$+ 210 n^3 z^3 K\ 816 n^2 z^4 K\ 1332 n^3 z^2 + 2172 n^2 z^3 K\ 2826 n z^4$$
$$K\ 1824 n^3 z K\ 15075 n^2 z^2 + 7446 n z^3 K\ 3240 z^4 + 768 n^3$$
$$K\ 19680 n^2 z K\ 56493 n z^2 + 8460 z^3 + 8832 n^2 K\ 70416 z n$$
$$K\ 70110 z^2 + 33744 n K\ 83520 z + 42840)\ b(n+3) + 9 (n$$
$$+ 5) (3 n + 14) (3 n + 13) (2 z K\ 1)\ b(n+4)$$

To fix ideas, let us numerically verify that this recurrence relation is indeed satisfied for a few values of n (from 10 to 15):

```
> seq(normal(eval(subs(b = unapply(coeff(ser_B, t, n), n),
  subs(n=i, rec_B)))), i=10..15);
```

$$0, 0, 0, 0, 0, 0 \tag{2.2.7}$$

Second step: compute a recurrence relation on the polynomials bb[n](z) = Sum(CF_B * z^k, k=0..n-1).

We use Zeilberger's algorithm for the computation of sums. On our instance, the output is a bit large. It is a pair [T(n,z,sn), C(n,k,z)] consisting of a skew polynomial T in Q(n,z)<sn; sn*n = (n+1)*sn>, the “telescoper”, and of an expression C(n,k,z) in Q(n,k,z)*CF_B*z^k, the “certificate”, satisfying a specific recurrence relation (to be provided below).

```
> zb_B := SumTools:-Hypergeometric:-Zeilberger(CF_B * z^k, n,
  k, sn):
  zb_B := applyop(collect, 1, zb_B, sn, factor):
  zb_B := applyop(factor, 2, zb_B);
```

$$zb\_B := \Big[3\,(3n+7)\,(n+3)\,(3n+8)\,(n^2 z^2 - 4n^2 z + 2n z^2 - 32 n^2 \qquad \textbf{(2.3.1)}$$

$$- 8 z n - 64 n - 30)\, sn^2 - (2n+3)\,(2n^4 z^5 - 11 n^4 z^4 + 12 n^3 z^5$$

$$+ 44 n^4 z^3 - 66 n^3 z^4 + 22 n^2 z^5 - 160 n^4 z^2 + 264 n^3 z^3 - 121 n^2 z^4$$

$$+ 12 n z^5 - 3584 n^4 z - 960 n^3 z^2 + 484 n^2 z^3 - 66 n z^4 - 4096 n^4$$

$$- 21504 n^3 z - 1790 n^2 z^2 + 264 n z^3 - 24576 n^3 - 44704 n^2 z$$

$$- 1050 n z^2 - 52736 n^2 - 37344 z n - 47616 n - 10080 z - 15120)$$

$$sn + 3 z^4 n\,(3n+2)\,(3n+1)\,(n^2 z^2 - 4 n^2 z + 4 n z^2 - 32 n^2 - 16 z n$$

$$+ 3 z^2 - 128 n - 12 z - 126), - \left(2 z^k \binom{4n+1-k}{n+1} \binom{n-1}{k} (\right.$$

$$-4n - 2 + k)\, k\, n\, (27 k^6 n^5 z^5 - 432 k^5 n^6 z^5 + 2835 k^4 n^7 z^5$$

$$- 9720 k^3 n^8 z^5 + 18225 k^2 n^9 z^5 - 17496 k n^{10} z^5 + 6561 n^{11} z^5$$

$$- 139 k^6 n^5 z^4 + 243 k^6 n^4 z^5 + 2248 k^5 n^6 z^4 - 4455 k^5 n^5 z^5$$

$$- 14898 k^4 n^7 z^4 + 32940 k^4 n^6 z^5 + 51532 k^3 n^8 z^4 - 125550 k^3 n^7 z^5$$

$$- 97375 k^2 n^9 z^4 + 258795 k^2 n^8 z^5 + 94092 k n^{10} z^4 - 270459 k n^9 z^5$$

$$\mathrm{K}\ 35460\, n^{11} z^{4} + 109350\, n^{10} z^{5} \mathrm{K}\ 703\, k^{6} n^{5} z^{3} \mathrm{K}\ 1245\, k^{6} n^{4} z^{4}$$

$$+ 843\, k^{6} n^{3} z^{5} + 11092\, k^{5} n^{6} z^{3} + 23078\, k^{5} n^{5} z^{4} \mathrm{K}\ 18591\, k^{5} n^{4} z^{5}$$

$$\mathrm{K}\ 71817\, k^{4} n^{7} z^{3} \mathrm{K}\ 172430\, k^{4} n^{6} z^{4} + 160065\, k^{4} n^{5} z^{5}$$

$$+ 243112\, k^{3} n^{8} z^{3} + 663582\, k^{3} n^{7} z^{4} \mathrm{K}\ 694440\, k^{3} n^{6} z^{5}$$

$$\mathrm{K}\ 450532\, k^{2} n^{9} z^{3} \mathrm{K}\ 1379681\, k^{2} n^{8} z^{4} + 1601775\, k^{2} n^{7} z^{5}$$

$$+ 428112\, k\, n^{10} z^{3} + 1452476\, k\, n^{9} z^{4} \mathrm{K}\ 1847529\, k\, n^{8} z^{5}$$

$$\mathrm{K}\ 159264\, n^{11} z^{3} \mathrm{K}\ 590532\, n^{10} z^{4} + 814293\, n^{9} z^{5} + 615\, k^{6} n^{5} z^{2}$$

$$\mathrm{K}\ 6366\, k^{6} n^{4} z^{3} \mathrm{K}\ 4295\, k^{6} n^{3} z^{4} + 1413\, k^{6} n^{2} z^{5} \mathrm{K}\ 9156\, k^{5} n^{6} z^{2}$$

$$+ 115075\, k^{5} n^{5} z^{3} + 95828\, k^{5} n^{4} z^{4} \mathrm{K}\ 40311\, k^{5} n^{3} z^{5} + 52020\, k^{4} n^{7} z^{2}$$

$$\mathrm{K}\ 838965\, k^{4} n^{6} z^{3} \mathrm{K}\ 834440\, k^{4} n^{5} z^{4} + 422715\, k^{4} n^{4} z^{5}$$

$$\mathrm{K}\ 132384\, k^{3} n^{8} z^{2} + 3154290\, k^{3} n^{7} z^{3} + 3658872\, k^{3} n^{6} z^{4}$$

$$\mathrm{K}\ 2152125\, k^{3} n^{5} z^{5} + 107040\, k^{2} n^{9} z^{2} \mathrm{K}\ 6419080\, k^{2} n^{8} z^{3}$$

$$\mathrm{K}\ 8521096\, k^{2} n^{7} z^{4} + 5678370\, k^{2} n^{6} z^{5} + 125568\, k\, n^{10} z^{2}$$

$$+ 6632496\, k\, n^{9} z^{3} + 9909924\, k\, n^{8} z^{4} \mathrm{K}\ 7350264\, k\, n^{7} z^{5}$$

$$\mathrm{K}\ 202752\, n^{11} z^{2} \mathrm{K}\ 2657808\, n^{10} z^{3} \mathrm{K}\ 4395081\, n^{9} z^{4} + 3578418\, n^{8} z^{5}$$

$$\mathrm{K}\ 268\, k^{6} n^{5} z + 5706\, k^{6} n^{4} z^{2} \mathrm{K}\ 23051\, k^{6} n^{3} z^{3} \mathrm{K}\ 7155\, k^{6} n^{2} z^{4}$$

$$+ 1146\, k^{6} n\, z^{5} + 1360\, k^{5} n^{6} z \mathrm{K}\ 98706\, k^{5} n^{5} z^{2} + 496334\, k^{5} n^{4} z^{3}$$

$$+ 206698\, k^{5} n^{3} z^{4} \mathrm{K}\ 48009\, k^{5} n^{2} z^{5} + 23136\, k^{4} n^{7} z + 648036\, k^{4} n^{6} z^{2}$$

$$\mathrm{K}\ 4185220\, k^{4} n^{5} z^{3} \mathrm{K}\ 2194366\, k^{4} n^{4} z^{4} + 656430\, k^{4} n^{3} z^{5}$$

$$\mathrm{K}\ 258560\, k^{3} n^{8} z \mathrm{K}\ 1933704\, k^{3} n^{7} z^{2} + 17819486\, k^{3} n^{6} z^{3}$$

$$+ 11303812\, k^3\, n^5\, z^4 \mathrm{K}\ 4092465\, k^3\, n^4\, z^5 + 1035776\, k^2\, n^9\, z$$

$$+ 2126352\, k^2\, n^8\, z^2 \mathrm{K}\ 40410593\, k^2\, n^7\, z^3 \mathrm{K}\ 30147636\, k^2\, n^6\, z^4$$

$$+ 12716973\, k^2\, n^5\, z^5 \mathrm{K}\ 1892352\, k\, n^{10}\, z + 1123776\, k\, n^9\, z^2$$

$$+ 45926332\, k\, n^8\, z^3 + 39388154\, k\, n^7\, z^4 \mathrm{K}\ 18870570\, k\, n^6\, z^5$$

$$+ 1327104\, n^{11}\, z \mathrm{K}\ 2964096\, n^{10}\, z^2 \mathrm{K}\ 20010090\, n^9\, z^3$$

$$\mathrm{K}\ 19309695\, n^8\, z^4 + 10315431\, n^7\, z^5 + 7328\, k^6\, n^5 \mathrm{K}\ 3144\, k^6\, n^4\, z$$

$$+ 21291\, k^6\, n^3\, z^2 \mathrm{K}\ 41616\, k^6\, n^2\, z^3 \mathrm{K}\ 5766\, k^6\, n\, z^4 + 360\, k^6\, z^5$$

$$\mathrm{K}\ 156032\, k^5\, n^6 + 31544\, k^5\, n^5\, z \mathrm{K}\ 444840\, k^5\, n^4\, z^2$$

$$+ 1137119\, k^5\, n^3\, z^3 + 244860\, k^5\, n^2\, z^4 \mathrm{K}\ 29826\, k^5\, n\, z^5$$

$$+ 1380864\, k^4\, n^7 + 96736\, k^4\, n^6\, z + 3463374\, k^4\, n^5\, z^2$$

$$\mathrm{K}\ 11539809\, k^4\, n^4\, z^3 \mathrm{K}\ 3393238\, k^4\, n^3\, z^4 + 600225\, k^4\, n^2\, z^5$$

$$\mathrm{K}\ 6502400\, k^3\, n^8 \mathrm{K}\ 2468544\, k^3\, n^7\, z \mathrm{K}\ 12298344\, k^3\, n^6\, z^2$$

$$+ 57173743\, k^3\, n^5\, z^3 + 21430598\, k^3\, n^4\, z^4 \mathrm{K}\ 4894335\, k^3\, n^3\, z^5$$

$$+ 17186816\, k^2\, n^9 + 12341632\, k^2\, n^8\, z + 17634022\, k^2\, n^7\, z^2$$

$$\mathrm{K}\ 147359149\, k^2\, n^6\, z^3 \mathrm{K}\ 67397932\, k^2\, n^5\, z^4 + 18669237\, k^2\, n^4\, z^5$$

$$\mathrm{K}\ 24182784\, k\, n^{10} \mathrm{K}\ 26034688\, k\, n^9\, z + 1343080\, k\, n^8\, z^2$$

$$+ 186978820\, k\, n^7\, z^3 + 101059452\, k\, n^6\, z^4 \mathrm{K}\ 32678271\, k\, n^5\, z^5$$

$$+ 14155776\, n^{11} + 20471808\, n^{10}\, z \mathrm{K}\ 18649184\, n^9\, z^2$$

$$\mathrm{K}\ 89633160\, n^8\, z^3 \mathrm{K}\ 55672122\, n^7\, z^4 + 20486898\, n^6\, z^5$$

$$+ 56256\, k^6\, n^4 \mathrm{K}\ 14474\, k^6\, n^3\, z + 39636\, k^6\, n^2\, z^2 \mathrm{K}\ 37368\, k^6\, n\, z^3$$

$K\ 1800\, k^{6}\, z^{4} K\ 1399616\, k^{5}\, n^{5} + 221864\, k^{5}\, n^{4}\, z K\ 1065570\, k^{5}\, n^{3}\, z^{2}$

$+ 1457172\, k^{5}\, n^{2}\, z^{3} + 151320\, k^{5}\, n\, z^{4} K\ 7560\, k^{5}\, z^{5} + 14197248\, k^{4}\, n^{6}$

$K\ 480746\, k^{4}\, n^{5}\, z + 10224396\, k^{4}\, n^{4}\, z^{2} K\ 18967997\, k^{4}\, n^{3}\, z^{3}$

$K\ 3089916\, k^{4}\, n^{2}\, z^{4} + 299550\, k^{4}\, n\, z^{5} K\ 75500544\, k^{3}\, n^{7}$

$K\ 8439792\, k^{3}\, n^{6}\, z K\ 44162004\, k^{3}\, n^{5}\, z^{2} + 113810448\, k^{3}\, n^{4}\, z^{3}$

$+ 25557218\, k^{3}\, n^{3}\, z^{4} K\ 3597495\, k^{3}\, n^{2}\, z^{5} + 222728192\, k^{2}\, n^{8}$

$+ 60974656\, k^{2}\, n^{7}\, z + 80980002\, k^{2}\, n^{6}\, z^{2} K\ 342616779\, k^{2}\, n^{5}\, z^{3}$

$K\ 98798734\, k^{2}\, n^{4}\, z^{4} + 17972907\, k^{2}\, n^{3}\, z^{5} K\ 346423296\, k\, n^{9}$

$K\ 155009152\, k\, n^{8}\, z K\ 23176360\, k\, n^{7}\, z^{2} + 495123980\, k\, n^{6}\, z^{3}$

$+ 174968354\, k\, n^{5}\, z^{4} K\ 38663145\, k\, n^{4}\, z^{5} + 222363648\, n^{10}$

$+ 139701248\, n^{9}\, z K\ 65114456\, n^{8}\, z^{2} K\ 265176948\, n^{7}\, z^{3}$

$K\ 110635026\, n^{6}\, z^{4} + 28608687\, n^{5}\, z^{5} + 166966\, k^{6}\, n^{3}$

$K\ 31824\, k^{6}\, n^{2}\, z + 36576\, k^{6}\, n\, z^{2} K\ 13320\, k^{6}\, z^{3} K\ 5102440\, k^{5}\, n^{4}$

$+ 710254\, k^{5}\, n^{3}\, z K\ 1423944\, k^{5}\, n^{2}\, z^{2} + 989064\, k^{5}\, n\, z^{3} + 38160\, k^{5}\, z^{4}$

$+ 61338272\, k^{4}\, n^{5} K\ 4069702\, k^{4}\, n^{4}\, z + 17926386\, k^{4}\, n^{3}\, z^{2}$

$K\ 18564264\, k^{4}\, n^{2}\, z^{3} K\ 1535976\, k^{4}\, n\, z^{4} + 63000\, k^{4}\, z^{5}$

$K\ 377226112\, k^{3}\, n^{6} K\ 8684348\, k^{3}\, n^{5}\, z K\ 97545636\, k^{3}\, n^{4}\, z^{2}$

$+ 143761085\, k^{3}\, n^{3}\, z^{3} + 18736878\, k^{3}\, n^{2}\, z^{4} K\ 1486710\, k^{3}\, n\, z^{5}$

$+ 1264119296\, k^{2}\, n^{7} + 158099784\, k^{2}\, n^{6}\, z + 228893257\, k^{2}\, n^{5}\, z^{2}$

$K\ 526115719\, k^{2}\, n^{4}\, z^{3} K\ 95008779\, k^{2}\, n^{3}\, z^{4} + 10943742\, k^{2}\, n^{2}\, z^{5}$

$\mathrm{K}\ 2203125760\,k\,n^{8}\,\mathrm{K}\ 520737696\,k\,n^{7}\,z\,\mathrm{K}\ 142331300\,k\,n^{6}\,z^{2}$

$+\,890070476\,k\,n^{5}\,z^{3}+207071358\,k\,n^{4}\,z^{4}\,\mathrm{K}\ 30861990\,k\,n^{3}\,z^{5}$

$+\,1567473664\,n^{9}+554337536\,n^{8}\,z\,\mathrm{K}\ 132924362\,n^{7}\,z^{2}$

$\mathrm{K}\ 543537648\,n^{6}\,z^{3}\,\mathrm{K}\ 154672833\,n^{5}\,z^{4}+28091502\,n^{4}\,z^{5}$

$+\,239256\,k^{6}\,n^{2}\,\mathrm{K}\ 33306\,k^{6}\,n\,z+13320\,k^{6}\,z^{2}\,\mathrm{K}\ 9682300\,k^{5}\,n^{3}$

$+\,1159584\,k^{5}\,n^{2}\,z\,\mathrm{K}\ 1003464\,k^{5}\,n\,z^{2}+277560\,k^{5}\,z^{3}$

$+\,144452256\,k^{4}\,n^{4}\,\mathrm{K}\ 11140768\,k^{4}\,n^{3}\,z+18616860\,k^{4}\,n^{2}\,z^{2}$

$\mathrm{K}\ 10007064\,k^{4}\,n\,z^{3}\,\mathrm{K}\ 321840\,k^{4}\,z^{4}\,\mathrm{K}\ 1059808448\,k^{3}\,n^{5}$

$+\,19542740\,k^{3}\,n^{4}\,z\,\mathrm{K}\ 135459636\,k^{3}\,n^{3}\,z^{2}+112405284\,k^{3}\,n^{2}\,z^{3}$

$+\,7725588\,k^{3}\,n\,z^{4}\,\mathrm{K}\ 264600\,k^{3}\,z^{5}+4125453824\,k^{2}\,n^{6}$

$+\,214483726\,k^{2}\,n^{5}\,z+415638732\,k^{2}\,n^{4}\,z^{2}\,\mathrm{K}\ 532952796\,k^{2}\,n^{3}\,z^{3}$

$\mathrm{K}\ 57812229\,k^{2}\,n^{2}\,z^{4}+3824904\,k^{2}\,n\,z^{5}\,\mathrm{K}\ 8192646144\,k\,n^{7}$

$\mathrm{K}\ 1072798360\,k\,n^{6}\,z\,\mathrm{K}\ 408817306\,k\,n^{5}\,z^{2}+1098823824\,k\,n^{4}\,z^{3}$

$+\,165431988\,k\,n^{3}\,z^{4}\,\mathrm{K}\ 15904908\,k\,n^{2}\,z^{5}+6545600512\,n^{8}$

$+\,1411617536\,n^{7}\,z\,\mathrm{K}\ 140465270\,n^{6}\,z^{2}\,\mathrm{K}\ 786875466\,n^{5}\,z^{3}$

$\mathrm{K}\ 152145999\,n^{4}\,z^{4}+19007028\,n^{3}\,z^{5}+165018\,k^{6}\,n\,\mathrm{K}\ 13320\,k^{6}\,z$

$\mathrm{K}\ 10082568\,k^{5}\,n^{2}+943530\,k^{5}\,n\,z\,\mathrm{K}\ 290880\,k^{5}\,z^{2}$

$+\,200296612\,k^{4}\,n^{3}\,\mathrm{K}\ 15078738\,k^{4}\,n^{2}\,z+10588752\,k^{4}\,n\,z^{2}$

$\mathrm{K}\ 2289960\,k^{4}\,z^{3}\,\mathrm{K}\ 1831620240\,k^{3}\,n^{4}+70450094\,k^{3}\,n^{3}\,z$

$\mathrm{K}\ 115410996\,k^{3}\,n^{2}\,z^{2}+49684392\,k^{3}\,n\,z^{3}+1372320\,k^{3}\,z^{4}$

$$+ 8532753280\,k^2\,n^5 + 92660342\,k^2\,n^4\,z + 487191787\,k^2\,n^3\,z^2$$

$$\mathrm{K}\;\; 343015320\,k^2\,n^2\,z^3\,\mathrm{K}\;\; 20202258\,k^2\,n\,z^4 + 584640\,k^2\,z^5$$

$$\mathrm{K}\;\; 19729116672\,k\,n^6\,\mathrm{K}\;\; 1360884228\,k\,n^5\,z\,\mathrm{K}\;\; 701290700\,k\,n^4\,z^2$$

$$+ 918790656\,k\,n^3\,z^3 + 85386438\,k\,n^2\,z^4\,\mathrm{K}\;\; 4777704\,k\,n\,z^5$$

$$+ 17991700480\,n^7 + 2395289936\,n^6\,z + 577858\,n^5\,z^2$$

$$\mathrm{K}\;\; 803772216\,n^4\,z^3\,\mathrm{K}\;\; 103199832\,n^3\,z^4 + 8438472\,n^2\,z^5 + 43560\,k^6$$

$$\mathrm{K}\;\; 5455044\,k^5\,n + 304200\,k^5\,z + 163469808\,k^4\,n^2$$

$$\mathrm{K}\;\; 10245846\,k^4\,n\,z + 2543040\,k^4\,z^2\,\mathrm{K}\;\; 1993923256\,k^3\,n^3$$

$$+ 89815812\,k^3\,n^2\,z\,\mathrm{K}\;\; 55151136\,k^3\,n\,z^2 + 9494280\,k^3\,z^3$$

$$+ 11599484432\,k^2\,n^4\,\mathrm{K}\;\; 144956542\,k^2\,n^3\,z + 356705112\,k^2\,n^2\,z^2$$

$$\mathrm{K}\;\; 127115568\,k^2\,n\,z^3\,\mathrm{K}\;\; 3089160\,k^2\,z^4\,\mathrm{K}\;\; 32148318080\,k\,n^5$$

$$\mathrm{K}\;\; 964021492\,k\,n^4\,z\,\mathrm{K}\;\; 756783610\,k\,n^3\,z^2 + 497351784\,k\,n^2\,z^3$$

$$+ 25708212\,k\,n\,z^4\,\mathrm{K}\;\; 635040\,k\,z^5 + 34177089280\,n^6$$

$$+ 2707830944\,n^5\,z + 221511670\,n^4\,z^2\,\mathrm{K}\;\; 567130104\,n^3\,z^3$$

$$\mathrm{K}\;\; 45968796\,n^2\,z^4 + 2211840\,n\,z^5\,\mathrm{K}\;\; 1194480\,k^5 + 72648444\,n\,k^4$$

$$\mathrm{K}\;\; 2796120\,k^4\,z\,\mathrm{K}\;\; 1334837328\,k^3\,n^2 + 54505758\,k^3\,n\,z$$

$$\mathrm{K}\;\; 11319120\,k^3\,z^2 + 10362649558\,k^2\,n^3\,\mathrm{K}\;\; 245644590\,k^2\,n^2\,z$$

$$+ 148402512\,k^2\,n\,z^2\,\mathrm{K}\;\; 20635920\,k^2\,z^3\,\mathrm{K}\;\; 35894877320\,k\,n^4$$

$$\mathrm{K}\;\; 197178788\,k\,n^3\,z\,\mathrm{K}\;\; 505255908\,k\,n^2\,z^2 + 157161744\,k\,n\,z^3$$

$$+ 3427920\,k\,z^4 + 45779005120\,n^5 + 1953070496\,n^4\,z$$

$$+ 321114408\, n^3 z^2 \mathrm{K}\ 262952640\, n^2 z^3 \mathrm{K}\ 12100320\, n z^4$$

$$+ 259200\, z^5 + 13542480\, k^4 \mathrm{K}\ 502178472\, k^3 n + 13143960\, k^3 z$$

$$+ 5865355080\, k^2 n^2 \mathrm{K}\ 147001248\, k^2 n z + 26776440\, k^2 z^2$$

$$\mathrm{K}\ 27112538956\, k n^3 + 228280908\, k n^2 z \mathrm{K}\ 191087640\, k n z^2$$

$$+ 21980160\, k z^3 + 43231470352\, n^4 + 779054592\, n^3 z$$

$$+ 229653720\, n^2 z^2 \mathrm{K}\ 72014400\, n z^3 \mathrm{K}\ 1425600\, z^4 \mathrm{K}\ 81224640\, k^3$$

$$+ 1907973498\, k^2 n \mathrm{K}\ 32916960\, k^2 z \mathrm{K}\ 13255890984\, k n^2$$

$$+ 180079272\, k n z \mathrm{K}\ 31330800\, k z^2 + 28201921488\, n^3$$

$$+ 71334288\, n^2 z + 86313600\, n z^2 \mathrm{K}\ 8812800\, z^3 + 271663560\, k^2$$

$$\mathrm{K}\ 3787310484\, k n + 40681440\, k z + 12100864176\, n^2$$

$$\mathrm{K}\ 63430560\, z n + 13608000\, z^2 \mathrm{K}\ 480044880\, k + 3072921840\, n$$

$$\mathrm{K}\ 18403200\, z + 349790400)) \Big/ ((\mathrm{K}\ 3 n \mathrm{K}\ 6 + k)\ (\mathrm{K}\ 3 n \mathrm{K}\ 5 + k)\ (
\mathrm{K}\ 3 n + k \mathrm{K}\ 4)\ (\mathrm{K}\ n + k \mathrm{K}\ 1)\ (\mathrm{K}\ n + k)\ (n + 2)\ (\mathrm{K}\ 3 n + k \mathrm{K}\ 1)\ (\mathrm{K}\ 3 n
+ k \mathrm{K}\ 2)\ (\mathrm{K}\ 3 n + k \mathrm{K}\ 3)\ (3 n + 5)\ (3 n + 4))]$$

```
> reczb_B := add(coeff(zb_B[1], sn, i) * bb[n+i,k], i=0..2);
```

$$reczb_B := 3 z^4 n (3 n + 2) (3 n + 1) (n^2 z^2 \mathrm{K}\ 4 n^2 z + 4 n z^2 \mathrm{K}\ 32 n^2
\mathrm{K}\ 16 z n + 3 z^2 \mathrm{K}\ 128 n \mathrm{K}\ 12 z \mathrm{K}\ 126)\ bb_{n,k} \mathrm{K}\ (2 n + 3)\ (2 n^4 z^5
\mathrm{K}\ 11 n^4 z^4 + 12 n^3 z^5 + 44 n^4 z^3 \mathrm{K}\ 66 n^3 z^4 + 22 n^2 z^5 \mathrm{K}\ 160 n^4 z^2
+ 264 n^3 z^3 \mathrm{K}\ 121 n^2 z^4 + 12 n z^5 \mathrm{K}\ 3584 n^4 z \mathrm{K}\ 960 n^3 z^2 + 484 n^2 z^3
\mathrm{K}\ 66 n z^4 \mathrm{K}\ 4096 n^4 \mathrm{K}\ 21504 n^3 z \mathrm{K}\ 1790 n^2 z^2 + 264 n z^3 \mathrm{K}\ 24576 n^3
\mathrm{K}\ 44704 n^2 z \mathrm{K}\ 1050 n z^2 \mathrm{K}\ 52736 n^2 \mathrm{K}\ 37344 z n \mathrm{K}\ 47616 n
\mathrm{K}\ 10080 z \mathrm{K}\ 15120)\ bb_{n+1,k} + 3 (3 n + 7) (n + 3) (3 n + 8) (n^2 z^2
\mathrm{K}\ 4 n^2 z + 2 n z^2 \mathrm{K}\ 32 n^2 \mathrm{K}\ 8 z n \mathrm{K}\ 64 n \mathrm{K}\ 30)\ bb_{n+2,k} \quad \textbf{(2.3.2)}$$

This provides the eta[i](n) ; only eta[2](n) is given in the article:

```
> for i from 2 to 0 by -1 do eta[i](n) = coeff(zb_B[1], sn, i)
  od;
```

$$\eta_2(n) = 3 (3 n + 7) (n + 3) (3 n + 8) (n^2 z^2 \mathrm{K}\ 4 n^2 z + 2 n z^2 \mathrm{K}\ 32 n^2$$

$$K\ 8\,z\,n K\ 64\,n K\ 30)$$

$$\eta_1(n) = K\ (2\,n+3)\ (2\,n^4\,z^5 K\ 11\,n^4\,z^4 + 12\,n^3\,z^5 + 44\,n^4\,z^3 K\ 66\,n^3\,z^4$$
$$+ 22\,n^2\,z^5 K\ 160\,n^4\,z^2 + 264\,n^3\,z^3 K\ 121\,n^2\,z^4 + 12\,n\,z^5 K\ 3584\,n^4\,z$$
$$K\ 960\,n^3\,z^2 + 484\,n^2\,z^3 K\ 66\,n\,z^4 K\ 4096\,n^4 K\ 21504\,n^3\,z$$
$$K\ 1790\,n^2\,z^2 + 264\,n\,z^3 K\ 24576\,n^3 K\ 44704\,n^2\,z K\ 1050\,n\,z^2$$
$$K\ 52736\,n^2 K\ 37344\,z\,n K\ 47616\,n K\ 10080\,z K\ 15120)$$

$$\eta_0(n) = 3\,z^4\,n\,(3\,n+2)\,(3\,n+1)\,(n^2\,z^2 K\ 4\,n^2\,z + 4\,n\,z^2 K\ 32\,n^2 K\ 16\,z\,n \quad \textbf{(2.3.3)}$$
$$+ 3\,z^2 K\ 128\,n K\ 12\,z K\ 126)$$

The algorithm justifies the following recurrence relation involving a forward finite operator Delta[k]:

```
> reczb_B = Delta[k](factor(zb_B[2]/CF_B/z^k) * bb[n,k]);
```

$$3\,z^4\,n\,(3\,n+2)\,(3\,n+1)\,(n^2\,z^2 K\ 4\,n^2\,z + 4\,n\,z^2 K\ 32\,n^2 K\ 16\,z\,n + 3\,z^2 \quad \textbf{(2.3.4)}$$
$$K\ 128\,n K\ 12\,z K\ 126)\ bb_{n,k} K\ (2\,n+3)\ (2\,n^4\,z^5 K\ 11\,n^4\,z^4 + 12\,n^3\,z^5$$
$$+ 44\,n^4\,z^3 K\ 66\,n^3\,z^4 + 22\,n^2\,z^5 K\ 160\,n^4\,z^2 + 264\,n^3\,z^3 K\ 121\,n^2\,z^4$$
$$+ 12\,n\,z^5 K\ 3584\,n^4\,z K\ 960\,n^3\,z^2 + 484\,n^2\,z^3 K\ 66\,n\,z^4 K\ 4096\,n^4$$
$$K\ 21504\,n^3\,z K\ 1790\,n^2\,z^2 + 264\,n\,z^3 K\ 24576\,n^3 K\ 44704\,n^2\,z$$
$$K\ 1050\,n\,z^2 K\ 52736\,n^2 K\ 37344\,z\,n K\ 47616\,n K\ 10080\,z K\ 15120)$$
$$bb_{n+1,k} + 3\,(3\,n+7)\,(n+3)\,(3\,n+8)\,(n^2\,z^2 K\ 4\,n^2\,z + 2\,n\,z^2 K\ 32\,n^2$$
$$K\ 8\,z\,n K\ 64\,n K\ 30)\ bb_{n+2,k} = \Delta_k(K\ ((K\ 4\,n K\ 2$$
$$+ k)\ k\,n\,(27\,k^6\,n^5\,z^5 K\ 432\,k^5\,n^6\,z^5 + 2835\,k^4\,n^7\,z^5 K\ 9720\,k^3\,n^8\,z^5$$
$$+ 18225\,k^2\,n^9\,z^5 K\ 17496\,k\,n^{10}\,z^5 + 6561\,n^{11}\,z^5 K\ 139\,k^6\,n^5\,z^4$$
$$+ 243\,k^6\,n^4\,z^5 + 2248\,k^5\,n^6\,z^4 K\ 4455\,k^5\,n^5\,z^5 K\ 14898\,k^4\,n^7\,z^4$$
$$+ 32940\,k^4\,n^6\,z^5 + 51532\,k^3\,n^8\,z^4 K\ 125550\,k^3\,n^7\,z^5 K\ 97375\,k^2\,n^9\,z^4$$
$$+ 258795\,k^2\,n^8\,z^5 + 94092\,k\,n^{10}\,z^4 K\ 270459\,k\,n^9\,z^5 K\ 35460\,n^{11}\,z^4$$
$$+ 109350\,n^{10}\,z^5 K\ 703\,k^6\,n^5\,z^3 K\ 1245\,k^6\,n^4\,z^4 + 843\,k^6\,n^3\,z^5$$
$$+ 11092\,k^5\,n^6\,z^3 + 23078\,k^5\,n^5\,z^4 K\ 18591\,k^5\,n^4\,z^5 K\ 71817\,k^4\,n^7\,z^3$$
$$K\ 172430\,k^4\,n^6\,z^4 + 160065\,k^4\,n^5\,z^5 + 243112\,k^3\,n^8\,z^3$$
$$+ 663582\,k^3\,n^7\,z^4 K\ 694440\,k^3\,n^6\,z^5 K\ 450532\,k^2\,n^9\,z^3$$
$$K\ 1379681\,k^2\,n^8\,z^4 + 1601775\,k^2\,n^7\,z^5 + 428112\,k\,n^{10}\,z^3$$
$$+ 1452476\,k\,n^9\,z^4 K\ 1847529\,k\,n^8\,z^5 K\ 159264\,n^{11}\,z^3$$
$$K\ 590532\,n^{10}\,z^4 + 814293\,n^9\,z^5 + 615\,k^6\,n^5\,z^2 K\ 6366\,k^6\,n^4\,z^3$$
$$K\ 4295\,k^6\,n^3\,z^4 + 1413\,k^6\,n^2\,z^5 K\ 9156\,k^5\,n^6\,z^2 + 115075\,k^5\,n^5\,z^3$$
$$+ 95828\,k^5\,n^4\,z^4 K\ 40311\,k^5\,n^3\,z^5 + 52020\,k^4\,n^7\,z^2 K\ 838965\,k^4\,n^6\,z^3$$
$$K\ 834440\,k^4\,n^5\,z^4 + 422715\,k^4\,n^4\,z^5 K\ 132384\,k^3\,n^8\,z^2$$

$$+ 3154290\,k^3 n^7 z^3 + 3658872\,k^3 n^6 z^4 - 2152125\,k^3 n^5 z^5$$
$$+ 107040\,k^2 n^9 z^2 - 6419080\,k^2 n^8 z^3 - 8521096\,k^2 n^7 z^4$$
$$+ 5678370\,k^2 n^6 z^5 + 125568\,k\,n^{10} z^2 + 6632496\,k\,n^9 z^3$$
$$+ 9909924\,k\,n^8 z^4 - 7350264\,k\,n^7 z^5 - 202752\,n^{11} z^2$$
$$- 2657808\,n^{10} z^3 - 4395081\,n^9 z^4 + 3578418\,n^8 z^5 - 268\,k^6 n^5 z$$
$$+ 5706\,k^6 n^4 z^2 - 23051\,k^6 n^3 z^3 - 7155\,k^6 n^2 z^4 + 1146\,k^6 n\,z^5$$
$$+ 1360\,k^5 n^6 z - 98706\,k^5 n^5 z^2 + 496334\,k^5 n^4 z^3 + 206698\,k^5 n^3 z^4$$
$$- 48009\,k^5 n^2 z^5 + 23136\,k^4 n^7 z + 648036\,k^4 n^6 z^2$$
$$- 4185220\,k^4 n^5 z^3 - 2194366\,k^4 n^4 z^4 + 656430\,k^4 n^3 z^5$$
$$- 258560\,k^3 n^8 z - 1933704\,k^3 n^7 z^2 + 17819486\,k^3 n^6 z^3$$
$$+ 11303812\,k^3 n^5 z^4 - 4092465\,k^3 n^4 z^5 + 1035776\,k^2 n^9 z$$
$$+ 2126352\,k^2 n^8 z^2 - 40410593\,k^2 n^7 z^3 - 30147636\,k^2 n^6 z^4$$
$$+ 12716973\,k^2 n^5 z^5 - 1892352\,k\,n^{10} z + 1123776\,k\,n^9 z^2$$
$$+ 45926332\,k\,n^8 z^3 + 39388154\,k\,n^7 z^4 - 18870570\,k\,n^6 z^5$$
$$+ 1327104\,n^{11} z - 2964096\,n^{10} z^2 - 20010090\,n^9 z^3$$
$$- 19309695\,n^8 z^4 + 10315431\,n^7 z^5 + 7328\,k^6 n^5 - 3144\,k^6 n^4 z$$
$$+ 21291\,k^6 n^3 z^2 - 41616\,k^6 n^2 z^3 - 5766\,k^6 n\,z^4 + 360\,k^6 z^5$$
$$- 156032\,k^5 n^6 + 31544\,k^5 n^5 z - 444840\,k^5 n^4 z^2$$
$$+ 1137119\,k^5 n^3 z^3 + 244860\,k^5 n^2 z^4 - 29826\,k^5 n\,z^5$$
$$+ 1380864\,k^4 n^7 + 96736\,k^4 n^6 z + 3463374\,k^4 n^5 z^2$$
$$- 11539809\,k^4 n^4 z^3 - 3393238\,k^4 n^3 z^4 + 600225\,k^4 n^2 z^5$$
$$- 6502400\,k^3 n^8 - 2468544\,k^3 n^7 z - 12298344\,k^3 n^6 z^2$$
$$+ 57173743\,k^3 n^5 z^3 + 21430598\,k^3 n^4 z^4 - 4894335\,k^3 n^3 z^5$$
$$+ 17186816\,k^2 n^9 + 12341632\,k^2 n^8 z + 17634022\,k^2 n^7 z^2$$
$$- 147359149\,k^2 n^6 z^3 - 67397932\,k^2 n^5 z^4 + 18669237\,k^2 n^4 z^5$$
$$- 24182784\,k\,n^{10} - 26034688\,k\,n^9 z + 1343080\,k\,n^8 z^2$$
$$+ 186978820\,k\,n^7 z^3 + 101059452\,k\,n^6 z^4 - 32678271\,k\,n^5 z^5$$
$$+ 14155776\,n^{11} + 20471808\,n^{10} z - 18649184\,n^9 z^2$$
$$- 89633160\,n^8 z^3 - 55672122\,n^7 z^4 + 20486898\,n^6 z^5$$
$$+ 56256\,k^6 n^4 - 14474\,k^6 n^3 z + 39636\,k^6 n^2 z^2 - 37368\,k^6 n\,z^3$$
$$- 1800\,k^6 z^4 - 1399616\,k^5 n^5 + 221864\,k^5 n^4 z - 1065570\,k^5 n^3 z^2$$
$$+ 1457172\,k^5 n^2 z^3 + 151320\,k^5 n\,z^4 - 7560\,k^5 z^5 + 14197248\,k^4 n^6$$

$$\mathrm{K}\ 480746\,k^4\,n^5\,z + 10224396\,k^4\,n^4\,z^2\,\mathrm{K}\ 18967997\,k^4\,n^3\,z^3$$
$$\mathrm{K}\ 3089916\,k^4\,n^2\,z^4 + 299550\,k^4\,n\,z^5\,\mathrm{K}\ 75500544\,k^3\,n^7$$
$$\mathrm{K}\ 8439792\,k^3\,n^6\,z\,\mathrm{K}\ 44162004\,k^3\,n^5\,z^2 + 113810448\,k^3\,n^4\,z^3$$
$$+ 25557218\,k^3\,n^3\,z^4\,\mathrm{K}\ 3597495\,k^3\,n^2\,z^5 + 222728192\,k^2\,n^8$$
$$+ 60974656\,k^2\,n^7\,z + 80980002\,k^2\,n^6\,z^2\,\mathrm{K}\ 342616779\,k^2\,n^5\,z^3$$
$$\mathrm{K}\ 98798734\,k^2\,n^4\,z^4 + 17972907\,k^2\,n^3\,z^5\,\mathrm{K}\ 346423296\,k\,n^9$$
$$\mathrm{K}\ 155009152\,k\,n^8\,z\,\mathrm{K}\ 23176360\,k\,n^7\,z^2 + 495123980\,k\,n^6\,z^3$$
$$+ 174968354\,k\,n^5\,z^4\,\mathrm{K}\ 38663145\,k\,n^4\,z^5 + 222363648\,n^{10}$$
$$+ 139701248\,n^9\,z\,\mathrm{K}\ 65114456\,n^8\,z^2\,\mathrm{K}\ 265176948\,n^7\,z^3$$
$$\mathrm{K}\ 110635026\,n^6\,z^4 + 28608687\,n^5\,z^5 + 166966\,k^6\,n^3$$
$$\mathrm{K}\ 31824\,k^6\,n^2\,z + 36576\,k^6\,n\,z^2\,\mathrm{K}\ 13320\,k^6\,z^3\,\mathrm{K}\ 5102440\,k^5\,n^4$$
$$+ 710254\,k^5\,n^3\,z\,\mathrm{K}\ 1423944\,k^5\,n^2\,z^2 + 989064\,k^5\,n\,z^3 + 38160\,k^5\,z^4$$
$$+ 61338272\,k^4\,n^5\,\mathrm{K}\ 4069702\,k^4\,n^4\,z + 17926386\,k^4\,n^3\,z^2$$
$$\mathrm{K}\ 18564264\,k^4\,n^2\,z^3\,\mathrm{K}\ 1535976\,k^4\,n\,z^4 + 63000\,k^4\,z^5$$
$$\mathrm{K}\ 377226112\,k^3\,n^6\,\mathrm{K}\ 8684348\,k^3\,n^5\,z\,\mathrm{K}\ 97545636\,k^3\,n^4\,z^2$$
$$+ 143761085\,k^3\,n^3\,z^3 + 18736878\,k^3\,n^2\,z^4\,\mathrm{K}\ 1486710\,k^3\,n\,z^5$$
$$+ 1264119296\,k^2\,n^7 + 158099784\,k^2\,n^6\,z + 228893257\,k^2\,n^5\,z^2$$
$$\mathrm{K}\ 526115719\,k^2\,n^4\,z^3\,\mathrm{K}\ 95008779\,k^2\,n^3\,z^4 + 10943742\,k^2\,n^2\,z^5$$
$$\mathrm{K}\ 2203125760\,k\,n^8\,\mathrm{K}\ 520737696\,k\,n^7\,z\,\mathrm{K}\ 142331300\,k\,n^6\,z^2$$
$$+ 890070476\,k\,n^5\,z^3 + 207071358\,k\,n^4\,z^4\,\mathrm{K}\ 30861990\,k\,n^3\,z^5$$
$$+ 1567473664\,n^9 + 554337536\,n^8\,z\,\mathrm{K}\ 132924362\,n^7\,z^2$$
$$\mathrm{K}\ 543537648\,n^6\,z^3\,\mathrm{K}\ 154672833\,n^5\,z^4 + 28091502\,n^4\,z^5$$
$$+ 239256\,k^6\,n^2\,\mathrm{K}\ 33306\,k^6\,n\,z + 13320\,k^6\,z^2\,\mathrm{K}\ 9682300\,k^5\,n^3$$
$$+ 1159584\,k^5\,n^2\,z\,\mathrm{K}\ 1003464\,k^5\,n\,z^2 + 277560\,k^5\,z^3$$
$$+ 144452256\,k^4\,n^4\,\mathrm{K}\ 11140768\,k^4\,n^3\,z + 18616860\,k^4\,n^2\,z^2$$
$$\mathrm{K}\ 10007064\,k^4\,n\,z^3\,\mathrm{K}\ 321840\,k^4\,z^4\,\mathrm{K}\ 1059808448\,k^3\,n^5$$
$$+ 19542740\,k^3\,n^4\,z\,\mathrm{K}\ 135459636\,k^3\,n^3\,z^2 + 112405284\,k^3\,n^2\,z^3$$
$$+ 7725588\,k^3\,n\,z^4\,\mathrm{K}\ 264600\,k^3\,z^5 + 4125453824\,k^2\,n^6$$
$$+ 214483726\,k^2\,n^5\,z + 415638732\,k^2\,n^4\,z^2\,\mathrm{K}\ 532952796\,k^2\,n^3\,z^3$$
$$\mathrm{K}\ 57812229\,k^2\,n^2\,z^4 + 3824904\,k^2\,n\,z^5\,\mathrm{K}\ 8192646144\,k\,n^7$$
$$\mathrm{K}\ 1072798360\,k\,n^6\,z\,\mathrm{K}\ 408817306\,k\,n^5\,z^2 + 1098823824\,k\,n^4\,z^3$$
$$+ 165431988\,k\,n^3\,z^4\,\mathrm{K}\ 15904908\,k\,n^2\,z^5 + 6545600512\,n^8$$

$$+ 1411617536\, n^{7}\, z \mathrm{K}\ 140465270\, n^{6}\, z^{2} \mathrm{K}\ 786875466\, n^{5}\, z^{3}$$
$$\mathrm{K}\ 152145999\, n^{4}\, z^{4} + 19007028\, n^{3}\, z^{5} + 165018\, k^{6}\, n \mathrm{K}\ 13320\, k^{6}\, z$$
$$\mathrm{K}\ 10082568\, k^{5}\, n^{2} + 943530\, k^{5}\, n\, z \mathrm{K}\ 290880\, k^{5}\, z^{2}$$
$$+ 200296612\, k^{4}\, n^{3} \mathrm{K}\ 15078738\, k^{4}\, n^{2}\, z + 10588752\, k^{4}\, n\, z^{2}$$
$$\mathrm{K}\ 2289960\, k^{4}\, z^{3} \mathrm{K}\ 1831620240\, k^{3}\, n^{4} + 70450094\, k^{3}\, n^{3}\, z$$
$$\mathrm{K}\ 115410996\, k^{3}\, n^{2}\, z^{2} + 49684392\, k^{3}\, n\, z^{3} + 1372320\, k^{3}\, z^{4}$$
$$+ 8532753280\, k^{2}\, n^{5} + 92660342\, k^{2}\, n^{4}\, z + 487191787\, k^{2}\, n^{3}\, z^{2}$$
$$\mathrm{K}\ 343015320\, k^{2}\, n^{2}\, z^{3} \mathrm{K}\ 20202258\, k^{2}\, n\, z^{4} + 584640\, k^{2}\, z^{5}$$
$$\mathrm{K}\ 19729116672\, k\, n^{6} \mathrm{K}\ 1360884228\, k\, n^{5}\, z \mathrm{K}\ 701290700\, k\, n^{4}\, z^{2}$$
$$+ 918790656\, k\, n^{3}\, z^{3} + 85386438\, k\, n^{2}\, z^{4} \mathrm{K}\ 4777704\, k\, n\, z^{5}$$
$$+ 17991700480\, n^{7} + 2395289936\, n^{6}\, z + 577858\, n^{5}\, z^{2}$$
$$\mathrm{K}\ 803772216\, n^{4}\, z^{3} \mathrm{K}\ 103199832\, n^{3}\, z^{4} + 8438472\, n^{2}\, z^{5} + 43560\, k^{6}$$
$$\mathrm{K}\ 5455044\, k^{5}\, n + 304200\, k^{5}\, z + 163469808\, k^{4}\, n^{2}$$
$$\mathrm{K}\ 10245846\, k^{4}\, n\, z + 2543040\, k^{4}\, z^{2} \mathrm{K}\ 1993923256\, k^{3}\, n^{3}$$
$$+ 89815812\, k^{3}\, n^{2}\, z \mathrm{K}\ 55151136\, k^{3}\, n\, z^{2} + 9494280\, k^{3}\, z^{3}$$
$$+ 11599484432\, k^{2}\, n^{4} \mathrm{K}\ 144956542\, k^{2}\, n^{3}\, z + 356705112\, k^{2}\, n^{2}\, z^{2}$$
$$\mathrm{K}\ 127115568\, k^{2}\, n\, z^{3} \mathrm{K}\ 3089160\, k^{2}\, z^{4} \mathrm{K}\ 32148318080\, k\, n^{5}$$
$$\mathrm{K}\ 964021492\, k\, n^{4}\, z \mathrm{K}\ 756783610\, k\, n^{3}\, z^{2} + 497351784\, k\, n^{2}\, z^{3}$$
$$+ 25708212\, k\, n\, z^{4} \mathrm{K}\ 635040\, k\, z^{5} + 34177089280\, n^{6}$$
$$+ 2707830944\, n^{5}\, z + 221511670\, n^{4}\, z^{2} \mathrm{K}\ 567130104\, n^{3}\, z^{3}$$
$$\mathrm{K}\ 45968796\, n^{2}\, z^{4} + 2211840\, n\, z^{5} \mathrm{K}\ 1194480\, k^{5} + 72648444\, n\, k^{4}$$
$$\mathrm{K}\ 2796120\, k^{4}\, z \mathrm{K}\ 1334837328\, k^{3}\, n^{2} + 54505758\, k^{3}\, n\, z$$
$$\mathrm{K}\ 11319120\, k^{3}\, z^{2} + 10362649558\, k^{2}\, n^{3} \mathrm{K}\ 245644590\, k^{2}\, n^{2}\, z$$
$$+ 148402512\, k^{2}\, n\, z^{2} \mathrm{K}\ 20635920\, k^{2}\, z^{3} \mathrm{K}\ 35894877320\, k\, n^{4}$$
$$\mathrm{K}\ 197178788\, k\, n^{3}\, z \mathrm{K}\ 505255908\, k\, n^{2}\, z^{2} + 157161744\, k\, n\, z^{3}$$
$$+ 3427920\, k\, z^{4} + 45779005120\, n^{5} + 1953070496\, n^{4}\, z$$
$$+ 321114408\, n^{3}\, z^{2} \mathrm{K}\ 262952640\, n^{2}\, z^{3} \mathrm{K}\ 12100320\, n\, z^{4}$$
$$+ 259200\, z^{5} + 13542480\, k^{4} \mathrm{K}\ 502178472\, k^{3}\, n + 13143960\, k^{3}\, z$$
$$+ 5865355080\, k^{2}\, n^{2} \mathrm{K}\ 147001248\, k^{2}\, n\, z + 26776440\, k^{2}\, z^{2}$$
$$\mathrm{K}\ 27112538956\, k\, n^{3} + 228280908\, k\, n^{2}\, z \mathrm{K}\ 191087640\, k\, n\, z^{2}$$
$$+ 21980160\, k\, z^{3} + 43231470352\, n^{4} + 779054592\, n^{3}\, z$$
$$+ 229653720\, n^{2}\, z^{2} \mathrm{K}\ 72014400\, n\, z^{3} \mathrm{K}\ 1425600\, z^{4} \mathrm{K}\ 81224640\, k^{3}$$

$$+ 1907973498\,k^2\,n - 32916960\,k^2\,z - 13255890984\,k\,n^2$$
$$+ 180079272\,k\,n\,z - 31330800\,k\,z^2 + 28201921488\,n^3$$
$$+ 71334288\,n^2\,z + 86313600\,n\,z^2 - 8812800\,z^3 + 271663560\,k^2$$
$$- 3787310484\,k\,n + 40681440\,k\,z + 12100864176\,n^2$$
$$- 63430560\,z\,n + 13608000\,z^2 - 480044880\,k + 3072921840\,n$$
$$- 18403200\,z + 349790400)\,(3\,n+2)\,(3\,n+1)\,bb_{n,k}\big) \big/ \big((-3\,n - 6$$
$$+ k)\,(-3\,n - 5 + k)\,(-3\,n + k - 4)\,(-n + k - 1)\,(-n + k)\,(n+2)\,($$
$$-3\,n + k - 1)\,(-3\,n + k - 2)\,(-3\,n + k - 3)\,(3\,n+5)\,(3\,n+4)\big)\big)$$

The set of poles to be avoided in the (ommitted) summation argument in the article is the following.

```
> Z = {solve(denom(zb_B[2]), k)};
```

$$Z = \{n, n+1, 3\,n+1, 3\,n+2, 3\,n+3, 3\,n+4, 3\,n+5, 3\,n+6\} \qquad \textbf{(2.3.5)}$$

We apply sound creative telescoping. Here, this justifies a homogeneous recurrence relation, not shown in the article:

```
> rhs = collect(expand(subs(k=n-1, zb_B[2]) - subs(k=-1, zb_B
  [2]) + add(coeff(zb_B[1], sn, i) * add(subs(n = n+i, k=n+j,
  CF_B * z^k), j=-1..i), i=0..2)), binomial, factor);
```

$$rhs = -\frac{1}{(3\,n+5)\,(n+2)\,(2\,n+1)\,(n+1)}\Big(3\,(3\,n+2)\,(2\,n+3)\,(3\,n \qquad \textbf{(2.3.6)}$$
$$+ 1)\,(2\,n^4\,z^5 - 11\,n^4\,z^4 + 12\,n^3\,z^5 + 44\,n^4\,z^3 - 66\,n^3\,z^4 + 22\,n^2\,z^5$$
$$- 160\,n^4\,z^2 + 264\,n^3\,z^3 - 121\,n^2\,z^4 + 12\,n\,z^5 - 3584\,n^4\,z - 960\,n^3\,z^2$$
$$+ 484\,n^2\,z^3 - 66\,n\,z^4 - 4096\,n^4 - 21504\,n^3\,z - 1790\,n^2\,z^2 + 264\,n\,z^3$$
$$- 24576\,n^3 - 44704\,n^2\,z - 1050\,n\,z^2 - 52736\,n^2 - 37344\,z\,n$$
$$- 47616\,n - 10080\,z - 15120)\,z^n\,z\,\binom{3\,n}{n}\binom{n}{-1}\Big)$$
$$+ \Bigg(\frac{1}{n+1}\Big(6\,(3\,n+1)\,(n^2\,z^2 - 4\,n^2\,z + 4\,n\,z^2 - 32\,n^2 - 16\,z\,n + 3\,z^2$$
$$- 128\,n - 12\,z - 126)\,z^n\,z^4\,n\binom{n-1}{-1}\Big)$$
$$+ \frac{1}{2\,(2\,n+1)\,(n+1)\,(2\,n+3)\,(n+2)}\Big(27\,z^2\,z^n\,(3\,n+5)\,(3\,n$$
$$+ 2)\,(3\,n+7)\,(3\,n+4)\,(3\,n+1)\,(n^2\,z^2 - 4\,n^2\,z + 2\,n\,z^2 - 32\,n^2$$
$$- 8\,z\,n - 64\,n - 30)\binom{n+1}{-1}\Big)\Bigg)\binom{3\,n}{n} + \Big(4\,n\,(4\,n+1)\,(2\,n$$

$$+1)\,(4\,n+3)\,\big(6561\,n^9 z^5 - 35460\,n^9 z^4 + 107163\,n^8 z^5$$

$$- 159264\,n^9 z^3 - 578244\,n^8 z^4 + 768366\,n^7 z^5 - 202752\,n^9 z^2$$

$$- 2608128\,n^8 z^3 - 4139280\,n^7 z^4 + 3175038\,n^6 z^5 + 1327104\,n^9 z$$

$$- 2481408\,n^8 z^2 - 18950206\,n^7 z^3 - 17076504\,n^6 z^4$$

$$+ 8334009\,n^5 z^5 + 14155776\,n^9 + 18382848\,n^8 z - 11816192\,n^7 z^2$$

$$- 80154810\,n^6 z^3 - 44751780\,n^5 z^4 + 14411547\,n^4 z^5$$

$$+ 204079104\,n^8 + 108968960\,n^7 z - 23787408\,n^6 z^2$$

$$- 217427626\,n^5 z^3 - 77267316\,n^4 z^4 + 16417944\,n^3 z^5$$

$$+ 1290534912\,n^7 + 358274304\,n^6 z + 4866352\,n^5 z^2$$

$$- 392097822\,n^4 z^3 - 87895080\,n^3 z^4 + 11880972\,n^2 z^5$$

$$+ 4698193920\,n^6 + 703060736\,n^5 z + 128777328\,n^4 z^2$$

$$- 469888744\,n^3 z^3 - 63518256\,n^2 z^4 + 4955040\,n\,z^5$$

$$+ 10849695744\,n^5 + 808992432\,n^4 z + 289948432\,n^3 z^2$$

$$- 360702840\,n^2 z^3 - 26457120\,n\,z^4 + 907200\,z^5$$

$$+ 16477764096\,n^4 + 458271920\,n^3 z + 319261728\,n^2 z^2$$

$$- 160876800\,n\,z^3 - 4838400\,z^4 + 16449946368\,n^3 - 6885744\,n^2 z$$

$$+ 181880640\,n\,z^2 - 31752000\,z^3 + 10401696000\,n^2$$

$$- 146925360\,z\,n + 42940800\,z^2 + 3776198400\,n - 54129600\,z$$

$$+ 598752000\big)\binom{n-1}{-1}\binom{4\,n}{n}\Big)\Big/\big(9\,(3\,n+4)^2\,(3\,n+2)\,(3\,n$$

$$+5)^2\,(n+2)^2\,(3\,n+7)\,(n+1)^2\,(3\,n+1)\,z\big)$$

```
> subs(seq(binomial(e,-1) = 0, e=[n-1,n,n+1]), %);
```

$$rhs = 0 \qquad \textbf{(2.3.7)}$$

We can also validate numerically that the recurrence is satisfied for a few values of n (in the range 10 to 15):

```
> seq(normal(eval(subs(bb = unapply(coeff(ser_B, t, n), n),
  subs(seq(bb[n+j,k] = bb(n+j,k), j=0..2), n=i, reczb_B)))),
  i=10..15);
```

$$0, 0, 0, 0, 0, 0 \quad \textbf{(2.3.8)}$$

So we have justified the following recurrence relation:

```
> rec2_B := subs(seq(bb[n+i,k] = bb(n+i), i=0..2), reczb_B);
```

$$rec2\_B := 3 z^4 n (3 n+2) (3 n+1) (n^2 z^2 - 4 n^2 z + 4 n z^2 - 32 n^2 - 16 z n + 3 z^2 - 128 n - 12 z - 126) bb(n) - (2 n+3) (2 n^4 z^5 - 11 n^4 z^4 + 12 n^3 z^5 + 44 n^4 z^3 - 66 n^3 z^4 + 22 n^2 z^5 - 160 n^4 z^2 + 264 n^3 z^3 - 121 n^2 z^4 + 12 n z^5 - 3584 n^4 z - 960 n^3 z^2 + 484 n^2 z^3 - 66 n z^4 - 4096 n^4 - 21504 n^3 z - 1790 n^2 z^2 + 264 n z^3 - 24576 n^3 - 44704 n^2 z - 1050 n z^2 - 52736 n^2 - 37344 z n - 47616 n - 10080 z - 15120) bb(n+1) + 3 (3 n+7) (n+3) (3 n+8) (n^2 z^2 - 4 n^2 z + 2 n z^2 - 32 n^2 - 8 z n - 64 n - 30) bb(n+2) \quad \textbf{(2.3.9)}$$

Third step: compute a common recurrence relation for b[n](z) and bb[n](z), and conclude.

We proceed to prove the equality between the solution b[n](z) of a fourth-order recurrence equation and the solution bb[n](z) of a second-order recurrence equation.

```
> indets(rec_B);
```

$$\{n, z, b(n), b(n+1), b(n+2), b(n+3), b(n+4)\} \quad \textbf{(2.4.1)}$$

```
> indets(rec2_B);
```

$$\{n, z, bb(n), bb(n+1), bb(n+2)\} \quad \textbf{(2.4.2)}$$

To this end, we first derive a common recurrence relation, which is a relation satisfied by any linear combination with constant coefficients of the two sequences. It will in particular be valid for b(n) = b(n) + 0 and for bb(n) = 0 + bb(n). The syntax of `rec+rec` uses the same name for all input and output equations.

```
> reclclm_B := collect(`rec+rec`(rec_B, subs(bb=b, rec2_B), b
  (n)), b, factor);
```

$$reclclm\_B := 9 n z^8 (3 n+2) (3 n+1) (2 z - 1) b(n) - 3 z^4 (26 n^3 z^4 - 70 n^3 z^3 + 118 n^2 z^4 + 444 n^3 z^2 - 326 n^2 z^3 + 172 n z^4 + 608 n^3 z + 1635 n^2 z^2 - 492 n z^3 + 80 z^4 - 256 n^3 + 2560 n^2 z + 1881 n z^2 - 240 z^3 - 896 n^2 + 3472 z n + 660 z^2 - 1008 n + 1520 z - 360) b(n + 1) + z (2 n+5) (4 n^2 z^6 - 18 n^2 z^5 + 20 n z^6 + 384 n^2 z^4 - 90 n z^5 + 24 z^6 - 355 n^2 z^3 + 1920 n z^4 - 108 z^5 - 288 n^2 z^2 - 1775 n z^3 \quad \textbf{(2.4.3)}$$

$$+ 2304 z^4 + 3840 n^2 z - 1440 n z^2 - 2244 z^3 + 4096 n^2 + 19200 z n - 1728 z^2 + 20480 n + 23760 z + 25344) b(n+2) + (-78 n^3 z^4 + 210 n^3 z^3 - 816 n^2 z^4 - 1332 n^3 z^2 + 2172 n^2 z^3 - 2826 n z^4 - 1824 n^3 z - 15075 n^2 z^2 + 7446 n z^3 - 3240 z^4 + 768 n^3 - 19680 n^2 z - 56493 n z^2 + 8460 z^3 + 8832 n^2 - 70416 z n - 70110 z^2 + 33744 n - 83520 z + 42840) b(n+3) + 9 (n + 5) (3 n + 14) (3 n + 13) (2 z - 1) b(n+4)$$

It turns out that this is the already known fourth-order equation. This is just by coincidence, and it would in general have a higher order.

```
> rec_B - reclclm_B;
```

$$0 \qquad \textbf{(2.4.4)}$$

The recurrence can be unrolled over all natural integers, as its leading coefficient has no nonnegative integer root.

```
> solve(coeff(reclclm_B, b(n+4)), n);
```

$$-5, -\frac{14}{3}, -\frac{13}{3} \qquad \textbf{(2.4.5)}$$

So the solution to the recurrence is identified by its first 4 terms, and to end the proof, it is sufficient to compare enough initial conditions.

```
> for i from 0 to 3 do
      coeff(ser_B, t, i) = eval(add(subs(n=i, k=j, CF_B*z^k),
  j=0..i-1))
  od;
```

$$0 = 0$$
$$1 = 1$$
$$2 z + 3 = 2 z + 3$$
$$6 z^2 + 18 z + 13 = 6 z^2 + 18 z + 13 \qquad \textbf{(2.4.6)}$$

[4.3 Proposition 16 and Proposition 17 by creative telescoping]

Here, we prove Proposition 17, which states that the sum of

```
> s := binomial(n+1,l+2)*binomial(r,l)*binomial(l,k);
```

$$s := \binom{n+1}{l+2} \binom{r}{l} \binom{l}{k} \qquad \textbf{(3.1)}$$

is

```
> closed_form := n*(n+1)/(r+1)/(r+2) * binomial(n-1,k) *
  binomial(r+n+1-k,n+1);
```

$$closed\_form := \frac{n (n+1) \binom{n-1}{k} \binom{r+n+1-k}{n+1}}{(r+1)(r+2)} \qquad \textbf{(3.2)}$$

that is,

```
> Sum(s, l=k..n-1) = closed_form;
```

$$\sum_{l=k}^{n-1} \binom{n+1}{l+2} \binom{r}{l} \binom{l}{k} = \frac{n\,(n+1)\binom{n-1}{k}\binom{r+n+1-k}{n+1}}{(r+1)\,(r+2)} \tag{3.3}$$

Reusing the plain Maple command provides a recurrence with respect to k, considering r as a free parameter.

```
> zbs := SumTools:-Hypergeometric:-Zeilberger(s, k, l, sk);
```

$$zbs := \Big[(-k^2+k\,n+k\,r+n+r+1)\,sk - k^2+k\,n+k\,r-n\,r-k+r,\ (l+2)\,(-l+k)\binom{n+1}{l+2}\binom{r}{l}\binom{l}{k}\Big] \tag{3.4}$$

```
> collect(zbs[1], sk, factor);
```

$$-(k+1)\,(-r-n-1+k)\,sk-(k-r)\,(-n+1+k) \tag{3.5}$$

The certificate introduces no pole, so the summation step of creative telescoping is sound.

```
> normal(zbs[2]/s);
```

$$(l+2)\,(-l+k) \tag{3.6}$$

Fixing the constant in the identity to be proved amount to verifying the following equality:

```
> subs(k=n-1, l=n-1, s) = subs(k=n-1, closed_form);
```

$$\binom{n+1}{n+1}\binom{r}{n-1}\binom{n-1}{n-1} = \frac{n\,(n+1)\binom{n-1}{n-1}\binom{r+2}{n+1}}{(r+1)\,(r+2)} \tag{3.7}$$

```
> normal(subs(n=n+1, %), expanded);
```

$$\binom{r}{n} = \binom{r}{n} \tag{3.8}$$

We can get more recurrences, that is, also recurrences with respect to r, by the more general command in the package Mgfun. The notation of the output is different.

```
> with(Mgfun);
```

$$[MG\_Internals, creative\_telescoping, dfinite\_expr\_to\_diffeq, dfinite\_expr\_to\_rec, dfinite\_expr\_to\_sys, diag\_of\_sys, int\_of\_sys, pol\_to\_sys, rational\_creative\_telescoping, sum\_of\_sys, sys{*}sys, sys+sys] \tag{3.9}$$

```
> cts := creative_telescoping(s, [n::shift, k::shift, r::shift],
  [l::shift]):
> map(q -> collect(q[1], _F, factor), cts);
```

$$[(-k+n+r+2)\,\_F(n,k,r)+(-n+k)\,\_F(n+1,k,r),\ (k-r)\,(-n+1+k)\,\_F(n,k,r)+(k+1)\,(-r-n-1+k)\,\_F(n,k+1,r),\ -(r+1)\,(k-n-r-2)\,\_F(n,k,r)+(r+3)\,(-r+k-1)\,\_F(n,k,r+1)] \tag{3.10}$$

The obtained certificates also have no poles.

```
> map(q -> factor(denom(q[2])), cts);
```
$$[-n+l, 1, -r+l-1] \tag{3.11}$$

The proof for Proposition 16 is the same with r replace by 3*n. We do the calculations with fewer comments.

```
> s2 := 2*binomial(n+1,l+2)*binomial(3*n,l)*binomial(l,k)/n/
  (n+1);
```
$$s2 := \frac{2 \binom{n+1}{l+2} \binom{3\,n}{l} \binom{l}{k}}{n\,(n+1)} \tag{3.12}$$

```
> zbs2 := SumTools:-Hypergeometric:-Zeilberger(s2, k, l, sk);
```
$$zbs2 := \left[(-k^2+4\,k\,n+4\,n+1)\,sk - k^2+4\,k\,n - 3\,n^2 - k+3\,n, \right. \tag{3.13}$$
$$\left. \frac{2\,(l+2)\,(-l+k) \binom{n+1}{l+2} \binom{3\,n}{l} \binom{l}{k}}{n\,(n+1)} \right]$$

```
> collect(zbs2[1], sk, factor);
```
$$-(-4\,n-1+k)\,sk\,(k+1) - (-n+1+k)\,(-3\,n+k) \tag{3.14}$$

```
> normal(zbs2[2]/s2);
```
$$(l+2)\,(-l+k) \tag{3.15}$$

```
> cts2 := creative_telescoping(s2, [n::shift, k::shift],
  [l::shift]):
> map(q -> collect(q[1], _F, factor), cts2);
```
$$[-(3\,n+2)\,(3\,n+1)\,(-4\,n-2+k)\,(-4\,n-3+k)\,(-4\,n-4+k)\,(-4\,n-5+k)\,n\,\_F(n,k) + (-n+k)\,(n+2)\,(-3\,n+k-1)\,(-3\,n+k-2)\,(-3\,n+k-3)\,(3\,n+5)\,(3\,n+4)\,\_F(n+1,k), (-n+1+k)\,(-3\,n+k)\,\_F(n,k) + (k+1)\,(-4\,n-1+k)\,\_F(n,k+1)] \tag{3.16}$$

```
> map(q -> factor(denom(q[2])), cts2);
```
$$[(l-3\,n-3)\,(-2+l-3\,n)\,(-n+l)\,(-3\,n+l-1), 1] \tag{3.17}$$

[5 Solving a holonomic recurrence system]

It is conceptually tempting to avoid recurrence equations and the technicalities induced by their domains of validity in the proof of the previous section. A solution might be to discuss only generating series and their differential equations.

To this end, we use a package by one of the authors for the initial calculation. The command creative_telescoping in the Mgfun package is a possible replacement of SumTools:-Hypergeometric:-Zeilberger used for summation in the previous section.

```
> with(Mgfun);
```
$$[MG\_Internals, creative\_telescoping, dfinite\_expr\_to\_diffeq, \tag{4.1}$$

*dfinite_expr_to_rec, dfinite_expr_to_sys, diag_of_sys, int_of_sys, pol_to_sys, rational_creative_telescoping, sum_of_sys, sys*sys, sys+sys*]

Here, we use the command dfinite_expr_to_sys that computes a system of linear functional equations for some d-finite (occasionally holonomic) expression. The output system has to be considered as having coefficients in the field Q(t,z).

```
> dsysro_A := collect(dfinite_expr_to_sys(RootOf(P, X), A
  (t::diff, z::diff)), {A, diff}, factor);
```

$$dsysro\_A := \Big\{12\,t\,(t\,z^4 + 18\,t\,z^3 + 198\,t\,z^2 \mathrm{K}\ 486\,t\,z \mathrm{K}\ 9\,z^2 \mathrm{K}\ 243\,t \tag{4.2}$$

$$+ 243)\,A(t,z) + 12\,t^2\,z^2\,(10\,t^2\,z^4 \mathrm{K}\ 110\,t^2\,z^3 + 334\,t^2\,z^2 \mathrm{K}\ 378\,t^2\,z \mathrm{K}\ t\,z^2$$

$$+ 144\,t^2 \mathrm{K}\ 108\,t\,z + 333\,t + 9)\left(\frac{\partial}{\partial t}\,A(t,z)\right) + (432\,t^3\,z^7 \mathrm{K}\ 4536\,t^3\,z^6$$

$$+ 14256\,t^3\,z^5 \mathrm{K}\ 60\,t^2\,z^6 \mathrm{K}\ 18414\,t^3\,z^4 + 672\,t^2\,z^5 + 7128\,t^3\,z^3 \mathrm{K}\ 6102\,t^2\,z^4$$

$$+ 5508\,t^3\,z^2 + 28080\,t^2\,z^3 \mathrm{K}\ 5832\,t^3\,z \mathrm{K}\ 22680\,t^2\,z^2 + 432\,t\,z^3 + 1458\,t^3$$

$$\mathrm{K}\ 11664\,t^2\,z \mathrm{K}\ 3240\,t\,z^2 \mathrm{K}\ 4374\,t^2 + 17496\,t\,z + 4374\,t \mathrm{K}\ 1458)\left(\frac{\partial}{\partial z}\right.$$

$$\left.A(t,z)\right) + 2\,z\,(189\,t^3\,z^7 \mathrm{K}\ 1890\,t^3\,z^6 + 5670\,t^3\,z^5 \mathrm{K}\ 26\,t^2\,z^6 \mathrm{K}\ 6831\,t^3\,z^4$$

$$+ 273\,t^2\,z^5 + 1809\,t^3\,z^3 \mathrm{K}\ 2889\,t^2\,z^4 + 3240\,t^3\,z^2 + 11124\,t^2\,z^3 \mathrm{K}\ 2916\,t^3\,z$$

$$\mathrm{K}\ 4698\,t^2\,z^2 + 270\,t\,z^3 + 729\,t^3 \mathrm{K}\ 3645\,t^2\,z \mathrm{K}\ 1458\,t\,z^2 \mathrm{K}\ 2187\,t^2$$

$$+ 6561\,t\,z + 2187\,t \mathrm{K}\ 729)\left(\frac{\partial^2}{\partial z^2}\,A(t,z)\right) + z^2\,(2\,t\,z^3 \mathrm{K}\ 11\,t\,z^2 + 9\,t$$

$$\mathrm{K}\ 9)\,(27\,t^2\,z^4 \mathrm{K}\ 108\,t^2\,z^3 + 162\,t^2\,z^2 \mathrm{K}\ 4\,t\,z^3 \mathrm{K}\ 108\,t^2\,z + 18\,t\,z^2 + 27\,t^2$$

$$\mathrm{K}\ 216\,t\,z \mathrm{K}\ 54\,t + 27)\left(\frac{\partial^3}{\partial z^3}\,A(t,z)\right), 18\,A(t,z) \mathrm{K}\ 18\,t\,(t\,z \mathrm{K}\ t + 1)\left(\frac{\partial}{\partial t}\right.$$

$$\left.A(t,z)\right) \mathrm{K}\ z\,(4\,t\,z^3 \mathrm{K}\ 22\,t\,z^2 + 36\,t\,z \mathrm{K}\ 18\,t \mathrm{K}\ 45)\left(\frac{\partial}{\partial z}\,A(t,z)\right) + t\,z\,(2\,t\,z^3$$

$$\mathrm{K}\ 11\,t\,z^2 + 9\,t \mathrm{K}\ 9)\left(\frac{\partial^2}{\partial t\,\partial z}\,A(t,z)\right) \mathrm{K}\ 2\,z^2\,(t\,z^3 \mathrm{K}\ 5\,t\,z^2 + 7\,t\,z \mathrm{K}\ 3\,t$$

$$\mathrm{K}\ 6)\left(\frac{\partial^2}{\partial z^2}\,A(t,z)\right), 24\,A(t,z) \mathrm{K}\ 24\,t\,(t\,z \mathrm{K}\ t + 1)\left(\frac{\partial}{\partial t}\,A(t,z)\right) + (\mathrm{K}\ 4\,t\,z^4$$

$$+ 20\,t\,z^3 \mathrm{K}\ 37\,t\,z^2 + 30\,t\,z \mathrm{K}\ 9\,t + 54\,z + 9)\left(\frac{\partial}{\partial z}\,A(t,z)\right) + t^2\,(2\,t\,z^3$$

$\mathrm{K}\ 11\, t z^2 + 9\, t \mathrm{K}\ 9)\ \left(\frac{\partial^2}{\partial t^2}\, A(t, z)\right) \mathrm{K}\ z\, (2\, t z^4 \mathrm{K}\ 9\, t z^3 + 15\, t z^2 \mathrm{K}\ 11\, t z + 3\, t$

$\mathrm{K}\ 13\, z \mathrm{K}\ 3)\ \left(\frac{\partial^2}{\partial z^2}\, A(t, z)\right)\}$

The structure of related derivatives is more apparent in the following outputs.

```
> map(indets, dsysro_A);
```

$$\left\{\left\{t, z, A(t, z), \frac{\partial}{\partial t} A(t, z), \frac{\partial}{\partial z} A(t, z), \frac{\partial^2}{\partial t^2} A(t, z), \frac{\partial^2}{\partial z^2} A(t, z)\right\}, \left\{t, z, A(t, z), \frac{\partial}{\partial t} A(t, z), \frac{\partial}{\partial z} A(t, z), \frac{\partial^2}{\partial t \partial z} A(t, z), \frac{\partial^2}{\partial z^2} A(t, z)\right\}, \left\{t, z, A(t, z), \frac{\partial}{\partial t} A(t, z), \frac{\partial}{\partial z} A(t, z), \frac{\partial^2}{\partial z^2} A(t, z), \frac{\partial^3}{\partial z^3} A(t, z)\right\}\right\} \quad \textbf{(4.3)}$$

```
> indets(dsysro_A);
```

$$\left\{t, z, A(t, z), \frac{\partial}{\partial t} A(t, z), \frac{\partial}{\partial z} A(t, z), \frac{\partial^2}{\partial t^2} A(t, z), \frac{\partial^2}{\partial t \partial z} A(t, z), \frac{\partial^2}{\partial z^2} A(t, z), \frac{\partial^3}{\partial z^3} A(t, z)\right\} \quad \textbf{(4.4)}$$

The linear PDEs have underlying linear differential operators in the noncommutative (Weyl) algebra Q<t,z,dt,dz; dt*t = t*dt+1, dz*z = z*dz+1>. They are the generators of some left ideal there.

```
> Gro_A := subs([seq(seq(diff(A(t,z), [t$i,z$j]) = dt^i*dz^j, j=
  0..3), i=0..3)], dsysro_A);
```

$$\begin{aligned}
Gro_A &:= \{18 \mathrm{K}\ 18\, t\, (t z \mathrm{K}\ t + 1)\, dt \mathrm{K}\ z\, (4\, t z^3 \mathrm{K}\ 22\, t z^2 + 36\, t z \mathrm{K}\ 18\, t \\
&\mathrm{K}\ 45)\, dz + t z\, (2\, t z^3 \mathrm{K}\ 11\, t z^2 + 9\, t \mathrm{K}\ 9)\, dt\, dz \mathrm{K}\ 2\, z^2\, (t z^3 \mathrm{K}\ 5\, t z^2 + 7\, t z \\
&\mathrm{K}\ 3\, t \mathrm{K}\ 6)\, dz^2, 24 \mathrm{K}\ 24\, t\, (t z \mathrm{K}\ t + 1)\, dt + (\mathrm{K}\ 4\, t z^4 + 20\, t z^3 \mathrm{K}\ 37\, t z^2 \\
&+ 30\, t z \mathrm{K}\ 9\, t + 54\, z + 9)\, dz + t^2\, (2\, t z^3 \mathrm{K}\ 11\, t z^2 + 9\, t \mathrm{K}\ 9)\, dt^2 \mathrm{K}\ z\, (2\, t z^4 \\
&\mathrm{K}\ 9\, t z^3 + 15\, t z^2 \mathrm{K}\ 11\, t z + 3\, t \mathrm{K}\ 13\, z \mathrm{K}\ 3)\, dz^2, 12\, t\, (t z^4 + 18\, t z^3 \\
&+ 198\, t z^2 \mathrm{K}\ 486\, t z \mathrm{K}\ 9\, z^2 \mathrm{K}\ 243\, t + 243) + 12\, t^2 z^2\, (10\, t^2 z^4 \mathrm{K}\ 110\, t^2 z^3 \\
&+ 334\, t^2 z^2 \mathrm{K}\ 378\, t^2 z \mathrm{K}\ t z^2 + 144\, t^2 \mathrm{K}\ 108\, t z + 333\, t + 9)\, dt + (432\, t^3 z^7 \\
&\mathrm{K}\ 4536\, t^3 z^6 + 14256\, t^3 z^5 \mathrm{K}\ 60\, t^2 z^6 \mathrm{K}\ 18414\, t^3 z^4 + 672\, t^2 z^5 + 7128\, t^3 z^3 \\
&\mathrm{K}\ 6102\, t^2 z^4 + 5508\, t^3 z^2 + 28080\, t^2 z^3 \mathrm{K}\ 5832\, t^3 z \mathrm{K}\ 22680\, t^2 z^2 \\
&+ 432\, t z^3 + 1458\, t^3 \mathrm{K}\ 11664\, t^2 z \mathrm{K}\ 3240\, t z^2 \mathrm{K}\ 4374\, t^2 + 17496\, t z \\
&+ 4374\, t \mathrm{K}\ 1458)\, dz + 2\, z\, (189\, t^3 z^7 \mathrm{K}\ 1890\, t^3 z^6 + 5670\, t^3 z^5 \mathrm{K}\ 26\, t^2 z^6 \\
&\mathrm{K}\ 6831\, t^3 z^4 + 273\, t^2 z^5 + 1809\, t^3 z^3 \mathrm{K}\ 2889\, t^2 z^4 + 3240\, t^3 z^2 \\
&+ 11124\, t^2 z^3 \mathrm{K}\ 2916\, t^3 z \mathrm{K}\ 4698\, t^2 z^2 + 270\, t z^3 + 729\, t^3 \mathrm{K}\ 3645\, t^2 z \\
&\mathrm{K}\ 1458\, t z^2 \mathrm{K}\ 2187\, t^2 + 6561\, t z + 2187\, t \mathrm{K}\ 729)\, dz^2 + z^2\, (2\, t z^3 \mathrm{K}\ 11\, t z^2 \\
&+ 9\, t \mathrm{K}\ 9)\, (27\, t^2 z^4 \mathrm{K}\ 108\, t^2 z^3 + 162\, t^2 z^2 \mathrm{K}\ 4\, t z^3 \mathrm{K}\ 108\, t^2 z + 18\, t z^2 \\
&+ 27\, t^2 \mathrm{K}\ 216\, t z \mathrm{K}\ 54\, t + 27)\, dz^3\}
\end{aligned} \quad \textbf{(4.5)}$$

```
> map(degree, convert(Gro_A, list), {t,z});
```

$$[6, 6, 12] \quad (4.6)$$

```
> map(degree, convert(Gro_A, list));
```

$$[8, 8, 15] \quad (4.7)$$

Computing with the left ideal is done with the packages Ore_algebra and Groebner.

```
> with(Ore_algebra);
```

[*Ore_to_DESol, Ore_to_RESol, Ore_to_diff, Ore_to_shift, annihilators, applyopr, diff_algebra, dual_algebra, dual_polynomial, poly_algebra, qshift_algebra, rand_skew_poly, reverse_algebra, reverse_polynomial, shift_algebra, skew_algebra, skew_elim, skew_gcdex, skew_pdiv, skew_power, skew_prem, skew_product*] **(4.8)**

```
> wAlg := diff_algebra([dt,t], [dz,z], polynom={t,z}):
> with(Groebner);
```

[*Basis, FGLM, HilbertDimension, HilbertPolynomial, HilbertSeries, Homogenize, InitialForm, InterReduce, IsBasis, IsProper, IsZeroDimensional, LeadingCoefficient, LeadingMonomial, LeadingTerm, MatrixOrder, MaximalIndependentSet, MonomialOrder, MultiplicationMatrix, MultivariateCyclicVector, NormalForm, NormalSet, RationalUnivariateRepresentation, Reduce, RememberBasis, SPolynomial, Solve, SuggestVariableOrder, Support, TestOrder, ToricIdealBasis, TrailingTerm, UnivariatePolynomial, Walk, WeightedDegree*] **(4.9)**

```
> wMOrd := MonomialOrder(wAlg, tdeg(dt,dz,t,z)):
```

To test holonomy of the left ideal, we compute a Gröbner basis of it for a total-degree ordering, in order to get its (Bernstein) dimension. In other words, we determine whether we have a holonomic presentation of the d-finite function A (t,z).

```
> Bro_A := Basis(Gro_A, wMOrd):
```

The following dimension equal to 2 is precisely the definition of holonomy. Having holonomy, more precisely a holonomic presentation, if very precious for the calculations to come.

```
> HilbertDimension(Bro_A, wMOrd);
```

$$2 \quad (4.10)$$

```
> map(indets, Bro_A);
```

$$[\{dt, dz, t, z\}, \{dt, dz, t, z\}, \{dt, dz, t, z\}, \{dt, dz, t, z\}, \{dt, dz, t, z\}, \{dt, dz, t, z\}, \{dt, dz, t, z\}] \quad (4.11)$$

```
> map(LeadingMonomial, Bro_A, wMOrd);
```

$$[dt^2\, t^2\, z, dt^2\, t^3, dt\, dz\, t^2\, z^4, dz^3\, t\, z^4, dt\, dz^2\, t^2\, z^3, dt\, dz^2\, t\, z^5, dz^4\, z^5] \quad (4.12)$$

```
> map(degree, Bro_A, dt);
```

$$[2, 2, 1, 1, 1, 2, 2] \quad (4.13)$$

```
> map(degree, Bro_A, dz);
```
$$[2, 2, 2, 3, 3, 3, 4] \quad (4.14)$$

The Weyl algebra can be localized by inverting t and z (not all nonzero polynomials). The obtained algebra is in bijection with the algebra Q<sn,tn,sk,tk,n,k; sn*n = (n+1)*sn, sk*k = (k+1)sk, sn*tn = 1, sk*tk = 1>, where tn is an inverse for sn and tk is an inverse for sk.

```
> sAlg := skew_algebra(`shift+dual_shift`=[sn,tn,n], `shift+
  dual_shift`=[sk,tk,k], polynom={n,k}):
```

A good property is that recurrence relations obtained by the following conversion are valid without any special proof for any relative integers n and k in Z. They hold on the bivariate coefficient sequence of A(t,z).

```
> conv := proc(L, $) local i,j;
     return add(add(skew_product(subs(t=tn, z=tk, coeff(coeff(L,
  dz, j), dt, i)), expand(skew_power((n+1)*sn, i, sAlg) *
  skew_power((k+1)*sk, j, sAlg)), sAlg), j=0..degree(L,dz)), i=0.
  .degree(L,dt))
  end:
> Gro_A_rec := [op(expand(map(conv, Bro_A))), sn*tn-1, sk*tk-1]:
> sMOrd := MonomialOrder(sAlg, lexdeg([sn,tn,sk,tk],[n,k])):
> N := 5:
  sort([seq(sn^i*sk^(N-i), i=0..N), seq(sn^i*tk^(N-i), i=0..N-1),
  seq(tn^i*sk^(N-i), i=1..N), seq(tn^i*tk^(N-i), i=1..N-1)], (u,
  v) -> TestOrder(u, v, lexdeg([sn,tn,sk,tk],[n,k])));
```
$$[tk^5, tk^4\, tn, sn\, tk^4, tk^3\, tn^2, sn^2\, tk^3, tn^3\, tk^2, sn^3\, tk^2, tn^4\, tk, sn^4\, tk, sk^5, tn\, sk^4, sk^4\, sn, tn^2\, sk^3, sk^3\, sn^2, tn^3\, sk^2, sn^3\, sk^2, tn^4\, sk, sn^4\, sk, tn^5, sn^5] \quad (4.15)$$

```
> map(m -> map2(degree, m, [sn,tn,sk,tk]), %);
```
$$[[0, 0, 0, 5], [0, 1, 0, 4], [1, 0, 0, 4], [0, 2, 0, 3], [2, 0, 0, 3], [0, 3, 0, 2], [3, 0, 0, 2], [0, 4, 0, 1], [4, 0, 0, 1], [0, 0, 5, 0], [0, 1, 4, 0], [1, 0, 4, 0], [0, 2, 3, 0], [2, 0, 3, 0], [0, 3, 2, 0], [3, 0, 2, 0], [0, 4, 1, 0], [4, 0, 1, 0], [0, 5, 0, 0], [5, 0, 0, 0]] \quad (4.16)$$

With the choice of an algebra in which n and k are polynomial variables, the following calculation of a Gröbner basis only considers equations valid over the same (and whole) domain, that is, only equations valid for n and k in Z.

```
> Bro_A_rec := Basis(Gro_A_rec, sMOrd):
> map(LeadingMonomial, Bro_A_rec, sMOrd);
```
$$[n^2\, tk, k^2\, sk, n^4\, tn, n^3\, sn, k^2\, n^2\, sn, k\, n\, tk^2, sk\, tk, k^2\, n\, tk\, tn, k^4\, tk\, tn, k^2\, n\, sn\, tk, k\, n\, sk\, tn, n^3\, sk\, tn, k\, n\, sk\, sn, sn\, tn, k^3\, tn\, tk^2, k^3\, sn\, tk^2] \quad (4.17)$$

```
> map(length, Bro_A_rec);
```
$$[82, 68, 193, 274, 345, 362, 19, 308, 503, 510, 497, 514, 364, 19, 576, 776] \quad (4.18)$$

The following is a usual hack to see the supports of the skew polynomials as operators in sn, tn, sk, tk.

```
> collect(Bro_A_rec, {sn,tn,sk,tk}, distributed, 1);
```
(4.19)

$$[1 + tk, 1 + sk, 1 + tn, 1 + sn + tk, 1 + sn + tk, tk^2 + sn + tk + 1, sk\, tk + 1, \quad \textbf{(4.19)}$$
$$tk\, tn + tn + 1, tk\, tn + tn + 1, sn\, tk + tk^2 + sn + tk + 1, tn\, sk + tk\, tn + sk$$
$$+ tn + 1, tn\, sk + tk\, tn + sk + tn + 1, sk\, sn + sk + sn + tk + 1, sn\, tn + 1,$$
$$tn\, tk^2 + tk\, tn + tk + tn + 1, sn\, tk^2 + tk^3 + sn^2 + sn\, tk + tk^2 + sn + tk$$
$$+ 1]$$

```
> collect(Bro_A_rec, {sn,tn,sk,tk}, distributed, factor);
```

$$[(-3n+k-1)(-n+k)\, tk - (k+2)\, k, (k+3)(k+1)\, sk - (-n+1 \quad \textbf{(4.20)}$$
$$+k)(-3n+k), 3n(n-1)(3n-1)(3n-2)\, tn - (-3n+2+k)($$
$$-3n+1+k)(-3n+k)(-n+1+k), -3(-n+k)(-3n+k-2)($$
$$-3n+k-3)\, sn + (k-1)(k+1)(-n+k)\, tk - (3n+2+k)(k^2$$
$$+9n^2+k+9n), k(k+2)(-3n+k-2)(-3n+k-3)\, sn + k(k$$
$$+1)(13k^2-40kn-7k-20n-6)\, tk - k(k+2)(13k^2+12kn$$
$$+9n^2+19k+15n+6), 3(k+2)(-3n+k)(-3n+k-3)\, sn - 2(k$$
$$-2)(-n+k-1)\, tk^2 + (39k^3-120k^2n-10k^2-72kn-27k$$
$$+12n-2)\, tk - 3(k+2)(13k^2+12kn+9n^2+9k+9n), sk\, tk$$
$$-1, -(k-1)(k+1)(-n+1+k)\, tk\, tn + (3n-1+k)(k^2+9n^2+k$$
$$-9n)\, tn + 3(-n+1+k)(-3n+1+k)(-3n+k), 3(k+1)(9k^3$$
$$+9k^2-18k+20n-20)\, tk\, tn + (-27k^4-108k^3n-351k^2n^2$$
$$-1080kn^3-27k^3+81k^2n+918kn^2-1620n^3+54k^2-6kn$$
$$+2160n^2+60k-540n)\, tn - (-3n+1+k)(-3n+k)(121k^2$$
$$-120kn+302k-180n+180), -4(k-1)(k+1)(-3n+k$$
$$-4)\, sn\, tk + (k+2)(307k^2-1086kn+549n^2-905k+555n)\, sn$$
$$+(-54k^3+108k^2+54k-120n-108)\, tk^2 + (-347k^3+1832k^2n$$
$$+294k^2+1380kn+107k+268n-54)\, tk + (k+2)(401k^2$$
$$-12kn-549n^2+161k-555n), 9(n-1)(k+3)\, tn\, sk - 9(n$$
$$-1)(k+3)\, sk + (81k^3+20k^2+61kn-162k+119n-119)\, tk\, tn$$
$$+(-81k^3-324k^2n-1053kn^2-3240n^3+61k^2+741kn$$
$$+4392n^2+110k-1206n+54)\, tn - 3(-3n+k)(121k^2-483kn$$
$$+360n^2+241k-482n+122), 3(n-1)(3n-1)(3n-2)\, tn\, sk$$
$$+(54kn^2-27n^3+108n^2+6k-33n+12)\, sk + (27k^3+6k^2$$
$$+21kn-54k+39n-39)\, tk\, tn + (-27k^3-108k^2n-351kn^2$$
$$-1107n^3+21k^2+252kn+1512n^2+33k-417n+12)\, tn - (-3n$$
$$+k)(117k^2-452kn+333n^2+230k-448n+115), 9n(k$$
$$+3)\, sk\, sn - 9n(k+3)\, sk + 3(-3n+k-3)(k^2-3kn-2k$$
$$-2n)\, sn + (39k^3-120k^2n-20k^2-61kn-19k+n)\, tk - 39k^3$$
$$-36k^2n-27kn^2-97k^2-75kn-72n^2-38k-18n, sn\, tn - 1,$$

$$(54\,k^3 - 108\,k^2 - 54\,k + 120\,n - 12)\,tn\,tk^2 + (-1485\,k^3 - 294\,k^2$$
$$- 1380\,k\,n + 3105\,k - 2100\,n + 2154)\,tk\,tn + 4\,(k - 1)\,(k+1)\,(-3\,n$$
$$+ k - 1)\,tk + (1431\,k^3 + 5508\,k^2\,n + 17037\,k\,n^2 + 49464\,n^3 - 975\,k^2$$
$$- 11511\,k\,n - 64854\,n^2 - 2184\,k + 15402\,n - 12)\,tn + 5189\,k^3$$
$$- 37386\,k^2\,n + 81891\,k\,n^2 - 49464\,n^3 + 10197\,k^2 - 52245\,k\,n$$
$$+ 64854\,n^2 + 5140\,k - 15402\,n + 12, -72\,k\,(k+2)\,(-3\,n - 6 + k)\,sn^2$$
$$+ 8\,k\,(2\,k - 1)\,(k - 2)\,sn\,tk^2 + (-84\,k^3 + 272\,k^2 - 852\,k\,n - 636\,k$$
$$- 1284\,n - 1712)\,sn\,tk + (k+2)\,(36143\,k^2 - 123762\,k\,n + 54423\,n^2$$
$$- 105805\,k + 55425\,n)\,sn + (-108\,k^3 + 540\,k^2 - 648\,k - 240\,n)\,tk^3$$
$$+ (-6048\,k^3 + 12474\,k^2 + 5022\,k - 13080\,n - 11556)\,tk^2 + ($$
$$-59305\,k^3 + 272200\,k^2\,n + 40434\,k^2 + 195708\,k\,n + 24649\,k$$
$$+ 27476\,n - 5778)\,tk + (k+2)\,(65461\,k^2 + 14700\,k\,n - 54423\,n^2$$
$$+ 30805\,k - 55425\,n)\,]$$

These operators denote the following recurrence equations (with a few tautologies describing the structure of the algebra, not properties of the specific sequence):

```
> for q in Bro_A_rec do applyopr(collect(q, {sn,tn,sk,tk}, distributed, factor), a(n,k), sAlg) = 0 od;
```

$$-(k+2)\,k\,a(n,k) + (-3\,n + k - 1)\,(-n + k)\,a(n, k-1) = 0$$

$$-(-n + 1 + k)\,(-3\,n + k)\,a(n,k) + (k+3)\,(k+1)\,a(n, k+1) = 0$$

$$-(-3\,n + 2 + k)\,(-3\,n + 1 + k)\,(-3\,n + k)\,(-n + 1 + k)\,a(n,k) + 3\,n\,(n - 1)\,(3\,n - 1)\,(3\,n - 2)\,a(n-1, k) = 0$$

$$-(3\,n + 2 + k)\,(k^2 + 9\,n^2 + k + 9\,n)\,a(n,k) - 3\,(-n + k)\,(-3\,n + k - 2)\,(-3\,n + k - 3)\,a(n+1, k) + (k - 1)\,(k+1)\,(-n + k)\,a(n, k-1) = 0$$

$$-k\,(k+2)\,(13\,k^2 + 12\,k\,n + 9\,n^2 + 19\,k + 15\,n + 6)\,a(n,k) + k\,(k+2)\,(-3\,n + k - 2)\,(-3\,n + k - 3)\,a(n+1, k) + k\,(k+1)\,(13\,k^2 - 40\,k\,n - 7\,k - 20\,n - 6)\,a(n, k-1) = 0$$

$$-3\,(k+2)\,(13\,k^2 + 12\,k\,n + 9\,n^2 + 9\,k + 9\,n)\,a(n,k) + 3\,(k+2)\,(-3\,n + k)\,(-3\,n + k - 3)\,a(n+1, k) + (39\,k^3 - 120\,k^2\,n - 10\,k^2 - 72\,k\,n - 27\,k + 12\,n - 2)\,a(n, k-1) - 2\,(k - 2)\,(-n + k - 1)\,a(n, k-2) = 0$$

$$0 = 0$$

$$3\,(-n + 1 + k)\,(-3\,n + 1 + k)\,(-3\,n + k)\,a(n,k) + (3\,n - 1 + k)\,(k^2 + 9\,n^2 + k - 9\,n)\,a(n-1, k) - (k - 1)\,(k+1)\,(-n + 1 + k)\,a(n-1, k-1) = 0$$

$$-(-3\,n + 1 + k)\,(-3\,n + k)\,(121\,k^2 - 120\,k\,n + 302\,k - 180\,n + 180)\,a(n,k) + (-27\,k^4 - 108\,k^3\,n - 351\,k^2\,n^2 - 1080\,k\,n^3 - 27\,k^3$$
$$+ 81\,k^2\,n + 918\,k\,n^2 - 1620\,n^3 + 54\,k^2 - 6\,k\,n + 2160\,n^2 + 60\,k$$

$$- 540\,n)\,a(n-1,k) + 3\,(k+1)\,(9\,k^3 + 9\,k^2 - 18\,k + 20\,n - 20)\,a(n-1,k-1) = 0$$

$$(k+2)\,(401\,k^2 - 12\,k\,n - 549\,n^2 + 161\,k - 555\,n)\,a(n,k) + (k+2)\,(307\,k^2 - 1086\,k\,n + 549\,n^2 - 905\,k + 555\,n)\,a(n+1,k) + (-347\,k^3 + 1832\,k^2\,n + 294\,k^2 + 1380\,k\,n + 107\,k + 268\,n - 54)\,a(n,k-1) + (-54\,k^3 + 108\,k^2 + 54\,k - 120\,n - 108)\,a(n,k-2) - 4\,(k-1)\,(k+1)\,(-3\,n + k - 4)\,a(n+1,k-1) = 0$$

$$-3\,(-3\,n + k)\,(121\,k^2 - 483\,k\,n + 360\,n^2 + 241\,k - 482\,n + 122)\,a(n,k) - 9\,(n-1)\,(k+3)\,a(n,k+1) + (-81\,k^3 - 324\,k^2\,n - 1053\,k\,n^2 - 3240\,n^3 + 61\,k^2 + 741\,k\,n + 4392\,n^2 + 110\,k - 1206\,n + 54)\,a(n-1,k) + (81\,k^3 + 20\,k^2 + 61\,k\,n - 162\,k + 119\,n - 119)\,a(n-1,k-1) + 9\,(n-1)\,(k+3)\,a(n-1,k+1) = 0$$

$$-(-3\,n + k)\,(117\,k^2 - 452\,k\,n + 333\,n^2 + 230\,k - 448\,n + 115)\,a(n,k) + (54\,k\,n^2 - 27\,n^3 + 108\,n^2 + 6\,k - 33\,n + 12)\,a(n,k+1) + (-27\,k^3 - 108\,k^2\,n - 351\,k\,n^2 - 1107\,n^3 + 21\,k^2 + 252\,k\,n + 1512\,n^2 + 33\,k - 417\,n + 12)\,a(n-1,k) + (27\,k^3 + 6\,k^2 + 21\,k\,n - 54\,k + 39\,n - 39)\,a(n-1,k-1) + 3\,(n-1)\,(3\,n-1)\,(3\,n-2)\,a(n-1,k+1) = 0$$

$$(-39\,k^3 - 36\,k^2\,n - 27\,k\,n^2 - 97\,k^2 - 75\,k\,n - 72\,n^2 - 38\,k - 18\,n)\,a(n,k) - 9\,n\,(k+3)\,a(n,k+1) + 3\,(-3\,n + k - 3)\,(k^2 - 3\,k\,n - 2\,k - 2\,n)\,a(n+1,k) + (39\,k^3 - 120\,k^2\,n - 20\,k^2 - 61\,k\,n - 19\,k + n)\,a(n,k-1) + 9\,n\,(k+3)\,a(n+1,k+1) = 0$$

$$0 = 0$$

$$(5189\,k^3 - 37386\,k^2\,n + 81891\,k\,n^2 - 49464\,n^3 + 10197\,k^2 - 52245\,k\,n + 64854\,n^2 + 5140\,k - 15402\,n + 12)\,a(n,k) + (1431\,k^3 + 5508\,k^2\,n + 17037\,k\,n^2 + 49464\,n^3 - 975\,k^2 - 11511\,k\,n - 64854\,n^2 - 2184\,k + 15402\,n - 12)\,a(n-1,k) + 4\,(k-1)\,(k+1)\,(-3\,n + k - 1)\,a(n,k-1) + (-1485\,k^3 - 294\,k^2 - 1380\,k\,n + 3105\,k - 2100\,n + 2154)\,a(n-1,k-1) + (54\,k^3 - 108\,k^2 - 54\,k + 120\,n - 12)\,a(n-1,k-2) = 0$$

$$(k+2)\,(65461\,k^2 + 14700\,k\,n - 54423\,n^2 + 30805\,k - 55425\,n)\,a(n,k) + (k+2)\,(36143\,k^2 - 123762\,k\,n + 54423\,n^2 - 105805\,k + 55425\,n)\,a(n+1,k) + (-59305\,k^3 + 272200\,k^2\,n + 40434\,k^2 + 195708\,k\,n + 24649\,k + 27476\,n - 5778)\,a(n,k-1) + (-108\,k^3 + 540\,k^2 - 648\,k - 240\,n)\,a(n,k-3) + (-6048\,k^3 + 12474\,k^2 + 5022\,k - 13080\,n - 11556)\,a(n,k-2) + 8\,k\,(2\,k-1)\,(k-2)\,a(n+1,k-2) - 72\,k\,(k+2)\,(-3\,n - 6 + k)\,a(n+2,k) + (-84\,k^3 + 272\,k^2 - 852\,k\,n - 636\,k - 1284\,n - 1712)\,a(n+1,k-1) = 0 \qquad \textbf{(4.21)}$$

We can check that they all cancel CF_A.

```
> normal(expand(subs(n=n+1, map(e -> e/CF_A, map(applyopr,
  Bro_A_rec, CF_A, sAlg)))));
```

$$[0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0] \quad (4.22)$$

```
> map(indets, Bro_A_rec);
```

$$[\{k, n, tk\}, \{k, n, sk\}, \{k, n, tn\}, \{k, n, sn, tk\}, \{k, n, sn, tk\}, \{k, n, sn, tk\}, \{sk, tk\}, \{k, n, tk, tn\}, \{k, n, tk, tn\}, \{k, n, sn, tk\}, \{k, n, sk, tk, tn\}, \{k, n, sk, tk, tn\}, \{k, n, sk, sn, tk\}, \{sn, tn\}, \{k, n, tk, tn\}, \{k, n, sn, tk\}] \quad (4.23)$$

```
> map(p -> map2(degree, p, [sn,tn,sk,tk]), Bro_A_rec);
```

$$[[0, 0, 0, 1], [0, 0, 1, 0], [0, 1, 0, 0], [1, 0, 0, 1], [1, 0, 0, 1], [1, 0, 0, 2], [0, 0, 1, 1], [0, 1, 0, 1], [0, 1, 0, 1], [1, 0, 0, 2], [0, 1, 1, 1], [0, 1, 1, 1], [1, 0, 1, 1], [1, 1, 0, 0], [0, 1, 0, 2], [2, 0, 0, 3]] \quad (4.24)$$

In case CF_A was not known, here is how we could solve the system of recurrences. Selecting two independent, low-order relations is sufficient.

```
> rec1_A := applyopr(collect(Bro_A_rec[1], tk, factor), a(n,k),
  sAlg);
```

$$rec1\_A := \mathrm{K}\ (k+2)\,k\,a(n,k) + (\mathrm{K}\ 3\,n+k\mathrm{K}\ 1)\,(\mathrm{K}\ n+k)\,a(n,k\mathrm{K}\ 1) \quad (4.25)$$

```
> rec2_A := applyopr(collect(Bro_A_rec[3], tn, factor), a(n,k),
  sAlg);
```

$$rec2\_A := \mathrm{K}\ (\mathrm{K}\ 3\,n+2+k)\,(\mathrm{K}\ 3\,n+1+k)\,(\mathrm{K}\ 3\,n+k)\,(\mathrm{K}\ n+1+k)\,a(n,k) + 3\,n\,(n\mathrm{K}\ 1)\,(3\,n\mathrm{K}\ 1)\,(3\,n\mathrm{K}\ 2)\,a(n\mathrm{K}\ 1,k) \quad (4.26)$$

We solve by considering n is the leading variable, so we first solve one of the recurrence equations up to some constant that is a function f of k.

```
> eval(subs(a = unapply(a(n), n, k), rec2_A));
```

$$\mathrm{K}\ (\mathrm{K}\ 3\,n+2+k)\,(\mathrm{K}\ 3\,n+1+k)\,(\mathrm{K}\ 3\,n+k)\,(\mathrm{K}\ n+1+k)\,a(n) + 3\,n\,(n\mathrm{K}\ 1)\,(3\,n\mathrm{K}\ 1)\,(3\,n\mathrm{K}\ 2)\,a(n\mathrm{K}\ 1) \quad (4.27)$$

```
> bivrec_sol := LREtools:-hypergeomsols(%, a(n), {a(1) = f(k)});
```

$$bivrec\_sol := \left(9\sqrt{3}\,f(k)\,\Gamma\left(\frac{4}{3}\mathrm{K}\ \frac{k}{3}\right)\Gamma(\mathrm{K}\ k+1)\,\Gamma\left(\frac{5}{3}\mathrm{K}\ \frac{k}{3}\right)\Gamma\left(2\mathrm{K}\ \frac{k}{3}\right)\Gamma\left(n+\frac{1}{3}\right)\Gamma\left(n+\frac{2}{3}\right)\Gamma(n)\,\Gamma(n+1)\right) \Big/ \left(4\,\pi\,\Gamma\left(n+\frac{1}{3}\mathrm{K}\ \frac{k}{3}\right)\Gamma(n\mathrm{K}\ k)\,\Gamma\left(n+\frac{2}{3}\mathrm{K}\ \frac{k}{3}\right)\Gamma\left(n+1\mathrm{K}\ \frac{k}{3}\right)\right) \quad (4.28)$$

Then, we use the other recurrence equation to determine f(k).

```
> factor(expand(eval(subs(a = unapply(bivrec_sol, n, k), rec1_A)
  )));
```

$$\mathrm{K}\ \left(\sqrt{3}\,\Gamma\left(\mathrm{K}\ \frac{k}{3}\right)\Gamma\left(\frac{2}{3}\mathrm{K}\ \frac{k}{3}\right)\Gamma(\mathrm{K}\ k)\,k^2\,\Gamma\left(\frac{1}{3}\mathrm{K}\ \frac{k}{3}\right)n\,\Gamma(n)^2\,\Gamma\left(n+\frac{2}{3}\right)\Gamma\left(n+\frac{1}{3}\right)(k\mathrm{K}\ 1)\,(k\mathrm{K}\ 2)\,(k\mathrm{K}\ 3)\,(k^2 f(k)\mathrm{K}\ f(k\mathrm{K}\ 1)\,k^2+2\,k\,f(k)+5\,f(k \quad (4.29)$$

$$-1)\,k-4\,f(k-1)\Big)\Big/\Big(12\,\pi\,\Gamma\Big(n+\frac{1}{3}-\frac{k}{3}\Big)\,\Gamma(n-k)\,\Gamma\Big(n+\frac{2}{3}-\frac{k}{3}\Big)\,\Gamma\Big(n-\frac{k}{3}\Big)\,(-3\,n+k)\Big)$$

```
> collect(select(has, %, f), f, factor);
```

$$(k+2)\,k\,f(k)-(k-1)\,(k-4)\,f(k-1) \tag{4.30}$$

In turn, f(k) depends on some constant with respect to n and k.

```
> LREtools:-hypergeomsols(%, f(k), {f(4) = c});
```

$$\frac{2880\,c}{(k+2)\,(k+1)\,k^2\,(k-1)\,(k-2)\,(k-3)} \tag{4.31}$$

The constants for such equations are 1-periodic functions, so c, the value of f at 4, is in fact the value of some 1-periodic function phi(k) at 4.

```
> bivrec_sol := subs(f(k) = subs(c=phi(k), %), bivrec_sol);
```

$$bivrec\_sol := \Big(6480\,\sqrt{3}\,\phi(k)\,\Gamma\Big(\frac{4}{3}-\frac{k}{3}\Big)\,\Gamma(-k+1)\,\Gamma\Big(\frac{5}{3}-\frac{k}{3}\Big)\,\Gamma\Big(2-\frac{k}{3}\Big)\,\Gamma\Big(n+\frac{1}{3}\Big)\,\Gamma\Big(n+\frac{2}{3}\Big)\,\Gamma(n)\,\Gamma(n+1)\Big)\Big/\Big(\pi\,(k+2)\,(k+1)\,k^2\,(k-1)\,(k-2)\,(k-3)\,\Gamma\Big(n+\frac{1}{3}-\frac{k}{3}\Big)\,\Gamma(n-k)\,\Gamma\Big(n+\frac{2}{3}-\frac{k}{3}\Big)\,\Gamma\Big(n+1-\frac{k}{3}\Big)\Big) \tag{4.32}$$

Some generalized duplication formula known to Gauss is used to simplify. We could use it as well as the reflection formula to derive the expression of CF_A. We shorten a bit the exposition by merely showing the existence of a 1-periodic phi(k) such that the solution bivrec_sol agrees with CF_A.

```
> DLMF_5_5_3 := proc(z, {inv:=true}, $);
   return `if`(inv, GAMMA(1-z) = Pi/sin(Pi*z)/GAMMA(z), GAMMA(z)
  *GAMMA(1-z) = Pi/sin(Pi*z))
  end:
> DLMF_5_5_6 := proc(n, z, {inv:=true}, $) local k;
     local l := GAMMA(n*z);
     local r := (2*Pi)^(1-n)/2*n^(n*z-1/2)*mul(GAMMA(z+k/n), k=
  0..n-1);
     return `if`(inv, GAMMA(z) * (1 = l/r), l = r)
  end:
> sol_phi := solve(normal(expand(simplify(bivrec_sol / CF_A = 1,
  GAMMA))), {phi(k)});
```

$$sol\_phi := \left\{\phi(k) = \frac{3^k\,\sin(\pi\,k)\,\sqrt{3}}{120\,\Gamma(k)\,\Gamma\Big(\frac{2}{3}-\frac{k}{3}\Big)\,\Gamma\Big(\frac{1}{3}-\frac{k}{3}\Big)\,\Gamma\Big(-\frac{k}{3}\Big)\,k}\right\} \tag{4.33}$$

```
> DLMF_5_5_6(3, -k/3);
```

(4.34)

$$\Gamma\left(-\frac{k}{3}\right) = \frac{8\,\Gamma(-k)\,\pi^2}{3^{-k-\frac{1}{2}}\,\Gamma\left(\frac{1}{3}-\frac{k}{3}\right)\Gamma\left(\frac{2}{3}-\frac{k}{3}\right)} \tag{4.34}$$

```
> sol_phi := simplify(subs(%, sol_phi));
```

$$sol\_phi := \left\{\phi(k) = -\frac{\sin(\pi k)^2}{960\,\pi^3}\right\} \tag{4.35}$$

Because this is a well-defined 1-periodic function, we have proven:

```
> subs(sol_phi, bivrec_sol) = CF_A;
```

$$-\left(27\sqrt{3}\,\sin(\pi k)^2\,\Gamma\left(\frac{4}{3}-\frac{k}{3}\right)\Gamma(-k+1)\,\Gamma\left(\frac{5}{3}-\frac{k}{3}\right)\Gamma\left(2-\frac{k}{3}\right)\Gamma\left(n+\frac{1}{3}\right)\Gamma\left(n+\frac{2}{3}\right)\Gamma(n)\,\Gamma(n+1)\right) \Big/ \left(4\,\pi^4\,(k+2)\,(k+1)\,k^2\,(k-1)\,(k-2)\,(k-3)\,\Gamma\left(n+\frac{1}{3}-\frac{k}{3}\right)\Gamma(n-k)\,\Gamma\left(n+\frac{2}{3}-\frac{k}{3}\right)\Gamma\left(n+1-\frac{k}{3}\right)\right) = \frac{2\binom{n+1}{k+2}\binom{3n}{k}}{n\,(n+1)} \tag{4.36}$$

To really go to the full proof, we perform further substitution:

```
> DLMF_5_5_6(3, n+1/3-k/3), DLMF_5_5_6(3, n), DLMF_5_5_6(3, 4/3-
  k/3), DLMF_5_5_3(k-3), DLMF_5_5_3(k);
```

$$\Gamma\left(n+\frac{1}{3}-\frac{k}{3}\right) = \frac{8\,\Gamma(3n+1-k)\,\pi^2}{3^{3n+\frac{1}{2}-k}\,\Gamma\left(n+\frac{2}{3}-\frac{k}{3}\right)\Gamma\left(n+1-\frac{k}{3}\right)}, \Gamma(n) = \frac{8\,\Gamma(3n)\,\pi^2}{3^{3n-\frac{1}{2}}\,\Gamma\left(n+\frac{1}{3}\right)\Gamma\left(n+\frac{2}{3}\right)}, \Gamma\left(\frac{4}{3}-\frac{k}{3}\right) = \frac{8\,\Gamma(-k+4)\,\pi^2}{3^{-k+\frac{7}{2}}\,\Gamma\left(\frac{5}{3}-\frac{k}{3}\right)\Gamma\left(2-\frac{k}{3}\right)}, \Gamma(-k+4) = \frac{\pi}{\sin((k-3)\,\pi)\,\Gamma(k-3)}, \Gamma(-k+1) = \frac{\pi}{\sin(\pi k)\,\Gamma(k)} \tag{4.37}$$

```
> subs(%, %%);
```

$$-\left(54\sqrt{3}\,\sin(\pi k)\,\Gamma(3n)\,\Gamma(n+1)\,3^{3n+\frac{1}{2}-k}\right) \Big/ \left((k+2)\,(k+1)\,k^2\,(k-1)\,(k-2)\,(k-3)\,\sin((k-3)\,\pi)\,\Gamma(k-3)\,3^{-k+\frac{7}{2}}\,\Gamma(k)\,3^{3n-\frac{1}{2}}\,\Gamma(3n+1-k)\,\Gamma(n-k)\right) = \frac{2\binom{n+1}{k+2}\binom{3n}{k}}{n\,(n+1)} \tag{4.38}$$

This has proved Theorem 1.

We now reproduce very similar calculations for Theorem 2, without reproducing all the explanations. The one big difference will be that we obtain second-order recurrence relations instead of first-order recurrence relations: the existence of hypergeometric solutions is not proved a priori.

```
> dsysro_B := collect(dfinite_expr_to_sys(RootOf(P_mod, X), B(t::diff, z::diff)), {B, diff}, factor);
```

$$dsysro\_B := \Big\{12\,t\,(t\,z^4 + 22\,t\,z^3 + 258\,t\,z^2 \mathrm{K}\ 32\,t\,z \mathrm{K}\ 9\,z^2 \mathrm{K}\ 512\,t \mathrm{K}\ 18\,z + 234)\,B(t,z) + 12\,t^2\,(z+1)^2\,(10\,t^2\,z^4 \mathrm{K}\ 70\,t^2\,z^3 + 64\,t^2\,z^2 \mathrm{K}\ t\,z^2 \mathrm{K}\ 110\,t\,z + 224\,t + 9)\left(\frac{\partial}{\partial t}\,B(t,z)\right) + (432\,t^3\,z^7 \mathrm{K}\ 1512\,t^3\,z^6 \mathrm{K}\ 3888\,t^3\,z^5 \mathrm{K}\ 60\,t^2\,z^6 \mathrm{K}\ 54\,t^3\,z^4 + 312\,t^2\,z^5 + 432\,t^3\,z^3 \mathrm{K}\ 3642\,t^2\,z^4 + 9192\,t^2\,z^3 + 30768\,t^2\,z^2 + 432\,t\,z^3 + 5808\,t^2\,z \mathrm{K}\ 1944\,t\,z^2 \mathrm{K}\ 16128\,t^2 + 12312\,t\,z + 19062\,t \mathrm{K}\ 1458)\left(\frac{\partial}{\partial z}\,B(t,z)\right) + 2\,(z+1)\,(189\,t^3\,z^7 \mathrm{K}\ 567\,t^3\,z^6 \mathrm{K}\ 1701\,t^3\,z^5 \mathrm{K}\ 26\,t^2\,z^6 \mathrm{K}\ 216\,t^3\,z^4 + 117\,t^2\,z^5 \mathrm{K}\ 1914\,t^2\,z^4 + 1778\,t^2\,z^3 + 13680\,t^2\,z^2 + 270\,t\,z^3 + 9984\,t^2\,z \mathrm{K}\ 648\,t\,z^2 \mathrm{K}\ 2048\,t^2 + 4455\,t\,z + 7560\,t \mathrm{K}\ 729)\left(\frac{\partial^2}{\partial z^2}\,B(t,z)\right) + (z+1)^2\,(2\,t\,z^3 \mathrm{K}\ 5\,t\,z^2 \mathrm{K}\ 16\,t\,z \mathrm{K}\ 9)\,(27\,t^2\,z^4 \mathrm{K}\ 4\,t\,z^3 + 6\,t\,z^2 \mathrm{K}\ 192\,t\,z \mathrm{K}\ 256\,t + 27)\left(\frac{\partial^3}{\partial z^3}\,B(t,z)\right), 18\,B(t,z) \mathrm{K}\ 18\,t\,(t\,z+1)\left(\frac{\partial}{\partial t}\,B(t,z)\right) \mathrm{K}\ (z+1)\,(4\,t\,z^3 \mathrm{K}\ 10\,t\,z^2 + 4\,t\,z \mathrm{K}\ 45)\left(\frac{\partial}{\partial z}\,B(t,z)\right) + t\,(z+1)\,(2\,t\,z^3 \mathrm{K}\ 5\,t\,z^2 \mathrm{K}\ 16\,t\,z \mathrm{K}\ 9)\left(\frac{\partial^2}{\partial t\,\partial z}\,B(t,z)\right) \mathrm{K}\ 2\,(z+1)^2\,(t\,z^3 \mathrm{K}\ 2\,t\,z^2 \mathrm{K}\ 6)\left(\frac{\partial^2}{\partial z^2}\,B(t,z)\right), 24\,B(t,z) \mathrm{K}\ 24\,t\,(t\,z+1)\left(\frac{\partial}{\partial t}\,B(t,z)\right) + (\mathrm{K}\ 4\,t\,z^4 + 4\,t\,z^3 \mathrm{K}\ t\,z^2 + 54\,z + 63)\left(\frac{\partial}{\partial z}\,B(t,z)\right) + t^2\,(2\,t\,z^3 \mathrm{K}\ 5\,t\,z^2 \mathrm{K}\ 16\,t\,z \mathrm{K}\ 9)\left(\frac{\partial^2}{\partial t^2}\,B(t,z)\right) \mathrm{K}\ (z+1)\,(2\,t\,z^4 \mathrm{K}\ t\,z^3 \mathrm{K}\ 13\,z \mathrm{K}\ 16)\left(\frac{\partial^2}{\partial z^2}\,B(t,z)\right)\Big\} \qquad \mathbf{(4.39)}$$

```
> map(indets, dsysro_B);
```

(4.40)

$$\left\{\left\{t, z, B(t,z), \frac{\partial}{\partial t} B(t,z), \frac{\partial}{\partial z} B(t,z), \frac{\partial^2}{\partial t^2} B(t,z), \frac{\partial^2}{\partial z^2} B(t,z)\right\}, \left\{t, z, B(t,z), \frac{\partial}{\partial t} B(t,z), \frac{\partial}{\partial z} B(t,z), \frac{\partial^2}{\partial t \partial z} B(t,z), \frac{\partial^2}{\partial z^2} B(t,z)\right\}, \left\{t, z, B(t,z), \frac{\partial}{\partial t} B(t,z), \frac{\partial}{\partial z} B(t,z), \frac{\partial^2}{\partial z^2} B(t,z), \frac{\partial^3}{\partial z^3} B(t,z)\right\}\right\} \quad \textbf{(4.40)}$$

```
> indets(dsysro_B);
```

$$\left\{t, z, B(t,z), \frac{\partial}{\partial t} B(t,z), \frac{\partial}{\partial z} B(t,z), \frac{\partial^2}{\partial t^2} B(t,z), \frac{\partial^2}{\partial t \partial z} B(t,z), \frac{\partial^2}{\partial z^2} B(t,z), \frac{\partial^3}{\partial z^3} B(t,z)\right\} \quad \textbf{(4.41)}$$

```
> Gro_B := subs([seq(seq(diff(B(t,z), [t$i,z$j]) = dt^i*dz^j, j=
  0..3), i=0..3)], dsysro_B);
```

$$Gro\_B := \{18 \mathrm{K}\ 18\, t\, (t\, z + 1)\, dt \mathrm{K}\ (z + 1)\, (4\, t\, z^3 \mathrm{K}\ 10\, t\, z^2 + 4\, t\, z \mathrm{K}\ 45)\, dz + t\, (z + 1)\, (2\, t\, z^3 \mathrm{K}\ 5\, t\, z^2 \mathrm{K}\ 16\, t\, z \mathrm{K}\ 9)\, dt\, dz \mathrm{K}\ 2\, (z + 1)^2\, (t\, z^3 \mathrm{K}\ 2\, t\, z^2 \mathrm{K}\ 6)\, dz^2, 24 \mathrm{K}\ 24\, t\, (t\, z + 1)\, dt + (\mathrm{K}\ 4\, t\, z^4 + 4\, t\, z^3 \mathrm{K}\ t\, z^2 + 54\, z + 63)\, dz + t^2\, (2\, t\, z^3 \mathrm{K}\ 5\, t\, z^2 \mathrm{K}\ 16\, t\, z \mathrm{K}\ 9)\, dt^2 \mathrm{K}\ (z + 1)\, (2\, t\, z^4 \mathrm{K}\ t\, z^3 \mathrm{K}\ 13\, z \mathrm{K}\ 16)\, dz^2, 12\, t\, (t\, z^4 + 22\, t\, z^3 + 258\, t\, z^2 \mathrm{K}\ 32\, t\, z \mathrm{K}\ 9\, z^2 \mathrm{K}\ 512\, t \mathrm{K}\ 18\, z + 234) + 12\, t^2\, (z + 1)^2\, (10\, t^2\, z^4 \mathrm{K}\ 70\, t^2\, z^3 + 64\, t^2\, z^2 \mathrm{K}\ t\, z^2 \mathrm{K}\ 110\, t\, z + 224\, t + 9)\, dt + (432\, t^3\, z^7 \mathrm{K}\ 1512\, t^3\, z^6 \mathrm{K}\ 3888\, t^3\, z^5 \mathrm{K}\ 60\, t^2\, z^6 \mathrm{K}\ 54\, t^3\, z^4 + 312\, t^2\, z^5 + 432\, t^3\, z^3 \mathrm{K}\ 3642\, t^2\, z^4 + 9192\, t^2\, z^3 + 30768\, t^2\, z^2 + 432\, t\, z^3 + 5808\, t^2\, z \mathrm{K}\ 1944\, t\, z^2 \mathrm{K}\ 16128\, t^2 + 12312\, t\, z + 19062\, t \mathrm{K}\ 1458)\, dz + 2\, (z + 1)\, (189\, t^3\, z^7 \mathrm{K}\ 567\, t^3\, z^6 \mathrm{K}\ 1701\, t^3\, z^5 \mathrm{K}\ 26\, t^2\, z^6 \mathrm{K}\ 216\, t^3\, z^4 + 117\, t^2\, z^5 \mathrm{K}\ 1914\, t^2\, z^4 + 1778\, t^2\, z^3 + 13680\, t^2\, z^2 + 270\, t\, z^3 + 9984\, t^2\, z \mathrm{K}\ 648\, t\, z^2 \mathrm{K}\ 2048\, t^2 + 4455\, t\, z + 7560\, t \mathrm{K}\ 729)\, dz^2 + (z + 1)^2\, (2\, t\, z^3 \mathrm{K}\ 5\, t\, z^2 \mathrm{K}\ 16\, t\, z \mathrm{K}\ 9)\, (27\, t^2\, z^4 \mathrm{K}\ 4\, t\, z^3 + 6\, t\, z^2 \mathrm{K}\ 192\, t\, z \mathrm{K}\ 256\, t + 27)\, dz^3\} \quad \textbf{(4.42)}$$

```
> map(degree, convert(Gro_B, list), {t,z});
```

$$[6, 6, 12] \quad \textbf{(4.43)}$$

```
> map(degree, convert(Gro_B, list));
```

$$[8, 8, 15] \quad \textbf{(4.44)}$$

```
> Bro_B := Basis(Gro_B, wMOrd):
```

Again, dimension is equal to 2, justifying holonomy, more precisely that we have a holonomic presentation at hand.

```
> HilbertDimension(Bro_B, wMOrd);
```

$$2 \quad \textbf{(4.45)}$$

```
> map(indets, Bro_B);
```

$$[\{dt, dz, t, z\}, \{dt, dz, t, z\}, \{dt, dz, t, z\}, \{dt, dz, t, z\}, \{dt, dz, t, z\}, \{dt, dz, t, z\}, \{dt, dz, t, z\}] \quad \textbf{(4.46)}$$

```
> map(LeadingMonomial, Bro_B, wMOrd);
```
$$[dt^2\, t^2\, z, dt^2\, t^3, dt\, dz\, t^2\, z^4, dz^3\, t\, z^4, dt\, dz^2\, t^2\, z^3, dt\, dz^2\, t\, z^5, dz^4\, z^5] \quad (4.47)$$

```
> map(degree, Bro_B, dt);
```
$$[2, 2, 1, 1, 1, 2, 2] \quad (4.48)$$

```
> map(degree, Bro_B, dz);
```
$$[2, 2, 2, 3, 3, 3, 4] \quad (4.49)$$

```
> Gro_B_rec := [op(expand(map(conv, Bro_B))), sn*tn-1, sk*tk-1]:
> Bro_B_rec := Basis(Gro_B_rec, sMOrd):
> map(LeadingMonomial, Bro_B_rec, sMOrd);
```
$$[k\, n\, sk, n^6\, tn, k\, n^5\, sn, n^6\, sn, n^2\, tk^2, k^3\, n\, tk^2, sk\, tk, n^2\, tk\, tn, k^5\, tk\, tn, k^4\, n\, tk\, tn, n^2\, sn\, tk, k^3\, n\, sn\, tk, k^3\, sk^2, n^3\, sk\, tn, k^2\, sk\, sn, n^4\, sk\, sn, sn\, tn, k\, n\, tk^3, k\, n\, tk^2\, tn, k^3\, tn\, tk^2, k^2\, n\, sn\, tk^2, k^4\, sn\, tk^2, k^2\, sk^3, k\, sk^2\, sn, k^3\, sn\, tk^3] \quad (4.50)$$

```
> map(length, Bro_B_rec);
```
$$[154, 1353, 836, 1072, 177, 955, 19, 245, 1520, 1464, 245, 1327, 1047, 1260, 1100, 1077, 19, 1193, 1129, 1336, 1466, 2418, 1119, 943, 2983] \quad (4.51)$$

```
> collect(Bro_B_rec, {sn,tn,sk,tk}, distributed, 1);
```
$$[1 + sk + tk, 1 + sk + tk + tn, 1 + sn + tk, 1 + sn + tk, tk^2 + tk + 1, tk^2 + sn + tk + 1, sk\, tk + 1, tk\, tn + tk + tn + 1, tk\, tn + sk + tk + tn + 1, tk\, tn + sk + tk + tn + 1, sn\, tk + sn + tk + 1, sn\, tk + tk^2 + sn + tk + 1, sk^2 + tk\, tn + sk + tk + tn + 1, sk^2 + sk\, tn + tk\, tn + sk + tk + tn + 1, sk\, sn + sn\, tk + tk^2 + sn + tk + 1, sk\, sn + tk^2 + sk + sn + tk + 1, sn\, tn + 1, tk^3 + sk\, sn + sn\, tk + tk^2 + sn + tk + 1, tk^2\, tn + sk^2 + tk\, tn + sk + tk + tn + 1, tk^2\, tn + sk^2 + tk\, tn + sk + tk + tn + 1, sn\, tk^2 + tk^3 + sk\, sn + sn\, tk + tk^2 + sn + tk + 1, sn\, tk^2 + tk^3 + sk\, sn + sn^2 + sn\, tk + tk^2 + sn + tk + 1, sk^3 + sk^2 + tk\, tn + sk + tk + tn + 1, sk^2\, sn + sk\, sn + sn\, tk + tk^2 + sk + sn + tk + 1, sn\, tk^3 + tk^4 + sk\, sn^2 + sn^2\, tk + sn\, tk^2 + tk^3 + sk\, sn + sn^2 + sn\, tk + tk^2 + sn + tk + 1] \quad (4.52)$$

```
> collect(Bro_B_rec, {sn,tn,sk,tk}, distributed, factor);
```
$$[\mathrm{K}\ (\mathrm{K}\ 4\, n \mathrm{K}\ 1 + k)\, sk\, (k + 1) \mathrm{K}\ (\mathrm{K}\ 3\, n + k \mathrm{K}\ 1)\, (\mathrm{K}\ n + k)\, tk \mathrm{K}\ 2\, k^2 + 8\, k\, n \mathrm{K}\ 3\, n^2 + k + 3\, n, \mathrm{K}\ 423\, k^5 \mathrm{K}\ 1024\, (n \mathrm{K}\ 1)\, (3\, n \mathrm{K}\ 1)\, (3\, n \mathrm{K}\ 2)\, (\mathrm{K}\ 4\, n + 2 + k)\, (\mathrm{K}\ 4\, n + 1 + k)\, (\mathrm{K}\ 4\, n + k)\, tn + 511456\, k\, n^4 \mathrm{K}\ 175212\, k^3\, n^3 + 18405\, k^4\, n^2 \mathrm{K}\ 334620\, k^2\, n^3 + 56660\, k^3\, n^2 \mathrm{K}\ 9\, k^4\, n + 122172\, k\, n^3 \mathrm{K}\ 88298\, k^2\, n^2 + 11012\, k^3\, n \mathrm{K}\ (\mathrm{K}\ n + k)\, (\mathrm{K}\ 3\, n + k \mathrm{K}\ 1)\, (9\, k^4 + 36\, k^3\, n \mathrm{K}\ 18288\, k^2\, n^2 + 111168\, k\, n^3 \mathrm{K}\ 175872\, n^4 + 216\, k^3 + 900\, k^2\, n \mathrm{K}\ 72032\, k\, n^2 + 200896\, n^3 \mathrm{K}\ 270\, k^2 \mathrm{K}\ 180\, k\, n \mathrm{K}\ 42032\, n^2 + 176\, k \mathrm{K}\ 8692\, n + 893)\, tk \mathrm{K}\ 18\, k^6 \mathrm{K}\ 62208\, n^6 + 15552\, n^5 \mathrm{K}\ 594624\, k\, n^5 + 564816\, k^2\, n^4 \mathrm{K}\ (k + 11)\, (3\, k + 5)\, (k + 7)\, (3\, k + 1)\, (k + 3)\, (k + 1)\, sk$$

$\mathrm{K}\ 23220\,n^{3}\mathrm{K}\ 31827\,n^{2}+42378\,k^{2}\,n\mathrm{K}\ 156668\,k\,n^{2}+18436\,k\,n$
$+92880\,n^{4}\mathrm{K}\ 1292\,k^{4}\mathrm{K}\ 6766\,k^{3}\mathrm{K}\ 7754\,k^{2}+8823\,n\mathrm{K}\ 1155\,k,\ k\,(n$
$+2)\,(3\,n+4)\,(3\,n+5)\,(\mathrm{K}\ 3\,n+k\mathrm{K}\ 2)\,(\mathrm{K}\ 3\,n+k\mathrm{K}\ 3)\,sn+k\,(18\,k^{3}\,n^{2}$
$\mathrm{K}\ 180\,k^{2}\,n^{3}+606\,k\,n^{4}\mathrm{K}\ 687\,n^{5}+36\,k^{3}\,n\mathrm{K}\ 484\,k^{2}\,n^{2}+2055\,k\,n^{3}$
$\mathrm{K}\ 2822\,n^{4}+20\,k^{3}\mathrm{K}\ 428\,k^{2}\,n+2499\,k\,n^{2}\mathrm{K}\ 4386\,n^{3}\mathrm{K}\ 120\,k^{2}+1266\,k\,n$
$\mathrm{K}\ 3171\,n^{2}+220\,k\mathrm{K}\ 1042\,n\mathrm{K}\ 120)\,tk+k\,(\mathrm{K}\ 4\,n\mathrm{K}\ 2+k)\,(18\,k^{2}\,n^{2}$
$\mathrm{K}\ 117\,k\,n^{3}+192\,n^{4}+36\,k^{2}\,n\mathrm{K}\ 331\,k\,n^{2}+704\,n^{3}+20\,k^{2}\mathrm{K}\ 314\,k\,n$
$+948\,n^{2}\mathrm{K}\ 100\,k+556\,n+120),\ (n+2)\,(3\,n+4)\,(3\,n+5)\,(3\,n$
$+2\,k)\,(\mathrm{K}\ 3\,n+k\mathrm{K}\ 2)\,(\mathrm{K}\ 3\,n+k\mathrm{K}\ 3)\,sn+(3\,n+2\,k)\,(18\,k^{3}\,n^{2}$
$\mathrm{K}\ 180\,k^{2}\,n^{3}+606\,k\,n^{4}\mathrm{K}\ 687\,n^{5}+36\,k^{3}\,n\mathrm{K}\ 484\,k^{2}\,n^{2}+2055\,k\,n^{3}$
$\mathrm{K}\ 2822\,n^{4}+20\,k^{3}\mathrm{K}\ 428\,k^{2}\,n+2499\,k\,n^{2}\mathrm{K}\ 4386\,n^{3}\mathrm{K}\ 120\,k^{2}+1266\,k\,n$
$\mathrm{K}\ 3171\,n^{2}+220\,k\mathrm{K}\ 1042\,n\mathrm{K}\ 120)\,tk+(3\,n+2\,k)\,(\mathrm{K}\ 4\,n\mathrm{K}\ 2$
$+k)\,(18\,k^{2}\,n^{2}\mathrm{K}\ 117\,k\,n^{3}+192\,n^{4}+36\,k^{2}\,n\mathrm{K}\ 331\,k\,n^{2}+704\,n^{3}+20\,k^{2}$
$\mathrm{K}\ 314\,k\,n+948\,n^{2}\mathrm{K}\ 100\,k+556\,n+120),\ (\mathrm{K}\ n+k\mathrm{K}\ 1)\,(\mathrm{K}\ 3\,n+k$
$\mathrm{K}\ 2)\,tk^{2}+(2\,k^{2}\mathrm{K}\ 8\,k\,n+3\,n^{2}\mathrm{K}\ 5\,k+5\,n+3)\,tk+k\,(\mathrm{K}\ 4\,n\mathrm{K}\ 2+k),$
$27\,(n+2)\,(3\,n+4)\,(3\,n+5)\,(\mathrm{K}\ 3\,n+k)\,(\mathrm{K}\ 3\,n+k\mathrm{K}\ 3)\,sn\mathrm{K}\ 2\,(k$
$+4)\,(k\mathrm{K}\ 2)\,(k+7)\,(\mathrm{K}\ n+k\mathrm{K}\ 1)\,tk^{2}+(486\,k^{3}\,n^{2}\mathrm{K}\ 4860\,k^{2}\,n^{3}$
$+16362\,k\,n^{4}\mathrm{K}\ 18549\,n^{5}\mathrm{K}\ 4\,k^{4}+976\,k^{3}\,n\mathrm{K}\ 12090\,k^{2}\,n^{2}+48699\,k\,n^{3}$
$\mathrm{K}\ 63828\,n^{4}+506\,k^{3}\mathrm{K}\ 9572\,k^{2}\,n+49191\,k\,n^{2}\mathrm{K}\ 75870\,n^{3}\mathrm{K}\ 2168\,k^{2}$
$+18260\,k\,n\mathrm{K}\ 35037\,n^{2}+1834\,k\mathrm{K}\ 4888\,n\mathrm{K}\ 168)\,tk\mathrm{K}\ (\mathrm{K}\ 4\,n\mathrm{K}\ 2$
$+k)\,(\mathrm{K}\ 486\,k^{2}\,n^{2}+3159\,k\,n^{3}\mathrm{K}\ 5184\,n^{4}+2\,k^{3}\mathrm{K}\ 966\,k^{2}\,n+7983\,k\,n^{2}$
$\mathrm{K}\ 15552\,n^{3}\mathrm{K}\ 518\,k^{2}+6612\,k\,n\mathrm{K}\ 15228\,n^{2}+1676\,k\mathrm{K}\ 4860\,n),\ sk\,tk$
$\mathrm{K}\ 1,\ (\mathrm{K}\ n+1+k)\,(\mathrm{K}\ 3\,n+2+k)\,tk\,tn\mathrm{K}\ (\mathrm{K}\ 3\,n+k\mathrm{K}\ 1)\,(\mathrm{K}\ n+k)\,tk$
$+k\,(\mathrm{K}\ 4\,n+2+k)\,tn\mathrm{K}\ k\,(\mathrm{K}\ 4\,n\mathrm{K}\ 2+k),\ \mathrm{K}\ 9\,(k+7)\,(3\,k+1)\,(9841\,k$
$+16395)\,(k+3)\,(k+1)\,sk+256\,(k+1)\,(729\,k^{4}+12818\,k^{3}$
$\mathrm{K}\ 11117\,k^{2}\,n+41074\,k^{2}\mathrm{K}\ 28823\,k\,n\mathrm{K}\ 14681\,k+43720\,n\mathrm{K}\ 43720)$
$tk\,tn\mathrm{K}\ (\mathrm{K}\ 3\,n+k\mathrm{K}\ 1)\,(\mathrm{K}\ n+k)\,(452331\,k^{3}+1809324\,k^{2}\,n$
$\mathrm{K}\ 129350736\,k\,n^{2}+432513216\,n^{3}+6362171\,k^{2}+22359192\,k\,n$
$\mathrm{K}\ 205711344\,n^{2}+4078137\,k\mathrm{K}\ 33773700\,n\mathrm{K}\ 1139999)\,tk+256\,($
$\mathrm{K}\ 4\,n+2+k)\,(729\,k^{4}+2916\,k^{3}\,n+9477\,k^{2}\,n^{2}\mathrm{K}\ 236520\,k\,n^{3}$
$+1416528\,n^{4}+11360\,k^{3}+29220\,k^{2}\,n+600309\,k\,n^{2}\mathrm{K}\ 2242836\,n^{3}$
$+18354\,k^{2}\mathrm{K}\ 327936\,k\,n+944352\,n^{2}+7651\,k\mathrm{K}\ 118044\,n)\,tn$
$\mathrm{K}\ 718038\,k^{5}+135790911\,k^{3}\,n^{2}\mathrm{K}\ 885090564\,k^{2}\,n^{3}+1513277424\,k\,n^{4}$
$+152985024\,n^{5}\mathrm{K}\ 9177227\,k^{4}+5379605\,k^{3}\,n+156798627\,k^{2}\,n^{2}$
$\mathrm{K}\ 227733624\,k\,n^{3}+63743760\,n^{4}\mathrm{K}\ 9301559\,k^{3}+158537241\,k^{2}\,n$
$\mathrm{K}\ 514212339\,k\,n^{2}\mathrm{K}\ 185919300\,n^{3}\mathrm{K}\ 17208457\,k^{2}+29687599\,k\,n$

$K\ 66842415\, n^2 K\ 14290975\, k + 36032931\, n,\ K\ 45\ (k+7)\ (3\,k+1)\ (k$

$+ 3)\ (164\,k + 273)\ (k+1)\ sk + 64\ (k+1)\ (729\,k^3\,n + 3341\,k^3$

$K\ 2369\,k^2\,n + 12643\,k^2 K\ 9140\,k\,n K\ 5204\,k + 14560\,n K\ 14560)\ tk\ tn$

$K\ (K\ 3\,n + k K\ 1)\ (K\ n + k)\ (22140\,k^3 + 135216\,k^2\,n K\ 10791360\,k\,n^2$

$+ 36009792\,n^3 + 501599\,k^2 + 1850028\,k\,n K\ 17126928\,n^2 + 329718\,k$

$K\ 2811900\,n K\ 94913)\ tk + 64\ (K\ 4\,n + 2 + k)\ (729\,k^3\,n + 2916\,k^2\,n^2$

$K\ 79056\,k\,n^3 + 471744\,n^4 + 3341\,k^3 + 9537\,k^2\,n + 200088\,k\,n^2$

$K\ 746928\,n^3 + 5961\,k^2 K\ 109236\,k\,n + 314496\,n^2 + 2548\,k$

$K\ 39312\,n)\ tn K\ 44280\,k^5 K\ 46656\,k^4\,n + 11265804\,k^3\,n^2$

$K\ 73774800\,k^2\,n^3 + 125983296\,k\,n^4 + 12737088\,n^5 K\ 767234\,k^4$

$+ 316100\,k^3\,n + 12970419\,k^2\,n^2 K\ 18971460\,k\,n^3 + 5307120\,n^4$

$K\ 821645\,k^3 + 13140081\,k^2\,n K\ 42813390\,k\,n^2 K\ 15479100\,n^3$

$K\ 1453000\,k^2 + 2473630\,k\,n K\ 5565105\,n^2 K\ 1189825\,k$

$+ 2999997\,n,\ (K\ 3\,n + k K\ 4)\ (K\ n + k K\ 1)\ sn\ tk + k\ (k K\ 4\,n K\ 6)\ sn$

$K\ (K\ 3\,n + k K\ 1)\ (K\ n + k)\ tk K\ k\ (K\ 4\,n K\ 2 + k),\ K\ 18\ (k K\ 1)\ (3\,k$

$+ 1)\ (k+1)\ (K\ 3\,n + k K\ 4)\ sn\ tk + (132840\,k^2\,n^3 K\ 797040\,k\,n^4$

$+ 1195560\,n^5 K\ 54\,k^4 + 162\,k^3\,n + 664362\,k^2\,n^2 K\ 4383558\,k\,n^3$

$+ 7173603\,n^4 + 252\,k^3 + 1092726\,k^2\,n K\ 8545392\,k\,n^2 + 15809175\,n^3$

$+ 590814\,k^2 K\ 6818436\,k\,n + 15145758\,n^2 K\ 1771092\,k$

$+ 5314680\,n)\ sn + (K\ 729\,k^4 K\ 9902\,k^3 + 11117\,k^2\,n + 4123\,k^2$

$+ 6589\,k\,n + 67880\,k K\ 61426\,n K\ 61372)\ tk^2 + (265680\,k^3\,n^2$

$K\ 2656800\,k^2\,n^3 + 8944560\,k\,n^4 K\ 10140120\,n^5 K\ 1404\,k^4$

$+ 531198\,k^3\,n K\ 6610293\,k^2\,n^2 + 26621190\,k\,n^3 K\ 34894701\,n^4$

$+ 274685\,k^3 K\ 5231759\,k^2\,n + 26890026\,k\,n^2 K\ 41480631\,n^3$

$K\ 1183968\,k^2 + 9982635\,k\,n K\ 19156959\,n^2 + 1002745\,k$

$K\ 2672746\,n K\ 92058)\ tk K\ (K\ 4\,n K\ 2 + k)\ (K\ 265680\,k^2\,n^2$

$+ 1726920\,k\,n^3 K\ 2833920\,n^4 + 675\,k^3 K\ 528498\,k^2\,n + 4363623\,k\,n^2$

$K\ 8502336\,n^3 K\ 283183\,k^2 + 3614136\,k\,n K\ 8325792\,n^2 + 916268\,k$

$K\ 2657340\,n),\ 576\ (k+4)\ (k+2)\ (k+1)\ sk^2 K\ 9\ (3\,k K\ 43)\ (k+3)\ (k$

$+ 7)\ (k+1)\ sk K\ 64\ (k+1)\ (364\,k^3 K\ 1093\,k^2\,n + 911\,k^2 K\ 547\,k\,n$

$+ 365\,k + 20\,n K\ 20)\ tk\ tn + (K\ n + k)\ (K\ 3\,n + k K\ 1)\ (23269\,k^2$

$+ 23124\,k\,n + 32976\,n^2 + 10650\,k K\ 15684\,n K\ 3007)\ tk K\ 64\ (K\ 4\,n$

$+ 2 + k)\ (364\,k^3 + 363\,k^2\,n + 360\,k\,n^2 + 432\,n^3 + 183\,k^2 K\ 177\,k\,n$

$K\ 684\,n^2 K\ k + 288\,n K\ 36)\ tn + 23242\,k^4 K\ 69952\,k^3\,n K\ 59601\,k^2\,n^2$

$K\ 127044\,k\,n^3 K\ 11664\,n^4 K\ 36401\,k^3 K\ 104667\,k^2\,n + 3426\,k\,n^2$

$K\ 4860\,n^3 K\ 26008\,k^2 + 44206\,k\,n + 12879\,n^2 + 4799\,k + 3645\,n,$

$2880\,(k+4)\,(k+2)\,sk^{2}\,\mathrm{K}\ 9216\,(\mathrm{K}\,4\,n+3+k)\,(k^{2}+4\,k\,n+16\,n^{2}\,\mathrm{K}\ k$

$\mathrm{K}\ 16\,n+2)\,sk\,tn+(\mathrm{K}\,27\,k^{4}+558\,k^{3}\,\mathrm{K}\ 62208\,n^{3}\,\mathrm{K}\ 72\,k^{2}\,\mathrm{K}\ 46656\,n^{2}$

$+18\,k+53568\,n+55971)\,sk+(\mathrm{K}\,23296\,k^{4}+69952\,k^{3}\,n$

$\mathrm{K}\ 205504\,k^{3}+477696\,k^{2}\,n\,\mathrm{K}\ 390912\,k^{2}+219072\,k\,n\,\mathrm{K}\ 147008\,k$

$\mathrm{K}\ 6400\,n+6400)\,tk\,tn+(\mathrm{K}\,n+k)\,(\mathrm{K}\,3\,n+k\,\mathrm{K}\ 1)\,(23269\,k^{2}$

$+23124\,k\,n+32976\,n^{2}+126958\,k+76812\,n\,\mathrm{K}\ 48803)\,tk+($

$\mathrm{K}\,23296\,k^{4}+69952\,k^{3}\,n+69888\,k^{2}\,n^{2}+64512\,k\,n^{3}+110592\,n^{4}$

$\mathrm{K}\ 191424\,k^{3}+384448\,k^{2}\,n+296448\,k\,n^{2}+617472\,n^{3}\,\mathrm{K}\ 236352\,k^{2}$

$\mathrm{K}\ 298112\,k\,n\,\mathrm{K}\ 1460736\,n^{2}+62848\,k+820224\,n\,\mathrm{K}\ 87552)\,tn$

$+23242\,k^{4}\,\mathrm{K}\ 69952\,k^{3}\,n\,\mathrm{K}\ 59601\,k^{2}\,n^{2}\,\mathrm{K}\ 127044\,k\,n^{3}\,\mathrm{K}\ 11664\,n^{4}$

$+80951\,k^{3}\,\mathrm{K}\ 477403\,k^{2}\,n\,\mathrm{K}\ 361698\,k\,n^{2}\,\mathrm{K}\ 47628\,n^{3}\,\mathrm{K}\ 305140\,k^{2}$

$+50498\,k\,n\,\mathrm{K}\ 4941\,n^{2}+76195\,k+64233\,n,\ 90\,(k+3)\,(k+1)\,sk\,sn$

$+540\,(k\,\mathrm{K}\ 1)\,(k+1)\,(\mathrm{K}\,3\,n+k\,\mathrm{K}\ 4)\,sn\,tk+(81\,k^{2}\,n^{3}\,\mathrm{K}\ 486\,k\,n^{4}$

$+729\,n^{5}+405\,k^{2}\,n^{2}\,\mathrm{K}\ 2835\,k\,n^{3}+4860\,n^{4}+540\,k^{3}\,\mathrm{K}\ 954\,k^{2}\,n$

$\mathrm{K}\ 7641\,k\,n^{2}+10125\,n^{3}\,\mathrm{K}\ 2250\,k^{2}\,\mathrm{K}\ 9810\,k\,n+5940\,n^{2}\,\mathrm{K}\ 4590\,k$

$\mathrm{K}\ 1674\,n)\,sn+(\mathrm{K}\,3640\,k^{3}+10930\,k^{2}\,n+12740\,k^{2}\,\mathrm{K}\ 16390\,k\,n$

$\mathrm{K}\ 12740\,k+5260\,n+3640)\,tk^{2}+(162\,k^{3}\,n^{2}\,\mathrm{K}\ 1620\,k^{2}\,n^{3}+5454\,k\,n^{4}$

$\mathrm{K}\ 6183\,n^{5}+324\,k^{3}\,n\,\mathrm{K}\ 4356\,k^{2}\,n^{2}+18495\,k\,n^{3}\,\mathrm{K}\ 25398\,n^{4}\,\mathrm{K}\ 7640\,k^{3}$

$+19628\,k^{2}\,n+31791\,k\,n^{2}\,\mathrm{K}\ 18864\,n^{3}+20850\,k^{2}\,\mathrm{K}\ 8976\,k\,n$

$+1161\,n^{2}\,\mathrm{K}\ 18670\,k+3502\,n+5460)\,tk+(\mathrm{K}\,4\,n\,\mathrm{K}\ 2+k)\,(162\,k^{2}\,n^{2}$

$\mathrm{K}\ 1053\,k\,n^{3}+1728\,n^{4}+324\,k^{2}\,n\,\mathrm{K}\ 2979\,k\,n^{2}+6336\,n^{3}\,\mathrm{K}\ 4000\,k^{2}$

$\mathrm{K}\ 6996\,k\,n+2772\,n^{2}+470\,k\,\mathrm{K}\ 756\,n),\ 9\,n\,(n+2)\,(3\,n+4)\,(3\,n$

$+5)\,sk\,sn+3\,(\mathrm{K}\,4\,n\,\mathrm{K}\ 1+k)\,(4\,n+4+k)\,(k^{2}+16\,n^{2}+3\,k+20\,n$

$+2)\,sk\,\mathrm{K}\ 3\,(n+2)\,(3\,n+4)\,(3\,n+5)\,(\mathrm{K}\,3\,n+k)\,(\mathrm{K}\,3\,n+k\,\mathrm{K}\ 3)\,sn$

$+2\,(k+4)\,(k\,\mathrm{K}\ 2)\,(\mathrm{K}\,n+k\,\mathrm{K}\ 1)\,tk^{2}+(\mathrm{K}\,54\,k^{3}\,n^{2}+540\,k^{2}\,n^{3}$

$\mathrm{K}\ 1818\,k\,n^{4}+2061\,n^{5}+3\,k^{4}\,\mathrm{K}\ 108\,k^{3}\,n+1353\,k^{2}\,n^{2}\,\mathrm{K}\ 5535\,k\,n^{3}$

$+7203\,n^{4}\,\mathrm{K}\ 41\,k^{3}+1067\,k^{2}\,n\,\mathrm{K}\ 5721\,k\,n^{2}+8667\,n^{3}+252\,k^{2}$

$\mathrm{K}\ 2196\,k\,n+4053\,n^{2}\,\mathrm{K}\ 238\,k+568\,n+24)\,tk\,\mathrm{K}\ 54\,k^{3}\,n^{2}+567\,k^{2}\,n^{3}$

$\mathrm{K}\ 1980\,k\,n^{4}+2304\,n^{5}+6\,k^{4}\,\mathrm{K}\ 108\,k^{3}\,n+1434\,k^{2}\,n^{2}\,\mathrm{K}\ 6102\,k\,n^{3}$

$+7488\,n^{4}\,\mathrm{K}\ 25\,k^{3}+1183\,k^{2}\,n\,\mathrm{K}\ 6660\,k\,n^{2}+8784\,n^{3}+334\,k^{2}$

$\mathrm{K}\ 2906\,k\,n+4608\,n^{2}\,\mathrm{K}\ 400\,k+1008\,n,\ sn\,tn\,\mathrm{K}\ 1,\ (\mathrm{K}\,720\,k$

$\mathrm{K}\ 720)\,sk\,sn\,\mathrm{K}\ 1620\,(k\,\mathrm{K}\ 1)\,(k+1)\,(\mathrm{K}\,3\,n+k\,\mathrm{K}\ 4)\,sn\,tk+(\mathrm{K}\,351\,k^{2}\,n^{3}$

$+2106\,k\,n^{4}\,\mathrm{K}\ 3159\,n^{5}\,\mathrm{K}\ 1755\,k^{2}\,n^{2}+12285\,k\,n^{3}\,\mathrm{K}\ 21060\,n^{4}\,\mathrm{K}\ 1620\,k^{3}$

$+1974\,k^{2}\,n+30951\,k\,n^{2}\,\mathrm{K}\ 47115\,n^{3}+6540\,k^{2}+36750\,k\,n$

$\mathrm{K}\ 37890\,n^{2}+16800\,k\,\mathrm{K}\ 4536\,n)\,sn\,\mathrm{K}\ 80\,(k\,\mathrm{K}\ 3)\,(\mathrm{K}\,n\,\mathrm{K}\ 2+k)\,tk^{3}$

$+(10920\,k^{3}\,\mathrm{K}\ 32790\,k^{2}\,n\,\mathrm{K}\ 38060\,k^{2}+48890\,k\,n+37980\,k$

$\mathrm{K}\ 15220\, n \mathrm{K}\ 11080)\ tk^2 + (\mathrm{K}\ 702\, k^3\, n^2 + 7020\, k^2\, n^3 \mathrm{K}\ 23634\, k\, n^4$
$+ 26793\, n^5 \mathrm{K}\ 1404\, k^3\, n + 18876\, k^2\, n^2 \mathrm{K}\ 80145\, k\, n^3 + 110058\, n^4$
$+ 22680\, k^3 \mathrm{K}\ 53748\, k^2\, n \mathrm{K}\ 125361\, k\, n^2 + 109224\, n^3 \mathrm{K}\ 60280\, k^2$
$+ 10936\, k\, n + 27699\, n^2 + 53260\, k \mathrm{K}\ 5452\, n \mathrm{K}\ 15660)\ tk \mathrm{K}\ (\mathrm{K}\ 4\, n \mathrm{K}\ 2$
$+ k)\ (702\, k^2\, n^2 \mathrm{K}\ 4563\, k\, n^3 + 7488\, n^4 + 1404\, k^2\, n \mathrm{K}\ 12909\, k\, n^2$
$+ 27456\, n^3 \mathrm{K}\ 11760\, k^2 \mathrm{K}\ 24756\, k\, n + 19692\, n^2 \mathrm{K}\ 380\, k + 2484\, n),$
$4608\, (k + 2)\, (k + 1)\, sk^2 + 9\, (k + 3)\, (k + 1)\, (3\, k^2 + 22\, k + 647)\, sk$
$\mathrm{K}\ 1536\, (k \mathrm{K}\ 2)\, (\mathrm{K}\ n + k)\, tk^2\, tn + (23296\, k^4 \mathrm{K}\ 69952\, k^3\, n + 112320\, k^3$
$\mathrm{K}\ 197888\, k^2\, n + 160768\, k^2 \mathrm{K}\ 84416\, k\, n + 60480\, k + 6656\, n \mathrm{K}\ 11264)$
$tk\, tn \mathrm{K}\ (\mathrm{K}\ 3\, n + k \mathrm{K}\ 1)\, (\mathrm{K}\ n + k)\, (23269\, k^2 + 23124\, k\, n + 32976\, n^2$
$+ 42126\, k + 17292\, n \mathrm{K}\ 6403)\, tk + 64\, (\mathrm{K}\ 4\, n + 2 + k)\, (364\, k^3$
$+ 363\, k^2\, n + 360\, k\, n^2 + 432\, n^3 + 663\, k^2 + 291\, k\, n \mathrm{K}\ 252\, n^2 \mathrm{K}\ 181\, k$
$\mathrm{K}\ 252\, n + 72)\, tn \mathrm{K}\ 23242\, k^4 + 69952\, k^3\, n + 59601\, k^2\, n^2 + 127044\, k\, n^3$
$+ 11664\, n^4 + 4745\, k^3 + 197595\, k^2\, n + 126210\, k\, n^2 + 16524\, n^3$
$+ 93076\, k^2 + 4382\, k\, n \mathrm{K}\ 8019\, n^2 \mathrm{K}\ 6083\, k \mathrm{K}\ 20169\, n,\ 1958400\, (k$
$+ 2)\, (k + 1)\, sk^2 \mathrm{K}\ 9\, (k + 3)\, (k + 1)\, (3093\, k^2 + 57658\, k \mathrm{K}\ 253295)\, sk$
$+ (124416\, k^3 \mathrm{K}\ 248832\, k^2 \mathrm{K}\ 124416\, k + 276480\, n \mathrm{K}\ 27648)\, tk^2\, tn + ($
$\mathrm{K}\ 24018176\, k^4 + 72120512\, k^3\, n \mathrm{K}\ 82940736\, k^3 + 104225536\, k^2\, n$
$\mathrm{K}\ 81133568\, k^2 + 31594048\, k\, n \mathrm{K}\ 20220864\, k \mathrm{K}\ 1616896\, n$
$+ 1990144)\, tk\, tn + (\mathrm{K}\ 3\, n + k \mathrm{K}\ 1)\, (23990339\, k^3 \mathrm{K}\ 149495\, k^2\, n$
$+ 10157412\, k\, n^2 \mathrm{K}\ 33998256\, n^3 + 10504770\, k^2 \mathrm{K}\ 24588270\, k\, n$
$+ 14092716\, n^2 \mathrm{K}\ 3317621\, k + 3317621\, n \mathrm{K}\ 9216)\, tk \mathrm{K}\ 64\, (\mathrm{K}\ 4\, n + 2$
$+ k)\, (375284\, k^3 + 374253\, k^2\, n + 371160\, k\, n^2 + 445392\, n^3$
$+ 172041\, k^2 \mathrm{K}\ 186699\, k\, n \mathrm{K}\ 677988\, n^2 \mathrm{K}\ 4811\, k + 262908\, n$
$\mathrm{K}\ 30312)\, tn + 23962502\, k^4 \mathrm{K}\ 72120512\, k^3\, n \mathrm{K}\ 61448631\, k^2\, n^2$
$\mathrm{K}\ 130982364\, k\, n^3 \mathrm{K}\ 12025584\, n^4 \mathrm{K}\ 38134015\, k^3 \mathrm{K}\ 103923453\, k^2\, n$
$\mathrm{K}\ 3411438\, k\, n^2 \mathrm{K}\ 5745492\, n^3 \mathrm{K}\ 26823212\, k^2 + 44595854\, k\, n$
$+ 12961701\, n^2 + 5045941\, k + 4781727\, n + 27648,\ (183600\, k$
$+ 183600)\, sk\, sn \mathrm{K}\ 288\, k\, (k \mathrm{K}\ 2)\, (\mathrm{K}\ 3\, n \mathrm{K}\ 5 + k)\, sn\, tk^2 + (412092\, k^3$
$\mathrm{K}\ 1235844\, k^2\, n \mathrm{K}\ 1646640\, k^2 \mathrm{K}\ 2160\, k\, n \mathrm{K}\ 417276\, k + 1238868\, n$
$+ 1651824)\, sn\, tk + (\mathrm{K}\ 28431\, k^2\, n^3 + 170586\, k\, n^4 \mathrm{K}\ 255879\, n^5$
$\mathrm{K}\ 142155\, k^2\, n^2 + 995085\, k\, n^3 \mathrm{K}\ 1705860\, n^4 + 412380\, k^3$
$\mathrm{K}\ 1470474\, k^2\, n + 875367\, k\, n^2 \mathrm{K}\ 5558139\, n^3 \mathrm{K}\ 2187684\, k^2$
$\mathrm{K}\ 1375218\, k\, n \mathrm{K}\ 9600930\, n^2 \mathrm{K}\ 1662336\, k \mathrm{K}\ 6705720\, n)\, sn + ($
$\mathrm{K}\ 3888\, k^3 + 19440\, k^2 \mathrm{K}\ 23328\, k \mathrm{K}\ 8640\, n)\, tk^3 + (\mathrm{K}\ 2751544\, k^3$
$+ 8297290\, k^2\, n + 9690260\, k^2 \mathrm{K}\ 12443638\, k\, n \mathrm{K}\ 9798308\, k$

$+ 4060876\, n + 2847928)\, tk^{2} + (\mathrm{K}\ 56862\, k^{3}\, n^{2} + 568620\, k^{2}\, n^{3}$
$\mathrm{K}\ 1914354\, k\, n^{4} + 2170233\, n^{5} \mathrm{K}\ 113724\, k^{3}\, n + 1528956\, k^{2}\, n^{2}$
$\mathrm{K}\ 6491745\, k\, n^{3} + 8914698\, n^{4} \mathrm{K}\ 5967272\, k^{3} + 19184204\, k^{2}\, n$
$+ 502095\, k\, n^{2} + 23620392\, n^{3} + 16961112\, k^{2} \mathrm{K}\ 16672344\, k\, n$
$+ 19839267\, n^{2} \mathrm{K}\ 15265732\, k + 5396788\, n + 4271892)\, tk\, \mathrm{K}\ (\mathrm{K}\ 4\, n$
$\mathrm{K}\ 2 + k)\, (56862\, k^{2}\, n^{2} \mathrm{K}\ 369603\, k\, n^{3} + 606528\, n^{4} + 113724\, k^{2}\, n$
$\mathrm{K}\ 1045629\, k\, n^{2} + 2223936\, n^{3} + 3219616\, k^{2} + 2098956\, k\, n$
$+ 5723820\, n^{2} \mathrm{K}\ 1326908\, k + 3297780\, n),\, (3129840\, k$
$+ 3129840)\, sk\, sn \mathrm{K}\ 26244\, (3\, n + 7)\, (n + 3)\, (3\, n + 8)\, (\mathrm{K}\ 3\, n + k \mathrm{K}\ 3)\, ($
$\mathrm{K}\ 3\, n \mathrm{K}\ 6 + k)\, sn^{2} + 648\, (k \mathrm{K}\ 2)\, (2\, k^{3} + 21\, k^{2} + 73\, k \mathrm{K}\ 84\, n$
$\mathrm{K}\ 168)\, sn\, tk^{2} + (324\, k^{5} + 972\, k^{4} \mathrm{K}\ 4645440\, k^{3} + 13844952\, k^{2}\, n$
$+ 3741588\, k^{2} + 43623360\, k\, n + 47778588\, k + 30734856\, n$
$+ 41303808)\, sn\, tk + (324\, k^{5} \mathrm{K}\ 972\, k^{3}\, n^{2} + 39235264137\, k^{2}\, n^{3}$
$\mathrm{K}\ 282479844528\, k\, n^{4} + 494321513373\, n^{5} \mathrm{K}\ 648\, k^{4} \mathrm{K}\ 3564\, k^{3}\, n$
$+ 196176090645\, k^{2}\, n^{2} \mathrm{K}\ 1530104388759\, k\, n^{3} + 2965931231274\, n^{4}$
$\mathrm{K}\ 4657752\, k^{3} + 322614219618\, k^{2}\, n \mathrm{K}\ 2911124127267\, k\, n^{2}$
$+ 6536071846863\, n^{3} + 174386834388\, k^{2} \mathrm{K}\ 2223183570678\, k\, n$
$+ 6261611044482\, n^{2} \mathrm{K}\ 523045203120\, k + 2197356250680\, n$
$+ 238085568)\, sn + (\mathrm{K}\ 8748\, k^{4} \mathrm{K}\ 52488\, k^{3} + 409212\, k^{2} \mathrm{K}\ 480168\, k$
$\mathrm{K}\ 272160\, n \mathrm{K}\ 116640)\, tk^{3} + (43044534\, k^{5} + 64555866\, k^{4}$
$\mathrm{K}\ 3690517018\, k^{3} + 3581683636\, k^{2}\, n \mathrm{K}\ 2645024722\, k^{2}$
$+ 6122574596\, k\, n + 31466347996\, k \mathrm{K}\ 25208029496\, n$
$\mathrm{K}\ 25238779904)\, tk^{2} + (86088744\, k^{5} + 78340457664\, k^{3}\, n^{2}$
$\mathrm{K}\ 879346060200\, k^{2}\, n^{3} + 3299056562268\, k\, n^{4} \mathrm{K}\ 4192389784359\, n^{5}$
$+ 172182834\, k^{4} + 156895189974\, k^{3}\, n \mathrm{K}\ 2142153315234\, k^{2}\, n^{2}$
$+ 9633677914683\, k\, n^{3} \mathrm{K}\ 14425751415276\, n^{4} + 79961482234\, k^{3}$
$\mathrm{K}\ 1649421916570\, k^{2}\, n + 9485635424331\, k\, n^{2} \mathrm{K}\ 17146537633134\, n^{3}$
$\mathrm{K}\ 357395365134\, k^{2} + 3376176232116\, k\, n \mathrm{K}\ 7917871596363\, n^{2}$
$+ 315033547898\, k \mathrm{K}\ 1104526277852\, n \mathrm{K}\ 37857936576)\, tk + (\mathrm{K}\ 4\, n$
$\mathrm{K}\ 2 + k)\, (43044210\, k^{4} + 172176840\, k^{3}\, n + 79029165996\, k^{2}\, n^{2}$
$\mathrm{K}\ 602464424157\, k\, n^{3} + 1171677292992\, n^{4} + 193706964\, k^{3}$
$+ 158014373130\, k^{2}\, n \mathrm{K}\ 1469742108309\, k\, n^{2} + 3514917538368\, n^{3}$
$+ 83996316158\, k^{2} \mathrm{K}\ 1166818896948\, k\, n + 3441519951012\, n^{2}$
$\mathrm{K}\ 274142702224\, k + 1098300205908\, n),\, 9216\, (k + 3)\, (k + 2)\, sk^{3}$
$+ 12096\, (k + 4)\, (k + 2)\, sk^{2} + (\mathrm{K}\ 27\, k^{4} \mathrm{K}\ 306\, k^{3} + 792\, k^{2} \mathrm{K}\ 5184\, n^{2}$
$+ 14706\, k \mathrm{K}\ 8640\, n + 27459)\, sk + (\mathrm{K}\ 23296\, k^{4} + 69952\, k^{3}\, n$

$\mathrm{K}\ 112320\,k^3 + 197888\,k^2\,n \mathrm{K}\ 160768\,k^2 + 82112\,k\,n \mathrm{K}\ 53568\,k$

$\mathrm{K}\ 4352\,n + 4352)\ tk\,tn + (\mathrm{K}\ n + k)\ (\mathrm{K}\ 3\,n + k \mathrm{K}\ 1)\ (23269\,k^2$

$+ 23124\,k\,n + 32976\,n^2 + 42126\,k + 17292\,n \mathrm{K}\ 7747)\ tk \mathrm{K}\ 64\ (\mathrm{K}\ 4\,n$

$+ 2 + k)\ (364\,k^3 + 363\,k^2\,n + 360\,k\,n^2 + 432\,n^3 + 663\,k^2 + 291\,k\,n$

$\mathrm{K}\ 252\,n^2 \mathrm{K}\ 205\,k \mathrm{K}\ 288\,n + 108)\ tn + 23242\,k^4 \mathrm{K}\ 69952\,k^3\,n$

$\mathrm{K}\ 59601\,k^2\,n^2 \mathrm{K}\ 127044\,k\,n^3 \mathrm{K}\ 11664\,n^4 \mathrm{K}\ 4745\,k^3 \mathrm{K}\ 197595\,k^2\,n$

$\mathrm{K}\ 126210\,k\,n^2 \mathrm{K}\ 16524\,n^3 \mathrm{K}\ 94420\,k^2 + 994\,k\,n + 8019\,n^2 + 8771\,k$

$+ 20169\,n,\ (540\,k + 1080)\ sk^2\,sn + (\mathrm{K}\ 3888\,n^3 \mathrm{K}\ 14580\,n^2 + 540\,k$

$\mathrm{K}\ 14148\,n + 540)\ sk\,sn \mathrm{K}\ 576\ (\mathrm{K}\ 4\,n \mathrm{K}\ 1 + k)\ (k^2 + 4\,k\,n + 16\,n^2 + 3\,k$

$+ 16\,n + 2)\ sk + 2430\ (k \mathrm{K}\ 1)\ (k + 1)\ (\mathrm{K}\ 3\,n + k \mathrm{K}\ 4)\ sn\,tk + (2430\,k^3$

$\mathrm{K}\ 7290\,k^2\,n \mathrm{K}\ 7290\,k\,n^2 \mathrm{K}\ 10935\,n^3 \mathrm{K}\ 12150\,k^2 \mathrm{K}\ 19440\,k\,n$

$\mathrm{K}\ 40095\,n^2 \mathrm{K}\ 14580\,k \mathrm{K}\ 36450\,n)\ sn + (\mathrm{K}\ 16348\,k^3 + 49153\,k^2\,n$

$+ 57422\,k^2 \mathrm{K}\ 73879\,k\,n \mathrm{K}\ 57830\,k + 24046\,n + 16756)\ tk^2 + ($

$\mathrm{K}\ 35702\,k^3 + 105596\,k^2\,n + 47802\,k\,n^2 + 87417\,n^3 + 97920\,k^2$

$\mathrm{K}\ 77001\,k\,n + 98019\,n^2 \mathrm{K}\ 87352\,k + 29767\,n + 25134)\ tk \mathrm{K}\ 19930\,k^3$

$+ 56443\,k^2\,n + 55092\,k\,n^2 + 131328\,n^3 + 41776\,k^2 + 1738\,k\,n$

$+ 155520\,n^2 \mathrm{K}\ 10744\,k + 39744\,n,\ (\mathrm{K}\ 25194240\,k \mathrm{K}\ 25194240)\ sk\,sn^2$

$+ (244032983280\,k + 244032983280)\ sk\,sn \mathrm{K}\ 56687040\ (k \mathrm{K}\ 1)\ (k$

$+ 1)\ (\mathrm{K}\ 3\,n \mathrm{K}\ 7 + k)\ sn^2\,tk + (1889568\,k^2\,n^3 \mathrm{K}\ 11337408\,k\,n^4$

$+ 17006112\,n^5 + 15116544\,k^2\,n^2 \mathrm{K}\ 111484512\,k\,n^3 + 198404640\,n^4$

$\mathrm{K}\ 56687040\,k^3 + 210161952\,k^2\,n \mathrm{K}\ 236825856\,k\,n^2 + 1086501600\,n^3$

$+ 488768256\,k^2 + 140877792\,k\,n + 3228012000\,n^2 + 550494144\,k$

$+ 4691167488\,n + 2368258560)\ sn^2 + 233280\ (k \mathrm{K}\ 1)\ (k \mathrm{K}\ 3)\ (2\,k$

$\mathrm{K}\ 3)\ sn\,tk^3 + (\mathrm{K}\ 1749600\,k^3 + 190006560\,k^2 \mathrm{K}\ 572819040\,k\,n$

$\mathrm{K}\ 750928320\,k \mathrm{K}\ 554273280\,n \mathrm{K}\ 944784000)\ sn\,tk^2 + (\mathrm{K}\ 2592\,k^5$

$\mathrm{K}\ 62208\,k^4 + 549736145964\,k^3 \mathrm{K}\ 1649127971844\,k^2\,n$

$\mathrm{K}\ 2198004864624\,k^2 \mathrm{K}\ 2712754800\,k\,n \mathrm{K}\ 552972903372\,k$

$+ 1646301143124\,n + 2194540018992)\ sn\,tk + (\mathrm{K}\ 2592\,k^5$

$+ 7776\,k^3\,n^2 \mathrm{K}\ 288756948519\,k^2\,n^3 + 2109087768762\,k\,n^4$

$\mathrm{K}\ 3728445625791\,n^5 \mathrm{K}\ 59616\,k^4 + 28512\,k^3\,n$

$\mathrm{K}\ 1443782676771\,k^2\,n^2 + 11486957823477\,k\,n^3$

$\mathrm{K}\ 22596391119924\,n^4 + 549738447660\,k^3 \mathrm{K}\ 4023345455802\,k^2\,n$

$+ 20399757957567\,k\,n^2 \mathrm{K}\ 52186279903011\,n^3 \mathrm{K}\ 4031297970804\,k^2$

$+ 12714931696878\,k\,n \mathrm{K}\ 55682028331794\,n^2 + 1127062246752\,k$

$\mathrm{K}\ 23982605350200\,n \mathrm{K}\ 7054387200)\ sn + (\mathrm{K}\ 3149280\,k^3$

$+ 25194240\,k^2 \mathrm{K}\ 59836320\,k \mathrm{K}\ 6998400\,n + 37791360)\ tk^4 + ($

$K\ 5158520640\, k^3 + 25825670640\, k^2 K\ 31107013200\, k$
$K\ 11431886400\, n + 170061120)\ tk^3 + (K\ 344356272\, k^5$
$K\ 860558256\, k^4 K\ 3632609099176\, k^3 + 11005595890666\, k^2\, n$
$+ 12903606348164\, k^2 K\ 16586264746198\, k\, n K\ 13243567228868\, k$
$+ 5570227231372\, n + 3958370191288)\ tk^2 + (K\ 688709952\, k^5$
$K\ 576473332158\, k^3\, n^2 + 6532265190060\, k^2\, n^3$
$K\ 24700691416626\, k\, n^4 + 31621230712161\, n^5 K\ 2065400208\, k^4$
$K\ 1154660861484\, k^3\, n + 16038072704268\, k^2\, n^2$
$K\ 73085161419369\, k\, n^3 + 110719731915882\, n^4$
$K\ 8442651013304\, k^3 + 36220189560476\, k^2\, n$
$K\ 62471456531529\, k\, n^2 + 148900114274040\, n^3$
$+ 24842762195544\, k^2 K\ 44431316181384\, k\, n + 80623069386219\, n^2$
$K\ 22334666715172\, k + 14740569964564\, n + 5937309643092)\ tk$
$K\ (K\ 4\, n K\ 2 + k)\ (344353680\, k^4 + 1377414720\, k^3\, n$
$+ 581982998814\, k^2\, n^2 K\ 4493088253755\, k\, n^3 + 8837414832960\, n^4$
$+ 1893551904\, k^3 + 1164989911500\, k^2\, n K\ 11080408282101\, k\, n^2$
$+ 27046067934528\, n^3 + 4818640108192\, k^2 K\ 4830603471060\, k\, n$
$+ 31192382524428\, n^2 K\ 3520916168252\, k + 11909158062804\, n)\ ]$

```
> for q in Bro_B_rec do applyopr(collect(q, {sn,tn,sk,tk},
  distributed, factor), b(n,k), sAlg) = 0 od;
```

$(K\ 2\, k^2 + 8\, k\, n K\ 3\, n^2 + k + 3\, n)\ b(n, k) K\ (k + 1)\ (K\ 4\, n K\ 1 + k)\ b(n, k$
$+ 1) K\ (K\ 3\, n + k K\ 1)\ (K\ n + k)\ b(n, k K\ 1) = 0$

$(K\ 18\, k^6 + 18405\, k^4\, n^2 K\ 175212\, k^3\, n^3 + 564816\, k^2\, n^4 K\ 594624\, k\, n^5$
$K\ 62208\, n^6 K\ 423\, k^5 K\ 9\, k^4\, n + 56660\, k^3\, n^2 K\ 334620\, k^2\, n^3$
$+ 511456\, k\, n^4 + 15552\, n^5 K\ 1292\, k^4 + 11012\, k^3\, n K\ 88298\, k^2\, n^2$
$+ 122172\, k\, n^3 + 92880\, n^4 K\ 6766\, k^3 + 42378\, k^2\, n K\ 156668\, k\, n^2$
$K\ 23220\, n^3 K\ 7754\, k^2 + 18436\, k\, n K\ 31827\, n^2 K\ 1155\, k + 8823\, n)$
$b(n, k) K\ (k + 11)\ (3\, k + 5)\ (k + 7)\ (3\, k + 1)\ (k + 3)\ (k + 1)\ b(n, k + 1)$
$K\ 1024\ (n K\ 1)\ (3\, n K\ 1)\ (3\, n K\ 2)\ (K\ 4\, n + 2 + k)\ (K\ 4\, n + 1 + k)\ (K\ 4\, n$
$+ k)\ b(n K\ 1, k) K\ (K\ n + k)\ (K\ 3\, n + k K\ 1)\ (9\, k^4 + 36\, k^3\, n$
$K\ 18288\, k^2\, n^2 + 111168\, k\, n^3 K\ 175872\, n^4 + 216\, k^3 + 900\, k^2\, n$
$K\ 72032\, k\, n^2 + 200896\, n^3 K\ 270\, k^2 K\ 180\, k\, n K\ 42032\, n^2 + 176\, k$
$K\ 8692\, n + 893)\ b(n, k K\ 1) = 0$

$k\ (K\ 4\, n K\ 2 + k)\ (18\, k^2\, n^2 K\ 117\, k\, n^3 + 192\, n^4 + 36\, k^2\, n K\ 331\, k\, n^2 + 704\, n^3$
$+ 20\, k^2 K\ 314\, k\, n + 948\, n^2 K\ 100\, k + 556\, n + 120)\ b(n, k) + k\ (n$
$+ 2)\ (3\, n + 4)\ (3\, n + 5)\ (K\ 3\, n + k K\ 2)\ (K\ 3\, n + k K\ 3)\ b(n + 1, k)$

$+ k\,(18\,k^3\,n^2 K\; 180\,k^2\,n^3 + 606\,k\,n^4 K\; 687\,n^5 + 36\,k^3\,n K\; 484\,k^2\,n^2$
$+ 2055\,k\,n^3 K\; 2822\,n^4 + 20\,k^3 K\; 428\,k^2\,n + 2499\,k\,n^2 K\; 4386\,n^3$
$K\; 120\,k^2 + 1266\,k\,n K\; 3171\,n^2 + 220\,k K\; 1042\,n K\; 120)\; b(n, k K\; 1) = 0$

$(3\,n + 2\,k)\;(K\,4\,n K\; 2 + k)\;(18\,k^2\,n^2 K\; 117\,k\,n^3 + 192\,n^4 + 36\,k^2\,n$
$K\; 331\,k\,n^2 + 704\,n^3 + 20\,k^2 K\; 314\,k\,n + 948\,n^2 K\; 100\,k + 556\,n + 120)$
$b(n, k) + (n + 2)\;(3\,n + 4)\;(3\,n + 5)\;(3\,n + 2\,k)\;(K\,3\,n + k K\; 2)\;(K\,3\,n$
$+ k K\; 3)\; b(n + 1, k) + (3\,n + 2\,k)\;(18\,k^3\,n^2 K\; 180\,k^2\,n^3 + 606\,k\,n^4$
$K\; 687\,n^5 + 36\,k^3\,n K\; 484\,k^2\,n^2 + 2055\,k\,n^3 K\; 2822\,n^4 + 20\,k^3 K\; 428\,k^2\,n$
$+ 2499\,k\,n^2 K\; 4386\,n^3 K\; 120\,k^2 + 1266\,k\,n K\; 3171\,n^2 + 220\,k K\; 1042\,n$
$K\; 120)\; b(n, k K\; 1) = 0$

$k\,(K\,4\,n K\; 2 + k)\; b(n, k) + (K\,n + k K\; 1)\;(K\,3\,n + k K\; 2)\; b(n, k K\; 2) + (2\,k^2$
$K\; 8\,k\,n + 3\,n^2 K\; 5\,k + 5\,n + 3)\; b(n, k K\; 1) = 0$

$K\;(K\,4\,n K\; 2 + k)\;(K\,486\,k^2\,n^2 + 3159\,k\,n^3 K\; 5184\,n^4 + 2\,k^3 K\; 966\,k^2\,n$
$+ 7983\,k\,n^2 K\; 15552\,n^3 K\; 518\,k^2 + 6612\,k\,n K\; 15228\,n^2 + 1676\,k$
$K\; 4860\,n)\; b(n, k) K\; 2\,(k + 4)\;(k K\; 2)\;(k + 7)\;(K\,n + k K\; 1)\; b(n, k K\; 2)$
$+ 27\,(n + 2)\;(3\,n + 4)\;(3\,n + 5)\;(K\,3\,n + k)\;(K\,3\,n + k K\; 3)\; b(n + 1, k)$
$+ (486\,k^3\,n^2 K\; 4860\,k^2\,n^3 + 16362\,k\,n^4 K\; 18549\,n^5 K\; 4\,k^4 + 976\,k^3\,n$
$K\; 12090\,k^2\,n^2 + 48699\,k\,n^3 K\; 63828\,n^4 + 506\,k^3 K\; 9572\,k^2\,n$
$+ 49191\,k\,n^2 K\; 75870\,n^3 K\; 2168\,k^2 + 18260\,k\,n K\; 35037\,n^2 + 1834\,k$
$K\; 4888\,n K\; 168)\; b(n, k K\; 1) = 0$

$$0 = 0$$

$K\,k\,(K\,4\,n K\; 2 + k)\; b(n, k) + k\,(K\,4\,n + 2 + k)\; b(n K\; 1, k) K\;(K\,3\,n + k$
$K\; 1)\;(K\,n + k)\; b(n, k K\; 1) + (K\,n + 1 + k)\;(K\,3\,n + 2 + k)\; b(n K\; 1, k$
$K\; 1) = 0$

$(K\,718038\,k^5 + 135790911\,k^3\,n^2 K\; 885090564\,k^2\,n^3 + 1513277424\,k\,n^4$
$+ 152985024\,n^5 K\; 9177227\,k^4 + 5379605\,k^3\,n + 156798627\,k^2\,n^2$
$K\; 227733624\,k\,n^3 + 63743760\,n^4 K\; 9301559\,k^3 + 158537241\,k^2\,n$
$K\; 514212339\,k\,n^2 K\; 185919300\,n^3 K\; 17208457\,k^2 + 29687599\,k\,n$
$K\; 66842415\,n^2 K\; 14290975\,k + 36032931\,n)\; b(n, k) K\; 9\,(k + 7)\;(3\,k$
$+ 1)\;(9841\,k + 16395)\;(k + 3)\;(k + 1)\; b(n, k + 1) + 256\,(K\,4\,n + 2$
$+ k)\;(729\,k^4 + 2916\,k^3\,n + 9477\,k^2\,n^2 K\; 236520\,k\,n^3 + 1416528\,n^4$
$+ 11360\,k^3 + 29220\,k^2\,n + 600309\,k\,n^2 K\; 2242836\,n^3 + 18354\,k^2$
$K\; 327936\,k\,n + 944352\,n^2 + 7651\,k K\; 118044\,n)\; b(n K\; 1, k) K\;(K\,3\,n$
$+ k K\; 1)\;(K\,n + k)\;(452331\,k^3 + 1809324\,k^2\,n K\; 129350736\,k\,n^2$
$+ 432513216\,n^3 + 6362171\,k^2 + 22359192\,k\,n K\; 205711344\,n^2$
$+ 4078137\,k K\; 33773700\,n K\; 1139999)\; b(n, k K\; 1) + 256\,(k$

$+ 1) (729 k^4 + 12818 k^3 \mathrm{K}\ 11117 k^2 n + 41074 k^2 \mathrm{K}\ 28823 k n$

$\mathrm{K}\ 14681 k + 43720 n \mathrm{K}\ 43720) b(n \mathrm{K}\ 1, k \mathrm{K}\ 1) = 0$

$(\mathrm{K}\ 44280 k^5 \mathrm{K}\ 46656 k^4 n + 11265804 k^3 n^2 \mathrm{K}\ 73774800 k^2 n^3$

$+ 125983296 k n^4 + 12737088 n^5 \mathrm{K}\ 767234 k^4 + 316100 k^3 n$

$+ 12970419 k^2 n^2 \mathrm{K}\ 18971460 k n^3 + 5307120 n^4 \mathrm{K}\ 821645 k^3$

$+ 13140081 k^2 n \mathrm{K}\ 42813390 k n^2 \mathrm{K}\ 15479100 n^3 \mathrm{K}\ 1453000 k^2$

$+ 2473630 k n \mathrm{K}\ 5565105 n^2 \mathrm{K}\ 1189825 k + 2999997 n) b(n, k)$

$\mathrm{K}\ 45 (k + 7) (3 k + 1) (k + 3) (164 k + 273) (k + 1) b(n, k + 1) + 64 ($

$\mathrm{K}\ 4 n + 2 + k) (729 k^3 n + 2916 k^2 n^2 \mathrm{K}\ 79056 k n^3 + 471744 n^4$

$+ 3341 k^3 + 9537 k^2 n + 200088 k n^2 \mathrm{K}\ 746928 n^3 + 5961 k^2$

$\mathrm{K}\ 109236 k n + 314496 n^2 + 2548 k \mathrm{K}\ 39312 n) b(n \mathrm{K}\ 1, k) \mathrm{K}\ (\mathrm{K}\ 3 n$

$+ k \mathrm{K}\ 1) (\mathrm{K}\ n + k) (22140 k^3 + 135216 k^2 n \mathrm{K}\ 10791360 k n^2$

$+ 36009792 n^3 + 501599 k^2 + 1850028 k n \mathrm{K}\ 17126928 n^2 + 329718 k$

$\mathrm{K}\ 2811900 n \mathrm{K}\ 94913) b(n, k \mathrm{K}\ 1) + 64 (k + 1) (729 k^3 n + 3341 k^3$

$\mathrm{K}\ 2369 k^2 n + 12643 k^2 \mathrm{K}\ 9140 k n \mathrm{K}\ 5204 k + 14560 n \mathrm{K}\ 14560) b(n$

$\mathrm{K}\ 1, k \mathrm{K}\ 1) = 0$

$\mathrm{K}\ k (\mathrm{K}\ 4 n \mathrm{K}\ 2 + k) b(n, k) + (\mathrm{K}\ 3 n + k \mathrm{K}\ 4) (\mathrm{K}\ n + k \mathrm{K}\ 1) b(n + 1, k \mathrm{K}\ 1)$

$+ k (k \mathrm{K}\ 4 n \mathrm{K}\ 6) b(n + 1, k) \mathrm{K}\ (\mathrm{K}\ 3 n + k \mathrm{K}\ 1) (\mathrm{K}\ n + k) b(n, k \mathrm{K}\ 1) = 0$

$\mathrm{K}\ (\mathrm{K}\ 4 n \mathrm{K}\ 2 + k) (\mathrm{K}\ 265680 k^2 n^2 + 1726920 k n^3 \mathrm{K}\ 2833920 n^4 + 675 k^3$

$\mathrm{K}\ 528498 k^2 n + 4363623 k n^2 \mathrm{K}\ 8502336 n^3 \mathrm{K}\ 283183 k^2$

$+ 3614136 k n \mathrm{K}\ 8325792 n^2 + 916268 k \mathrm{K}\ 2657340 n) b(n, k) \mathrm{K}\ 18 (k$

$\mathrm{K}\ 1) (3 k + 1) (k + 1) (\mathrm{K}\ 3 n + k \mathrm{K}\ 4) b(n + 1, k \mathrm{K}\ 1) + (\mathrm{K}\ 729 k^4$

$\mathrm{K}\ 9902 k^3 + 11117 k^2 n + 4123 k^2 + 6589 k n + 67880 k \mathrm{K}\ 61426 n$

$\mathrm{K}\ 61372) b(n, k \mathrm{K}\ 2) + (132840 k^2 n^3 \mathrm{K}\ 797040 k n^4 + 1195560 n^5$

$\mathrm{K}\ 54 k^4 + 162 k^3 n + 664362 k^2 n^2 \mathrm{K}\ 4383558 k n^3 + 7173603 n^4$

$+ 252 k^3 + 1092726 k^2 n \mathrm{K}\ 8545392 k n^2 + 15809175 n^3 + 590814 k^2$

$\mathrm{K}\ 6818436 k n + 15145758 n^2 \mathrm{K}\ 1771092 k + 5314680 n) b(n + 1, k)$

$+ (265680 k^3 n^2 \mathrm{K}\ 2656800 k^2 n^3 + 8944560 k n^4 \mathrm{K}\ 10140120 n^5$

$\mathrm{K}\ 1404 k^4 + 531198 k^3 n \mathrm{K}\ 6610293 k^2 n^2 + 26621190 k n^3$

$\mathrm{K}\ 34894701 n^4 + 274685 k^3 \mathrm{K}\ 5231759 k^2 n + 26890026 k n^2$

$\mathrm{K}\ 41480631 n^3 \mathrm{K}\ 1183968 k^2 + 9982635 k n \mathrm{K}\ 19156959 n^2$

$+ 1002745 k \mathrm{K}\ 2672746 n \mathrm{K}\ 92058) b(n, k \mathrm{K}\ 1) = 0$

$(23242 k^4 \mathrm{K}\ 69952 k^3 n \mathrm{K}\ 59601 k^2 n^2 \mathrm{K}\ 127044 k n^3 \mathrm{K}\ 11664 n^4$

$\mathrm{K}\ 36401 k^3 \mathrm{K}\ 104667 k^2 n + 3426 k n^2 \mathrm{K}\ 4860 n^3 \mathrm{K}\ 26008 k^2$

$+ 44206 k n + 12879 n^2 + 4799 k + 3645 n) b(n, k) \mathrm{K}\ 9 (3 k \mathrm{K}\ 43) (k$

$+ 3) (k + 7) (k + 1) b(n, k + 1) \mathrm{K}\ 64 (\mathrm{K}\ 4 n + 2 + k) (364 k^3 + 363 k^2 n$

$$+ 360\,k\,n^2 + 432\,n^3 + 183\,k^2 - 177\,k\,n - 684\,n^2 - k + 288\,n - 36)\,b(n - 1, k) + (-n + k)\,(-3\,n + k - 1)\,(23269\,k^2 + 23124\,k\,n + 32976\,n^2 + 10650\,k - 15684\,n - 3007)\,b(n, k - 1) + 576\,(k + 4)\,(k + 2)\,(k + 1)\,b(n, k + 2) - 64\,(k + 1)\,(364\,k^3 - 1093\,k^2\,n + 911\,k^2 - 547\,k\,n + 365\,k + 20\,n - 20)\,b(n - 1, k - 1) = 0$$

$$(23242\,k^4 - 69952\,k^3\,n - 59601\,k^2\,n^2 - 127044\,k\,n^3 - 11664\,n^4 + 80951\,k^3 - 477403\,k^2\,n - 361698\,k\,n^2 - 47628\,n^3 - 305140\,k^2 + 50498\,k\,n - 4941\,n^2 + 76195\,k + 64233\,n)\,b(n, k) - 9216\,(-4\,n + 3 + k)\,(k^2 + 4\,k\,n + 16\,n^2 - k - 16\,n + 2)\,b(n - 1, k + 1) + (-27\,k^4 + 558\,k^3 - 62208\,n^3 - 72\,k^2 - 46656\,n^2 + 18\,k + 53568\,n + 55971)\,b(n, k + 1) + (-23296\,k^4 + 69952\,k^3\,n + 69888\,k^2\,n^2 + 64512\,k\,n^3 + 110592\,n^4 - 191424\,k^3 + 384448\,k^2\,n + 296448\,k\,n^2 + 617472\,n^3 - 236352\,k^2 - 298112\,k\,n - 1460736\,n^2 + 62848\,k + 820224\,n - 87552)\,b(n - 1, k) + (-n + k)\,(-3\,n + k - 1)\,(23269\,k^2 + 23124\,k\,n + 32976\,n^2 + 126958\,k + 76812\,n - 48803)\,b(n, k - 1) + 2880\,(k + 4)\,(k + 2)\,b(n, k + 2) + (-23296\,k^4 + 69952\,k^3\,n - 205504\,k^3 + 477696\,k^2\,n - 390912\,k^2 + 219072\,k\,n - 147008\,k - 6400\,n + 6400)\,b(n - 1, k - 1) = 0$$

$$(-4\,n - 2 + k)\,(162\,k^2\,n^2 - 1053\,k\,n^3 + 1728\,n^4 + 324\,k^2\,n - 2979\,k\,n^2 + 6336\,n^3 - 4000\,k^2 - 6996\,k\,n + 2772\,n^2 + 470\,k - 756\,n)\,b(n, k) + 540\,(k - 1)\,(k + 1)\,(-3\,n + k - 4)\,b(n + 1, k - 1) + (-3640\,k^3 + 10930\,k^2\,n + 12740\,k^2 - 16390\,k\,n - 12740\,k + 5260\,n + 3640)\,b(n, k - 2) + 90\,(k + 3)\,(k + 1)\,b(n + 1, k + 1) + (81\,k^2\,n^3 - 486\,k\,n^4 + 729\,n^5 + 405\,k^2\,n^2 - 2835\,k\,n^3 + 4860\,n^4 + 540\,k^3 - 954\,k^2\,n - 7641\,k\,n^2 + 10125\,n^3 - 2250\,k^2 - 9810\,k\,n + 5940\,n^2 - 4590\,k - 1674\,n)\,b(n + 1, k) + (162\,k^3\,n^2 - 1620\,k^2\,n^3 + 5454\,k\,n^4 - 6183\,n^5 + 324\,k^3\,n - 4356\,k^2\,n^2 + 18495\,k\,n^3 - 25398\,n^4 - 7640\,k^3 + 19628\,k^2\,n + 31791\,k\,n^2 - 18864\,n^3 + 20850\,k^2 - 8976\,k\,n + 1161\,n^2 - 18670\,k + 3502\,n + 5460)\,b(n, k - 1) = 0$$

$$(-54\,k^3\,n^2 + 567\,k^2\,n^3 - 1980\,k\,n^4 + 2304\,n^5 + 6\,k^4 - 108\,k^3\,n + 1434\,k^2\,n^2 - 6102\,k\,n^3 + 7488\,n^4 - 25\,k^3 + 1183\,k^2\,n - 6660\,k\,n^2 + 8784\,n^3 + 334\,k^2 - 2906\,k\,n + 4608\,n^2 - 400\,k + 1008\,n)\,b(n, k) + 2\,(k + 4)\,(k - 2)\,(-n + k - 1)\,b(n, k - 2) + 9\,n\,(n + 2)\,(3\,n + 4)\,(3\,n + 5)\,b(n + 1, k + 1) + 3\,(-4\,n - 1 + k)\,(4\,n + 4 + k)\,(k^2 + 16\,n^2 + 3\,k + 20\,n + 2)\,b(n, k + 1) - 3\,(n + 2)\,(3\,n + 4)\,(3\,n + 5)\,(-3\,n + k)\,(-3\,n + k - 3)\,b(n + 1, k) + (-54\,k^3\,n^2 + 540\,k^2\,n^3 - 1818\,k\,n^4 + 2061\,n^5$$

$$+ 3 k^4 \mathrm{K}\ 108 k^3 n + 1353 k^2 n^2 \mathrm{K}\ 5535 k n^3 + 7203 n^4 \mathrm{K}\ 41 k^3$$
$$+ 1067 k^2 n \mathrm{K}\ 5721 k n^2 + 8667 n^3 + 252 k^2 \mathrm{K}\ 2196 k n + 4053 n^2$$
$$\mathrm{K}\ 238 k + 568 n + 24)\ b(n, k \mathrm{K}\ 1) = 0$$
$$0 = 0$$
$$\mathrm{K}\ (\mathrm{K}\ 4 n \mathrm{K}\ 2 + k)\ (702 k^2 n^2 \mathrm{K}\ 4563 k n^3 + 7488 n^4 + 1404 k^2 n$$
$$\mathrm{K}\ 12909 k n^2 + 27456 n^3 \mathrm{K}\ 11760 k^2 \mathrm{K}\ 24756 k n + 19692 n^2 \mathrm{K}\ 380 k$$
$$+ 2484 n)\ b(n, k) \mathrm{K}\ 1620\ (k \mathrm{K}\ 1)\ (k + 1)\ (\mathrm{K}\ 3 n + k \mathrm{K}\ 4)\ b(n + 1, k \mathrm{K}\ 1)$$
$$+ (10920 k^3 \mathrm{K}\ 32790 k^2 n \mathrm{K}\ 38060 k^2 + 48890 k n + 37980 k$$
$$\mathrm{K}\ 15220 n \mathrm{K}\ 11080)\ b(n, k \mathrm{K}\ 2) \mathrm{K}\ 80\ (k \mathrm{K}\ 3)\ (\mathrm{K}\ n \mathrm{K}\ 2 + k)\ b(n, k \mathrm{K}\ 3)$$
$$+ (\mathrm{K}\ 720 k \mathrm{K}\ 720)\ b(n + 1, k + 1) + (\mathrm{K}\ 351 k^2 n^3 + 2106 k n^4 \mathrm{K}\ 3159 n^5$$
$$\mathrm{K}\ 1755 k^2 n^2 + 12285 k n^3 \mathrm{K}\ 21060 n^4 \mathrm{K}\ 1620 k^3 + 1974 k^2 n$$
$$+ 30951 k n^2 \mathrm{K}\ 47115 n^3 + 6540 k^2 + 36750 k n \mathrm{K}\ 37890 n^2 + 16800 k$$
$$\mathrm{K}\ 4536 n)\ b(n + 1, k) + (\mathrm{K}\ 702 k^3 n^2 + 7020 k^2 n^3 \mathrm{K}\ 23634 k n^4$$
$$+ 26793 n^5 \mathrm{K}\ 1404 k^3 n + 18876 k^2 n^2 \mathrm{K}\ 80145 k n^3 + 110058 n^4$$
$$+ 22680 k^3 \mathrm{K}\ 53748 k^2 n \mathrm{K}\ 125361 k n^2 + 109224 n^3 \mathrm{K}\ 60280 k^2$$
$$+ 10936 k n + 27699 n^2 + 53260 k \mathrm{K}\ 5452 n \mathrm{K}\ 15660)\ b(n, k \mathrm{K}\ 1) = 0$$
$$(\mathrm{K}\ 23242 k^4 + 69952 k^3 n + 59601 k^2 n^2 + 127044 k n^3 + 11664 n^4$$
$$+ 4745 k^3 + 197595 k^2 n + 126210 k n^2 + 16524 n^3 + 93076 k^2$$
$$+ 4382 k n \mathrm{K}\ 8019 n^2 \mathrm{K}\ 6083 k \mathrm{K}\ 20169 n)\ b(n, k) \mathrm{K}\ 1536\ (k \mathrm{K}\ 2)\ (\mathrm{K}\ n$$
$$+ k)\ b(n \mathrm{K}\ 1, k \mathrm{K}\ 2) + 9\ (k + 3)\ (k + 1)\ (3 k^2 + 22 k + 647)\ b(n, k + 1)$$
$$+ 64\ (\mathrm{K}\ 4 n + 2 + k)\ (364 k^3 + 363 k^2 n + 360 k n^2 + 432 n^3 + 663 k^2$$
$$+ 291 k n \mathrm{K}\ 252 n^2 \mathrm{K}\ 181 k \mathrm{K}\ 252 n + 72)\ b(n \mathrm{K}\ 1, k) \mathrm{K}\ (\mathrm{K}\ 3 n + k$$
$$\mathrm{K}\ 1)\ (\mathrm{K}\ n + k)\ (23269 k^2 + 23124 k n + 32976 n^2 + 42126 k + 17292 n$$
$$\mathrm{K}\ 6403)\ b(n, k \mathrm{K}\ 1) + 4608\ (k + 2)\ (k + 1)\ b(n, k + 2) + (23296 k^4$$
$$\mathrm{K}\ 69952 k^3 n + 112320 k^3 \mathrm{K}\ 197888 k^2 n + 160768 k^2 \mathrm{K}\ 84416 k n$$
$$+ 60480 k + 6656 n \mathrm{K}\ 11264)\ b(n \mathrm{K}\ 1, k \mathrm{K}\ 1) = 0$$
$$(23962502 k^4 \mathrm{K}\ 72120512 k^3 n \mathrm{K}\ 61448631 k^2 n^2 \mathrm{K}\ 130982364 k n^3$$
$$\mathrm{K}\ 12025584 n^4 \mathrm{K}\ 38134015 k^3 \mathrm{K}\ 103923453 k^2 n \mathrm{K}\ 3411438 k n^2$$
$$\mathrm{K}\ 5745492 n^3 \mathrm{K}\ 26823212 k^2 + 44595854 k n + 12961701 n^2$$
$$+ 5045941 k + 4781727 n + 27648)\ b(n, k) + (124416 k^3 \mathrm{K}\ 248832 k^2$$
$$\mathrm{K}\ 124416 k + 276480 n \mathrm{K}\ 27648)\ b(n \mathrm{K}\ 1, k \mathrm{K}\ 2) \mathrm{K}\ 9\ (k + 3)\ (k$$
$$+ 1)\ (3093 k^2 + 57658 k \mathrm{K}\ 253295)\ b(n, k + 1) \mathrm{K}\ 64\ (\mathrm{K}\ 4 n + 2$$
$$+ k)\ (375284 k^3 + 374253 k^2 n + 371160 k n^2 + 445392 n^3$$
$$+ 172041 k^2 \mathrm{K}\ 186699 k n \mathrm{K}\ 677988 n^2 \mathrm{K}\ 4811 k + 262908 n$$
$$\mathrm{K}\ 30312)\ b(n \mathrm{K}\ 1, k) + (\mathrm{K}\ 3 n + k \mathrm{K}\ 1)\ (23990339 k^3 \mathrm{K}\ 149495 k^2 n$$
$$+ 10157412 k n^2 \mathrm{K}\ 33998256 n^3 + 10504770 k^2 \mathrm{K}\ 24588270 k n$$

$+\ 14092716\, n^{2} \mathrm{K}\ 3317621\, k + 3317621\, n \mathrm{K}\ 9216)\ b(n, k \mathrm{K}\ 1)$
$+\ 1958400\ (k+2)\ (k+1)\ b(n, k+2) + (\mathrm{K}\ 24018176\, k^{4}$
$+\ 72120512\, k^{3}\, n \mathrm{K}\ 82940736\, k^{3} + 104225536\, k^{2}\, n \mathrm{K}\ 81133568\, k^{2}$
$+\ 31594048\, k\, n \mathrm{K}\ 20220864\, k \mathrm{K}\ 1616896\, n + 1990144)\ b(n \mathrm{K}\ 1, k$
$\mathrm{K}\ 1) = 0$

$\mathrm{K}\ (\mathrm{K}\ 4\, n \mathrm{K}\ 2 + k)\ (56862\, k^{2}\, n^{2} \mathrm{K}\ 369603\, k\, n^{3} + 606528\, n^{4} + 113724\, k^{2}\, n$
$\mathrm{K}\ 1045629\, k\, n^{2} + 2223936\, n^{3} + 3219616\, k^{2} + 2098956\, k\, n$
$+\ 5723820\, n^{2} \mathrm{K}\ 1326908\, k + 3297780\, n)\ b(n, k) + (412092\, k^{3}$
$\mathrm{K}\ 1235844\, k^{2}\, n \mathrm{K}\ 1646640\, k^{2} \mathrm{K}\ 2160\, k\, n \mathrm{K}\ 417276\, k + 1238868\, n$
$+\ 1651824)\ b(n+1, k \mathrm{K}\ 1) + (\mathrm{K}\ 2751544\, k^{3} + 8297290\, k^{2}\, n$
$+\ 9690260\, k^{2} \mathrm{K}\ 12443638\, k\, n \mathrm{K}\ 9798308\, k + 4060876\, n + 2847928)$
$b(n, k \mathrm{K}\ 2) + (\mathrm{K}\ 3888\, k^{3} + 19440\, k^{2} \mathrm{K}\ 23328\, k \mathrm{K}\ 8640\, n)\ b(n, k \mathrm{K}\ 3)$
$+\ (183600\, k + 183600)\ b(n+1, k+1) \mathrm{K}\ 288\, k\ (k \mathrm{K}\ 2)\ (\mathrm{K}\ 3\, n \mathrm{K}\ 5$
$+\ k)\ b(n+1, k \mathrm{K}\ 2) + (\mathrm{K}\ 28431\, k^{2}\, n^{3} + 170586\, k\, n^{4} \mathrm{K}\ 255879\, n^{5}$
$\mathrm{K}\ 142155\, k^{2}\, n^{2} + 995085\, k\, n^{3} \mathrm{K}\ 1705860\, n^{4} + 412380\, k^{3}$
$\mathrm{K}\ 1470474\, k^{2}\, n + 875367\, k\, n^{2} \mathrm{K}\ 5558139\, n^{3} \mathrm{K}\ 2187684\, k^{2}$
$\mathrm{K}\ 1375218\, k\, n \mathrm{K}\ 9600930\, n^{2} \mathrm{K}\ 1662336\, k \mathrm{K}\ 6705720\, n)\ b(n+1, k)$
$+\ (\mathrm{K}\ 56862\, k^{3}\, n^{2} + 568620\, k^{2}\, n^{3} \mathrm{K}\ 1914354\, k\, n^{4} + 2170233\, n^{5}$
$\mathrm{K}\ 113724\, k^{3}\, n + 1528956\, k^{2}\, n^{2} \mathrm{K}\ 6491745\, k\, n^{3} + 8914698\, n^{4}$
$\mathrm{K}\ 5967272\, k^{3} + 19184204\, k^{2}\, n + 502095\, k\, n^{2} + 23620392\, n^{3}$
$+\ 16961112\, k^{2} \mathrm{K}\ 16672344\, k\, n + 19839267\, n^{2} \mathrm{K}\ 15265732\, k$
$+\ 5396788\, n + 4271892)\ b(n, k \mathrm{K}\ 1) = 0$

$(\mathrm{K}\ 4\, n \mathrm{K}\ 2 + k)\ (43044210\, k^{4} + 172176840\, k^{3}\, n + 79029165996\, k^{2}\, n^{2}$
$\mathrm{K}\ 602464424157\, k\, n^{3} + 1171677292992\, n^{4} + 193706964\, k^{3}$
$+\ 158014373130\, k^{2}\, n \mathrm{K}\ 1469742108309\, k\, n^{2} + 3514917538368\, n^{3}$
$+\ 83996316158\, k^{2} \mathrm{K}\ 1166818896948\, k\, n + 3441519951012\, n^{2}$
$\mathrm{K}\ 274142702224\, k + 1098300205908\, n)\ b(n, k) + (324\, k^{5} + 972\, k^{4}$
$\mathrm{K}\ 4645440\, k^{3} + 13844952\, k^{2}\, n + 3741588\, k^{2} + 43623360\, k\, n$
$+\ 47778588\, k + 30734856\, n + 41303808)\ b(n+1, k \mathrm{K}\ 1)$
$+\ (43044534\, k^{5} + 64555866\, k^{4} \mathrm{K}\ 3690517018\, k^{3} + 3581683636\, k^{2}\, n$
$\mathrm{K}\ 2645024722\, k^{2} + 6122574596\, k\, n + 31466347996\, k$
$\mathrm{K}\ 25208029496\, n \mathrm{K}\ 25238779904)\ b(n, k \mathrm{K}\ 2) + (\mathrm{K}\ 8748\, k^{4}$
$\mathrm{K}\ 52488\, k^{3} + 409212\, k^{2} \mathrm{K}\ 480168\, k \mathrm{K}\ 272160\, n \mathrm{K}\ 116640)\ b(n, k \mathrm{K}\ 3)$
$+\ (3129840\, k + 3129840)\ b(n+1, k+1) + 648\ (k \mathrm{K}\ 2)\ (2\, k^{3} + 21\, k^{2}$
$+\ 73\, k \mathrm{K}\ 84\, n \mathrm{K}\ 168)\ b(n+1, k \mathrm{K}\ 2) \mathrm{K}\ 26244\ (3\, n+7)\ (n+3)\ (3\, n$
$+\ 8)\ (\mathrm{K}\ 3\, n + k \mathrm{K}\ 3)\ (\mathrm{K}\ 3\, n \mathrm{K}\ 6 + k)\ b(n+2, k) + (324\, k^{5} \mathrm{K}\ 972\, k^{3}\, n^{2}$

$$+ 39235264137\, k^2\, n^3 \mathrm{K}\ 282479844528\, k\, n^4 + 494321513373\, n^5$$
$$\mathrm{K}\ 648\, k^4 \mathrm{K}\ 3564\, k^3\, n + 196176090645\, k^2\, n^2 \mathrm{K}\ 1530104388759\, k\, n^3$$
$$+ 2965931231274\, n^4 \mathrm{K}\ 4657752\, k^3 + 322614219618\, k^2\, n$$
$$\mathrm{K}\ 2911124127267\, k\, n^2 + 6536071846863\, n^3 + 174386834388\, k^2$$
$$\mathrm{K}\ 2223183570678\, k\, n + 6261611044482\, n^2 \mathrm{K}\ 523045203120\, k$$
$$+ 2197356250680\, n + 238085568)\ b(n+1, k) + (86088744\, k^5$$
$$+ 78340457664\, k^3\, n^2 \mathrm{K}\ 879346060200\, k^2\, n^3 + 3299056562268\, k\, n^4$$
$$\mathrm{K}\ 4192389784359\, n^5 + 172182834\, k^4 + 156895189974\, k^3\, n$$
$$\mathrm{K}\ 2142153315234\, k^2\, n^2 + 9633677914683\, k\, n^3$$
$$\mathrm{K}\ 14425751415276\, n^4 + 79961482234\, k^3 \mathrm{K}\ 1649421916570\, k^2\, n$$
$$+ 9485635424331\, k\, n^2 \mathrm{K}\ 17146537633134\, n^3 \mathrm{K}\ 357395365134\, k^2$$
$$+ 3376176232116\, k\, n \mathrm{K}\ 7917871596363\, n^2 + 315033547898\, k$$
$$\mathrm{K}\ 1104526277852\, n \mathrm{K}\ 37857936576)\ b(n, k \mathrm{K}\ 1) = 0$$
$$(23242\, k^4 \mathrm{K}\ 69952\, k^3\, n \mathrm{K}\ 59601\, k^2\, n^2 \mathrm{K}\ 127044\, k\, n^3 \mathrm{K}\ 11664\, n^4 \mathrm{K}\ 4745\, k^3$$
$$\mathrm{K}\ 197595\, k^2\, n \mathrm{K}\ 126210\, k\, n^2 \mathrm{K}\ 16524\, n^3 \mathrm{K}\ 94420\, k^2 + 994\, k\, n$$
$$+ 8019\, n^2 + 8771\, k + 20169\, n)\ b(n, k) + (\mathrm{K}\ 27\, k^4 \mathrm{K}\ 306\, k^3 + 792\, k^2$$
$$\mathrm{K}\ 5184\, n^2 + 14706\, k \mathrm{K}\ 8640\, n + 27459)\ b(n, k+1) \mathrm{K}\ 64\ (\mathrm{K}\ 4\, n + 2$$
$$+ k)\ (364\, k^3 + 363\, k^2\, n + 360\, k\, n^2 + 432\, n^3 + 663\, k^2 + 291\, k\, n \mathrm{K}\ 252\, n^2$$
$$\mathrm{K}\ 205\, k \mathrm{K}\ 288\, n + 108)\ b(n \mathrm{K}\ 1, k) + (\mathrm{K}\ n + k)\ (\mathrm{K}\ 3\, n + k$$
$$\mathrm{K}\ 1)\ (23269\, k^2 + 23124\, k\, n + 32976\, n^2 + 42126\, k + 17292\, n$$
$$\mathrm{K}\ 7747)\ b(n, k \mathrm{K}\ 1) + 12096\ (k+4)\ (k+2)\ b(n, k+2) + (\mathrm{K}\ 23296\, k^4$$
$$+ 69952\, k^3\, n \mathrm{K}\ 112320\, k^3 + 197888\, k^2\, n \mathrm{K}\ 160768\, k^2 + 82112\, k\, n$$
$$\mathrm{K}\ 53568\, k \mathrm{K}\ 4352\, n + 4352)\ b(n \mathrm{K}\ 1, k \mathrm{K}\ 1) + 9216\ (k+3)\ (k$$
$$+ 2)\ b(n, k+3) = 0$$
$$(\mathrm{K}\ 19930\, k^3 + 56443\, k^2\, n + 55092\, k\, n^2 + 131328\, n^3 + 41776\, k^2$$
$$+ 1738\, k\, n + 155520\, n^2 \mathrm{K}\ 10744\, k + 39744\, n)\ b(n, k) + 2430\ (k$$
$$\mathrm{K}\ 1)\ (k+1)\ (\mathrm{K}\ 3\, n + k \mathrm{K}\ 4)\ b(n+1, k \mathrm{K}\ 1) + (\mathrm{K}\ 16348\, k^3 + 49153\, k^2\, n$$
$$+ 57422\, k^2 \mathrm{K}\ 73879\, k\, n \mathrm{K}\ 57830\, k + 24046\, n + 16756)\ b(n, k \mathrm{K}\ 2)$$
$$+ (540\, k + 1080)\ b(n+1, k+2) + (\mathrm{K}\ 3888\, n^3 \mathrm{K}\ 14580\, n^2 + 540\, k$$
$$\mathrm{K}\ 14148\, n + 540)\ b(n+1, k+1) \mathrm{K}\ 576\ (\mathrm{K}\ 4\, n \mathrm{K}\ 1 + k)\ (k^2 + 4\, k\, n$$
$$+ 16\, n^2 + 3\, k + 16\, n + 2)\ b(n, k+1) + (2430\, k^3 \mathrm{K}\ 7290\, k^2\, n$$
$$\mathrm{K}\ 7290\, k\, n^2 \mathrm{K}\ 10935\, n^3 \mathrm{K}\ 12150\, k^2 \mathrm{K}\ 19440\, k\, n \mathrm{K}\ 40095\, n^2 \mathrm{K}\ 14580\, k$$
$$\mathrm{K}\ 36450\, n)\ b(n+1, k) + (\mathrm{K}\ 35702\, k^3 + 105596\, k^2\, n + 47802\, k\, n^2$$
$$+ 87417\, n^3 + 97920\, k^2 \mathrm{K}\ 77001\, k\, n + 98019\, n^2 \mathrm{K}\ 87352\, k + 29767\, n$$
$$+ 25134)\ b(n, k \mathrm{K}\ 1) = 0$$
$$\mathrm{K}\ (\mathrm{K}\ 4\, n \mathrm{K}\ 2 + k)\ (344353680\, k^4 + 1377414720\, k^3\, n \tag{4.54}$$

$+ 581982998814\, k^2\, n^2 \mathrm{K}\; 4493088253755\, k\, n^3 + 8837414832960\, n^4$
$+ 1893551904\, k^3 + 1164989911500\, k^2\, n \mathrm{K}\; 11080408282101\, k\, n^2$
$+ 27046067934528\, n^3 + 4818640108192\, k^2 \mathrm{K}\; 4830603471060\, k\, n$
$+ 31192382524428\, n^2 \mathrm{K}\; 3520916168252\, k + 11909158062804\, n)$
$b(n, k) + (\mathrm{K}\, 2592\, k^5 \mathrm{K}\; 62208\, k^4 + 549736145964\, k^3$
$\mathrm{K}\; 1649127971844\, k^2\, n \mathrm{K}\; 2198004864624\, k^2 \mathrm{K}\; 2712754800\, k\, n$
$\mathrm{K}\; 552972903372\, k + 1646301143124\, n + 2194540018992)\; b(n+1,$
$k \mathrm{K}\; 1) + (\mathrm{K}\, 344356272\, k^5 \mathrm{K}\; 860558256\, k^4 \mathrm{K}\; 3632609099176\, k^3$
$+ 11005595890666\, k^2\, n + 12903606348164\, k^2$
$\mathrm{K}\; 16586264746198\, k\, n \mathrm{K}\; 13243567228868\, k + 5570227231372\, n$
$+ 3958370191288)\; b(n, k \mathrm{K}\; 2) + (\mathrm{K}\, 5158520640\, k^3$
$+ 25825670640\, k^2 \mathrm{K}\; 31107013200\, k \mathrm{K}\; 11431886400\, n$
$+ 170061120)\; b(n, k \mathrm{K}\; 3) + (\mathrm{K}\, 3149280\, k^3 + 25194240\, k^2$
$\mathrm{K}\; 59836320\, k \mathrm{K}\; 6998400\, n + 37791360)\; b(n, k \mathrm{K}\; 4)$
$+ (244032983280\, k + 244032983280)\; b(n+1, k+1) + ($
$\mathrm{K}\, 1749600\, k^3 + 190006560\, k^2 \mathrm{K}\; 572819040\, k\, n \mathrm{K}\; 750928320\, k$
$\mathrm{K}\; 554273280\, n \mathrm{K}\; 944784000)\; b(n+1, k \mathrm{K}\; 2) + (1889568\, k^2\, n^3$
$\mathrm{K}\; 11337408\, k\, n^4 + 17006112\, n^5 + 15116544\, k^2\, n^2 \mathrm{K}\; 111484512\, k\, n^3$
$+ 198404640\, n^4 \mathrm{K}\; 56687040\, k^3 + 210161952\, k^2\, n \mathrm{K}\; 236825856\, k\, n^2$
$+ 1086501600\, n^3 + 488768256\, k^2 + 140877792\, k\, n + 3228012000\, n^2$
$+ 550494144\, k + 4691167488\, n + 2368258560)\; b(n+2, k) + ($
$\mathrm{K}\, 25194240\, k \mathrm{K}\; 25194240)\; b(n+2, k+1) \mathrm{K}\; 56687040\, (k \mathrm{K}\; 1)\, (k$
$+ 1)\, (\mathrm{K}\, 3\, n \mathrm{K}\; 7 + k)\; b(n+2, k \mathrm{K}\; 1) + 233280\, (k \mathrm{K}\; 1)\, (k \mathrm{K}\; 3)\, (2\, k$
$\mathrm{K}\; 3)\; b(n+1, k \mathrm{K}\; 3) + (\mathrm{K}\, 2592\, k^5 + 7776\, k^3\, n^2 \mathrm{K}\; 288756948519\, k^2\, n^3$
$+ 2109087768762\, k\, n^4 \mathrm{K}\; 3728445625791\, n^5 \mathrm{K}\; 59616\, k^4 + 28512\, k^3\, n$
$\mathrm{K}\; 1443782676771\, k^2\, n^2 + 11486957823477\, k\, n^3$
$\mathrm{K}\; 22596391119924\, n^4 + 549738447660\, k^3 \mathrm{K}\; 4023345455802\, k^2\, n$
$+ 20399757957567\, k\, n^2 \mathrm{K}\; 52186279903011\, n^3 \mathrm{K}\; 4031297970804\, k^2$
$+ 12714931696878\, k\, n \mathrm{K}\; 55682028331794\, n^2 + 1127062246752\, k$
$\mathrm{K}\; 23982605350200\, n \mathrm{K}\; 7054387200)\; b(n+1, k) + (\mathrm{K}\, 688709952\, k^5$
$\mathrm{K}\; 576473332158\, k^3\, n^2 + 6532265190060\, k^2\, n^3$
$\mathrm{K}\; 24700691416626\, k\, n^4 + 31621230712161\, n^5 \mathrm{K}\; 2065400208\, k^4$
$\mathrm{K}\; 1154660861484\, k^3\, n + 16038072704268\, k^2\, n^2$
$\mathrm{K}\; 73085161419369\, k\, n^3 + 110719731915882\, n^4$
$\mathrm{K}\; 8442651013304\, k^3 + 36220189560476\, k^2\, n$
$\mathrm{K}\; 62471456531529\, k\, n^2 + 148900114274040\, n^3$

$+ 24842762195544\,k^2 \mathrm{K}\; 44431316181384\,k\,n + 80623069386219\,n^2$
$\mathrm{K}\; 22334666715172\,k + 14740569964564\,n + 5937309643092)\; b(n,$
$k \mathrm{K}\; 1) = 0$

```
> normal(expand(subs(n=n+2, map(e -> e/CF_B, map(applyopr,
  Bro_B_rec, CF_B, sAlg)))));
```

$$[0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0] \quad \textbf{(4.55)}$$

```
> map(indets, Bro_B_rec);
```

$$[\{k, n, sk, tk\}, \{k, n, sk, tk, tn\}, \{k, n, sn, tk\}, \{k, n, sn, tk\}, \{k, n, tk\}, \{k, n, sn, tk\}, \{sk, tk\}, \{k, n, tk, tn\}, \{k, n, sk, tk, tn\}, \{k, n, sk, tk, tn\}, \{k, n, sn, tk\}, \{k, n, sn, tk\}, \{k, n, sk, tk, tn\}, \{k, n, sk, tk, tn\}, \{k, n, sk, sn, tk\}, \{k, n, sk, sn, tk\}, \{sn, tn\}, \{k, n, sk, sn, tk\}, \{k, n, sk, tk, tn\}, \{k, n, sk, tk, tn\}, \{k, n, sk, sn, tk\}, \{k, n, sk, sn, tk\}, \{k, n, sk, tk, tn\}, \{k, n, sk, sn, tk\}, \{k, n, sk, sn, tk\}] \quad \textbf{(4.56)}$$

```
> map(p -> map2(degree, p, [sn,tn,sk,tk]), Bro_B_rec);
```

$$[[0, 0, 1, 1], [0, 1, 1, 1], [1, 0, 0, 1], [1, 0, 0, 1], [0, 0, 0, 2], [1, 0, 0, 2], [0, 0, 1, 1], [0, 1, 0, 1], [0, 1, 1, 1], [0, 1, 1, 1], [1, 0, 0, 1], [1, 0, 0, 2], [0, 1, 2, 1], [0, 1, 2, 1], [1, 0, 1, 2], [1, 0, 1, 2], [1, 1, 0, 0], [1, 0, 1, 3], [0, 1, 2, 2], [0, 1, 2, 2], [1, 0, 1, 3], [2, 0, 1, 3], [0, 1, 3, 1], [1, 0, 2, 2], [2, 0, 1, 4]] \quad \textbf{(4.57)}$$

Selecting two independent, low-order relations is sufficient for solving. One can choose:

```
> rec1_B := applyopr(collect(Bro_B_rec[1], {sn,tn,sk,tk},
  factor), b(n,k), sAlg);
```

$$rec1\_B := (\mathrm{K}\, 2\,k^2 + 8\,k\,n \mathrm{K}\; 3\,n^2 + k + 3\,n)\; b(n, k) \mathrm{K}\; (k+1)\; (\mathrm{K}\, 4\,n \mathrm{K}\; 1 + k)\; b(n, k+1) \mathrm{K}\; (\mathrm{K}\, 3\,n + k \mathrm{K}\; 1)\; (\mathrm{K}\, n + k)\; b(n, k \mathrm{K}\; 1) \quad \textbf{(4.58)}$$

```
> rec2_B := applyopr(collect(Bro_B_rec[3], {sn,tn,sk,tk},
  factor), b(n,k), sAlg);
```

$$rec2\_B := k\; (\mathrm{K}\, 4\,n \mathrm{K}\; 2 + k)\; (18\,k^2\,n^2 \mathrm{K}\; 117\,k\,n^3 + 192\,n^4 + 36\,k^2\,n \mathrm{K}\; 331\,k\,n^2 + 704\,n^3 + 20\,k^2 \mathrm{K}\; 314\,k\,n + 948\,n^2 \mathrm{K}\; 100\,k + 556\,n + 120)\; b(n, k) + k\; (n+2)\; (3\,n+4)\; (3\,n+5)\; (\mathrm{K}\, 3\,n + k \mathrm{K}\; 2)\; (\mathrm{K}\, 3\,n + k \mathrm{K}\; 3)\; b(n+1, k) + k\; (18\,k^3\,n^2 \mathrm{K}\; 180\,k^2\,n^3 + 606\,k\,n^4 \mathrm{K}\; 687\,n^5 + 36\,k^3\,n \mathrm{K}\; 484\,k^2\,n^2 + 2055\,k\,n^3 \mathrm{K}\; 2822\,n^4 + 20\,k^3 \mathrm{K}\; 428\,k^2\,n + 2499\,k\,n^2 \mathrm{K}\; 4386\,n^3 \mathrm{K}\; 120\,k^2 + 1266\,k\,n \mathrm{K}\; 3171\,n^2 + 220\,k \mathrm{K}\; 1042\,n \mathrm{K}\; 120)\; b(n, k \mathrm{K}\; 1) \quad \textbf{(4.59)}$$

Because these are not two first-order recurrence equations, it is not known a priori that all solutions are bivariate hypergeometric. But we can still search for hypergeometric solutions and see if we can identify our series as having such a solution as its coefficient.

```
> eval(subs(b = unapply(b(k), n, k), rec1_B));
```

$$(\mathrm{K}\, 2\,k^2 + 8\,k\,n \mathrm{K}\; 3\,n^2 + k + 3\,n)\; b(k) + (\mathrm{K}\, k^2 + 4\,k\,n + 4\,n + 1)\; b(k+1) \mathrm{K}\; (\mathrm{K}\, 3\,n + k \mathrm{K}\; 1)\; (\mathrm{K}\, n + k)\; b(k \mathrm{K}\; 1) \quad \textbf{(4.60)}$$

```
> LREtools:-hypergeomsols(%, b(k), {b(0) = f(n), b(1) = g(n)});
```

K (Γ(K 4 n) Γ(K 4 nK 1) (3 f(n) n^2 + 3 g(n) n^2 + 5 f(n) n + 5 g(n) n
+ g(n)) (K 1)^k Γ(K n + 1 + k) Γ(K 3 n + k)) / ((3 Γ(K 4 nK 1) n^2 Γ(K n
+ 2) Γ(K 3 n + 1) K 3 Γ(K 4 n) n^2 Γ(K n + 1) Γ(K 3 n) + 5 Γ(K 4 n
K 1) n Γ(K n + 2) Γ(K 3 n + 1) K 5 Γ(K 4 n) n Γ(K n + 1) Γ(K 3 n) + Γ(
K 4 nK 1) Γ(K n + 2) Γ(K 3 n + 1)) Γ(K 4 nK 1 + k) Γ(k + 1))
K ((f(n) Γ(K 4 nK 1) Γ(K n + 2) Γ(K 3 n + 1) + g(n) Γ(K 4 n) Γ(K n + 1) Γ(K 3 n)) (K 1
K 1) n Γ(K n + 2) Γ(K 3 n + 1) K 5 Γ(K 4 n) n Γ(K n + 1) Γ(K 3 n) + Γ(
K 4 nK 1) Γ(K n + 2) Γ(K 3 n + 1))

Then, we use the other recurrence equation to determine f(n) and g(n).

```
> factor(expand(eval(subs(b = unapply(%, n, k), rec2_B))));
```

((K 82944 Γ(K 4 n) Γ(K 3 n + k) Γ(K n + k) f(n + 1) n^10 + 786432 Γ(**(4.62)**
K 4 n) Γ(K 3 n + k) Γ(K n + k) g(n) n^10 + 786432 Γ(K 4 n) Γ(K 3 n
+ k) Γ(K n + k) f(n) n^10 K 698112 Γ(K 4 n) Γ(K 3 n + k) Γ(K n + k) g(n
+ 1) n^9 K 698112 Γ(K 4 n) Γ(K 3 n + k) Γ(K n + k) f(n + 1) n^9
+ 5439488 Γ(K 4 n) Γ(K 3 n + k) Γ(K n + k) g(n) n^9 + 5439488 Γ(
K 4 n) Γ(K 3 n + k) f(n) Γ(K n + k) n^9 K 2463552 Γ(K 4 n) Γ(K n + k) Γ(
K 3 n + k) g(n + 1) n^8 K 2435904 Γ(K 4 n) Γ(K n + k) Γ(K 3 n + k) f(n
+ 1) n^8 + 16285696 Γ(K 4 n) Γ(K 3 n + k) Γ(K n + k) g(n) n^8
+ 16023552 Γ(K 4 n) Γ(K 3 n + k) f(n) Γ(K n + k) n^8 K 4807056 Γ(
K 4 n) Γ(K n + k) Γ(K 3 n + k) g(n + 1) n^7 K 4675728 Γ(K 4 n) Γ(K n
+ k) Γ(K 3 n + k) f(n + 1) n^7 + 27746304 Γ(K 4 n) Γ(K 3 n + k) Γ(K n
+ k) g(n) n^7 + 26370048 Γ(K 4 n) Γ(K 3 n + k) f(n) Γ(K n + k) n^7
K 5726952 Γ(K 4 n) Γ(K n + k) Γ(K 3 n + k) g(n + 1) n^6 K 5470248 Γ(
K 4 n) Γ(K n + k) Γ(K 3 n + k) f(n + 1) n^6 + 29740032 Γ(K 4 n) Γ(K 3 n
+ k) Γ(K n + k) g(n) n^6 + 26692608 Γ(K 4 n) Γ(K 3 n + k) f(n) Γ(K n
+ k) n^6 K 4321128 Γ(K 4 n) Γ(K n + k) Γ(K 3 n + k) g(n + 1) n^5
K 4054008 Γ(K 4 n) Γ(K n + k) Γ(K 3 n + k) f(n + 1) n^5 K 82944 Γ(
K 4 n) Γ(K 3 n + k) Γ(K n + k) g(n + 1) n^10 K 24 k^5 Γ(K 4 n) Γ(K 3 n
+ k) Γ(K n + k) g(n) + 20921088 Γ(K 4 n) Γ(K 3 n + k) Γ(K n
+ k) g(n) n^5 + 17210112 Γ(K 4 n) Γ(K 3 n + k) f(n) Γ(K n + k) n^5
K 2063088 Γ(K 4 n) Γ(K n + k) Γ(K 3 n + k) g(n + 1) n^4 K 1903656 Γ(
K 4 n) Γ(K n + k) Γ(K 3 n + k) f(n + 1) n^4 + 360 k^4 Γ(K 4 n) Γ(K 3 n
+ k) Γ(K n + k) g(n) + 9767424 Γ(K 4 n) Γ(K 3 n + k) Γ(K n
+ k) g(n) n^4 + 7054848 Γ(K 4 n) Γ(K 3 n + k) f(n) Γ(K n + k) n^4

$$\mathrm{K}\ 600024\,\Gamma(\mathrm{K}\ 4\,n)\,\Gamma(\mathrm{K}\ n+k)\,\Gamma(\mathrm{K}\ 3\,n+k)\,g(n+1)\,n^3\,\mathrm{K}\ 545592\,\Gamma($$
$$\mathrm{K}\ 4\,n)\,\Gamma(\mathrm{K}\ n+k)\,\Gamma(\mathrm{K}\ 3\,n+k)\,f(n+1)\,n^3\,\mathrm{K}\ 2040\,k^3\,\Gamma(\mathrm{K}\ 4\,n)\,\Gamma(\mathrm{K}\ 3\,n$$
$$+k)\,\Gamma(\mathrm{K}\ n+k)\,g(n)+2984896\,\Gamma(\mathrm{K}\ 4\,n)\,\Gamma(\mathrm{K}\ 3\,n+k)\,g(n)\,n^3\,\Gamma(\mathrm{K}\ n$$
$$+k)+1769152\,\Gamma(\mathrm{K}\ 4\,n)\,\Gamma(\mathrm{K}\ 3\,n+k)\,f(n)\,n^3\,\Gamma(\mathrm{K}\ n+k)\,\mathrm{K}\ 96264\,\Gamma($$
$$\mathrm{K}\ 4\,n)\,\Gamma(\mathrm{K}\ n+k)\,\Gamma(\mathrm{K}\ 3\,n+k)\,g(n+1)\,n^2\,\mathrm{K}\ 86448\,\Gamma(\mathrm{K}\ 4\,n)\,\Gamma(\mathrm{K}\ n$$
$$+k)\,\Gamma(\mathrm{K}\ 3\,n+k)\,f(n+1)\,n^2+5400\,k^2\,\Gamma(\mathrm{K}\ 4\,n)\,\Gamma(\mathrm{K}\ 3\,n+k)\,\Gamma(\mathrm{K}\ n$$
$$+k)\,g(n)+571136\,\Gamma(\mathrm{K}\ 4\,n)\,\Gamma(\mathrm{K}\ 3\,n+k)\,g(n)\,n^2\,\Gamma(\mathrm{K}\ n+k)$$
$$+245760\,\Gamma(\mathrm{K}\ 4\,n)\,\Gamma(\mathrm{K}\ 3\,n+k)\,f(n)\,n^2\,\Gamma(\mathrm{K}\ n+k)\,\mathrm{K}\ 6480\,\Gamma(\mathrm{K}\ 4\,n)\,\Gamma($$
$$\mathrm{K}\ n+k)\,\Gamma(\mathrm{K}\ 3\,n+k)\,g(n+1)\,n\,\mathrm{K}\ 5760\,\Gamma(\mathrm{K}\ 4\,n)\,\Gamma(\mathrm{K}\ n+k)\,\Gamma(\mathrm{K}\ 3\,n$$
$$+k)\,f(n+1)\,n\,\mathrm{K}\ 6576\,\Gamma(\mathrm{K}\ 4\,n)\,\Gamma(\mathrm{K}\ 3\,n+k)\,g(n)\,\Gamma(\mathrm{K}\ n+k)\,k$$
$$+61824\,\Gamma(\mathrm{K}\ 4\,n)\,\Gamma(\mathrm{K}\ 3\,n+k)\,g(n)\,n\,\Gamma(\mathrm{K}\ n+k)+14400\,\Gamma(\mathrm{K}\ 4\,n)\,\Gamma($$
$$\mathrm{K}\ 3\,n+k)\,f(n)\,n\,\Gamma(\mathrm{K}\ n+k)+81\,k^5\,\Gamma(\mathrm{K}\ 4\,n)\,\Gamma(\mathrm{K}\ 3\,n+k)\,\Gamma(\mathrm{K}\ n+k)\,g(n$$
$$+1)\,n^5\,\mathrm{K}\ 1620\,k^4\,\Gamma(\mathrm{K}\ 4\,n)\,\Gamma(\mathrm{K}\ 3\,n+k)\,\Gamma(\mathrm{K}\ n+k)\,g(n+1)\,n^6$$
$$+12960\,k^3\,\Gamma(\mathrm{K}\ 4\,n)\,\Gamma(\mathrm{K}\ 3\,n+k)\,\Gamma(\mathrm{K}\ n+k)\,g(n+1)\,n^7\,\mathrm{K}\ 5715\,\Gamma($$
$$\mathrm{K}\ 4\,n)\,\Gamma(\mathrm{K}\ n+k)\,\Gamma(\mathrm{K}\ 3\,n+k)\,k^4\,g(n+1)\,n^2+130230\,\Gamma(\mathrm{K}\ 4\,n)\,\Gamma(\mathrm{K}\ n$$
$$+k)\,\Gamma(\mathrm{K}\ 3\,n+k)\,k^3\,g(n+1)\,n^3\,\mathrm{K}\ 1133730\,\Gamma(\mathrm{K}\ 4\,n)\,\Gamma(\mathrm{K}\ n+k)\,\Gamma(\mathrm{K}\ 3\,n$$
$$+k)\,k^2\,g(n+1)\,n^4+4234914\,\Gamma(\mathrm{K}\ 4\,n)\,\Gamma(\mathrm{K}\ n+k)\,\Gamma(\mathrm{K}\ 3\,n+k)\,k\,g(n$$
$$+1)\,n^5+48\,k^5\,\Gamma(\mathrm{K}\ 4\,n)\,\Gamma(\mathrm{K}\ 3\,n+k)\,\Gamma(\mathrm{K}\ n+k)\,f(n+1)\,n\,\mathrm{K}\ 5190\,\Gamma($$
$$\mathrm{K}\ 4\,n)\,\Gamma(\mathrm{K}\ n+k)\,\Gamma(\mathrm{K}\ 3\,n+k)\,k^4\,f(n+1)\,n^2+120495\,\Gamma(\mathrm{K}\ 4\,n)\,\Gamma(\mathrm{K}\ n$$
$$+k)\,\Gamma(\mathrm{K}\ 3\,n+k)\,k^3\,f(n+1)\,n^3\,\mathrm{K}\ 1063470\,\Gamma(\mathrm{K}\ 4\,n)\,\Gamma(\mathrm{K}\ n+k)\,\Gamma(\mathrm{K}\ 3\,n$$
$$+k)\,k^2\,f(n+1)\,n^4+4013394\,\Gamma(\mathrm{K}\ 4\,n)\,\Gamma(\mathrm{K}\ n+k)\,\Gamma(\mathrm{K}\ 3\,n+k)\,k\,f(n$$
$$+1)\,n^5\,\mathrm{K}\ 296\,k^5\,\Gamma(\mathrm{K}\ 4\,n)\,\Gamma(\mathrm{K}\ 3\,n+k)\,\Gamma(\mathrm{K}\ n+k)\,g(n)\,n+25960\,k^4\,\Gamma($$
$$\mathrm{K}\ 4\,n)\,\Gamma(\mathrm{K}\ 3\,n+k)\,\Gamma(\mathrm{K}\ n+k)\,g(n)\,n^2\,\mathrm{K}\ 597840\,k^3\,\Gamma(\mathrm{K}\ 4\,n)\,\Gamma(\mathrm{K}\ 3\,n$$
$$+k)\,\Gamma(\mathrm{K}\ n+k)\,g(n)\,n^3+5418560\,k^2\,\Gamma(\mathrm{K}\ 4\,n)\,\Gamma(\mathrm{K}\ 3\,n+k)\,\Gamma(\mathrm{K}\ n$$
$$+k)\,g(n)\,n^4\,\mathrm{K}\ 21129472\,k\,\Gamma(\mathrm{K}\ 4\,n)\,\Gamma(\mathrm{K}\ 3\,n+k)\,\Gamma(\mathrm{K}\ n+k)\,g(n)\,n^5$$
$$\mathrm{K}\ 120\,k^5\,\Gamma(\mathrm{K}\ 4\,n)\,\Gamma(\mathrm{K}\ 3\,n+k)\,f(n)\,\Gamma(\mathrm{K}\ n+k)\,n+16680\,k^4\,\Gamma(\mathrm{K}\ 4\,n)\,\Gamma($$
$$\mathrm{K}\ 3\,n+k)\,f(n)\,\Gamma(\mathrm{K}\ n+k)\,n^2\,\mathrm{K}\ 455760\,k^3\,\Gamma(\mathrm{K}\ 4\,n)\,\Gamma(\mathrm{K}\ 3\,n+k)\,f(n)\,\Gamma($$
$$\mathrm{K}\ n+k)\,n^3+4491840\,k^2\,\Gamma(\mathrm{K}\ 4\,n)\,\Gamma(\mathrm{K}\ 3\,n+k)\,f(n)\,\Gamma(\mathrm{K}\ n+k)\,n^4$$
$$\mathrm{K}\ 18385152\,k\,\Gamma(\mathrm{K}\ 4\,n)\,\Gamma(\mathrm{K}\ 3\,n+k)\,f(n)\,\Gamma(\mathrm{K}\ n+k)\,n^5\,\mathrm{K}\ 810\,\Gamma($$
$$\mathrm{K}\ 4\,n)\,\Gamma(\mathrm{K}\ n+k)\,\Gamma(\mathrm{K}\ 3\,n+k)\,k^4\,g(n+1)\,n+39225\,\Gamma(\mathrm{K}\ 4\,n)\,\Gamma(\mathrm{K}\ n$$
$$+k)\,\Gamma(\mathrm{K}\ 3\,n+k)\,k^3\,g(n+1)\,n^2\,\mathrm{K}\ 518490\,\Gamma(\mathrm{K}\ 4\,n)\,\Gamma(\mathrm{K}\ n+k)\,\Gamma(\mathrm{K}\ 3\,n$$
$$+k)\,k^2\,g(n+1)\,n^3+2576052\,\Gamma(\mathrm{K}\ 4\,n)\,\Gamma(\mathrm{K}\ n+k)\,\Gamma(\mathrm{K}\ 3\,n+k)\,k\,g(n$$
$$+1)\,n^4\,\mathrm{K}\ 720\,\Gamma(\mathrm{K}\ 4\,n)\,\Gamma(\mathrm{K}\ n+k)\,\Gamma(\mathrm{K}\ 3\,n+k)\,k^4\,f(n+1)\,n+35490\,\Gamma($$
$$\mathrm{K}\ 4\,n)\,\Gamma(\mathrm{K}\ n+k)\,\Gamma(\mathrm{K}\ 3\,n+k)\,k^3\,f(n+1)\,n^2\,\mathrm{K}\ 476235\,\Gamma(\mathrm{K}\ 4\,n)\,\Gamma(\mathrm{K}\ n$$

$+ k)\ \Gamma(\mathrm{K}\ 3\,n + k)\ k^{2} f(n+1)\ n^{3} + 2394252\,\Gamma(\mathrm{K}\ 4\,n)\ \Gamma(\mathrm{K}\ n + k)\ \Gamma(\mathrm{K}\ 3\,n$
$+ k)\ k\,f(n+1)\ n^{4} + 4920\,k^{4}\,\Gamma(\mathrm{K}\ 4\,n)\ \Gamma(\mathrm{K}\ 3\,n + k)\ \Gamma(\mathrm{K}\ n + k)\ g(n)\ n$
$\mathrm{K}\ 188440\,k^{3}\,\Gamma(\mathrm{K}\ 4\,n)\ \Gamma(\mathrm{K}\ 3\,n + k)\ \Gamma(\mathrm{K}\ n + k)\ g(n)\ n^{2} + 2412720\,k^{2}\,\Gamma($
$\mathrm{K}\ 4\,n)\ \Gamma(\mathrm{K}\ 3\,n + k)\ \Gamma(\mathrm{K}\ n + k)\ g(n)\ n^{3}\,\mathrm{K}\ 12151808\,k\,\Gamma(\mathrm{K}\ 4\,n)\ \Gamma(\mathrm{K}\ 3\,n$
$+ k)\ \Gamma(\mathrm{K}\ n + k)\ g(n)\ n^{4} + 1800\,k^{4}\,\Gamma(\mathrm{K}\ 4\,n)\ \Gamma(\mathrm{K}\ 3\,n + k)\ f(n)\ \Gamma(\mathrm{K}\ n$
$+ k)\ n\,\mathrm{K}\ 109720\,k^{3}\,\Gamma(\mathrm{K}\ 4\,n)\ \Gamma(\mathrm{K}\ 3\,n + k)\ f(n)\ \Gamma(\mathrm{K}\ n + k)\ n^{2}$
$+ 1694640\,k^{2}\,\Gamma(\mathrm{K}\ 4\,n)\ \Gamma(\mathrm{K}\ 3\,n + k)\ f(n)\ \Gamma(\mathrm{K}\ n + k)\ n^{3}\,\mathrm{K}\ 9417728\,k\,\Gamma($
$\mathrm{K}\ 4\,n)\ \Gamma(\mathrm{K}\ 3\,n + k)\ f(n)\ \Gamma(\mathrm{K}\ n + k)\ n^{4} + 4590\,\Gamma(\mathrm{K}\ 4\,n)\ \Gamma(\mathrm{K}\ n + k)\ \Gamma(\mathrm{K}\ 3\,n$
$+ k)\ k^{3}\,g(n+1)\ n\,\mathrm{K}\ 124605\,\Gamma(\mathrm{K}\ 4\,n)\ \Gamma(\mathrm{K}\ n + k)\ \Gamma(\mathrm{K}\ 3\,n + k)\ k^{2}\,g(n$
$+ 1)\ n^{2} + 929412\,\Gamma(\mathrm{K}\ 4\,n)\ \Gamma(\mathrm{K}\ n + k)\ \Gamma(\mathrm{K}\ 3\,n + k)\ k\,g(n+1)\ n^{3}$
$+ 4080\,\Gamma(\mathrm{K}\ 4\,n)\ \Gamma(\mathrm{K}\ n + k)\ \Gamma(\mathrm{K}\ 3\,n + k)\ k^{3}\,f(n+1)\ n\,\mathrm{K}\ 112410\,\Gamma($
$\mathrm{K}\ 4\,n)\ \Gamma(\mathrm{K}\ n + k)\ \Gamma(\mathrm{K}\ 3\,n + k)\ k^{2}\,f(n+1)\ n^{2} + 848934\,\Gamma(\mathrm{K}\ 4\,n)\ \Gamma(\mathrm{K}\ n$
$+ k)\ \Gamma(\mathrm{K}\ 3\,n + k)\ k\,f(n+1)\ n^{3}\,\mathrm{K}\ 30920\,k^{3}\,\Gamma(\mathrm{K}\ 4\,n)\ \Gamma(\mathrm{K}\ 3\,n + k)\ \Gamma(\mathrm{K}\ n$
$+ k)\ g(n)\ n + 637080\,k^{2}\,\Gamma(\mathrm{K}\ 4\,n)\ \Gamma(\mathrm{K}\ 3\,n + k)\ \Gamma(\mathrm{K}\ n + k)\ g(n)\ n^{2}$
$\mathrm{K}\ 4445536\,k\,\Gamma(\mathrm{K}\ 4\,n)\ \Gamma(\mathrm{K}\ 3\,n + k)\ \Gamma(\mathrm{K}\ n + k)\ g(n)\ n^{3}\,\mathrm{K}\ 10200\,k^{3}\,\Gamma($
$\mathrm{K}\ 4\,n)\ \Gamma(\mathrm{K}\ 3\,n + k)\ f(n)\ \Gamma(\mathrm{K}\ n + k)\ n + 336600\,k^{2}\,\Gamma(\mathrm{K}\ 4\,n)\ \Gamma(\mathrm{K}\ 3\,n$
$+ k)\ f(n)\ \Gamma(\mathrm{K}\ n + k)\ n^{2}\,\mathrm{K}\ 2873952\,k\,\Gamma(\mathrm{K}\ 4\,n)\ \Gamma(\mathrm{K}\ 3\,n + k)\ f(n)\ \Gamma(\mathrm{K}\ n$
$+ k)\ n^{3}\,\mathrm{K}\ 12150\,\Gamma(\mathrm{K}\ 4\,n)\ \Gamma(\mathrm{K}\ n + k)\ \Gamma(\mathrm{K}\ 3\,n + k)\ k^{2}\,g(n+1)\ n$
$+ 181866\,\Gamma(\mathrm{K}\ 4\,n)\ \Gamma(\mathrm{K}\ n + k)\ \Gamma(\mathrm{K}\ 3\,n + k)\ k\,g(n+1)\ n^{2}\,\mathrm{K}\ 10800\,\Gamma($
$\mathrm{K}\ 4\,n)\ \Gamma(\mathrm{K}\ n + k)\ \Gamma(\mathrm{K}\ 3\,n + k)\ k^{2}\,f(n+1)\ n + 163668\,\Gamma(\mathrm{K}\ 4\,n)\ \Gamma(\mathrm{K}\ n$
$+ k)\ \Gamma(\mathrm{K}\ 3\,n + k)\ k\,f(n+1)\ n^{2} + 91080\,k^{2}\,\Gamma(\mathrm{K}\ 4\,n)\ \Gamma(\mathrm{K}\ 3\,n + k)\ \Gamma(\mathrm{K}\ n$
$+ k)\ g(n)\ n\,\mathrm{K}\ 996784\,\Gamma(\mathrm{K}\ 4\,n)\ \Gamma(\mathrm{K}\ n + k)\ \Gamma(\mathrm{K}\ 3\,n + k)\ g(n)\ n^{2}\,k$
$+ 27000\,k^{2}\,\Gamma(\mathrm{K}\ 4\,n)\ \Gamma(\mathrm{K}\ 3\,n + k)\ f(n)\ \Gamma(\mathrm{K}\ n + k)\ n\,\mathrm{K}\ 476848\,\Gamma($
$\mathrm{K}\ 4\,n)\ \Gamma(\mathrm{K}\ n + k)\ \Gamma(\mathrm{K}\ 3\,n + k)\ f(n)\ n^{2}\,k + 14796\,\Gamma(\mathrm{K}\ 4\,n)\ \Gamma(\mathrm{K}\ n + k)\ \Gamma($
$\mathrm{K}\ 3\,n + k)\ k\,g(n+1)\ n + 13152\,\Gamma(\mathrm{K}\ 4\,n)\ \Gamma(\mathrm{K}\ n + k)\ \Gamma(\mathrm{K}\ 3\,n + k)\ k\,f(n$
$+ 1)\ n\,\mathrm{K}\ 124304\,\Gamma(\mathrm{K}\ 4\,n)\ \Gamma(\mathrm{K}\ 3\,n + k)\ g(n)\ n\,\Gamma(\mathrm{K}\ n + k)\ k\,\mathrm{K}\ 32880\,\Gamma($
$\mathrm{K}\ 4\,n)\ \Gamma(\mathrm{K}\ n + k)\ \Gamma(\mathrm{K}\ 3\,n + k)\ f(n)\ n\,k + 186885\,\Gamma(\mathrm{K}\ 4\,n)\ \Gamma(\mathrm{K}\ n + k)\ \Gamma($
$\mathrm{K}\ 3\,n + k)\ k^{3}\,g(n+1)\ n^{5}\,\mathrm{K}\ 984060\,\Gamma(\mathrm{K}\ 4\,n)\ \Gamma(\mathrm{K}\ n + k)\ \Gamma(\mathrm{K}\ 3\,n$
$+ k)\ k^{2}\,g(n+1)\ n^{6} + 2496240\,\Gamma(\mathrm{K}\ 4\,n)\ \Gamma(\mathrm{K}\ n + k)\ \Gamma(\mathrm{K}\ 3\,n + k)\ k\,g(n$
$+ 1)\ n^{7} + 531\,k^{5}\,\Gamma(\mathrm{K}\ 4\,n)\ \Gamma(\mathrm{K}\ 3\,n + k)\ \Gamma(\mathrm{K}\ n + k)\ f(n+1)\ n^{3}$
$\mathrm{K}\ 16290\,\Gamma(\mathrm{K}\ 4\,n)\ \Gamma(\mathrm{K}\ n + k)\ \Gamma(\mathrm{K}\ 3\,n + k)\ k^{4}\,f(n+1)\ n^{4} + 182565\,\Gamma($
$\mathrm{K}\ 4\,n)\ \Gamma(\mathrm{K}\ n + k)\ \Gamma(\mathrm{K}\ 3\,n + k)\ k^{3}\,f(n+1)\ n^{5}\,\mathrm{K}\ 966780\,\Gamma(\mathrm{K}\ 4\,n)\ \Gamma(\mathrm{K}\ n$
$+ k)\ \Gamma(\mathrm{K}\ 3\,n + k)\ k^{2}\,f(n+1)\ n^{6} + 2461680\,\Gamma(\mathrm{K}\ 4\,n)\ \Gamma(\mathrm{K}\ n + k)\ \Gamma(\mathrm{K}\ 3\,n$
$+ k)\ k\,f(n+1)\ n^{7}\,\mathrm{K}\ 2704\,k^{5}\,\Gamma(\mathrm{K}\ 4\,n)\ \Gamma(\mathrm{K}\ 3\,n + k)\ \Gamma(\mathrm{K}\ n + k)\ g(n)\ n^{3}$

$$+ 90560\,k^4\,\Gamma(\mathrm{K}\,4\,n)\,\Gamma(\mathrm{K}\,3\,n+k)\,\Gamma(\mathrm{K}\,n+k)\,g(n)\,n^4\,\mathrm{K}\;1081600\,k^3\,\Gamma($$
$$\mathrm{K}\,4\,n)\,\Gamma(\mathrm{K}\,3\,n+k)\,\Gamma(\mathrm{K}\,n+k)\,g(n)\,n^5 + 6016000\,k^2\,\Gamma(\mathrm{K}\,4\,n)\,\Gamma(\mathrm{K}\,3\,n$$
$$+ k)\,\Gamma(\mathrm{K}\,n+k)\,g(n)\,n^6\,\mathrm{K}\;15933440\,k\,\Gamma(\mathrm{K}\,4\,n)\,\Gamma(\mathrm{K}\,3\,n+k)\,\Gamma(\mathrm{K}\,n$$
$$+ k)\,g(n)\,n^7\,\mathrm{K}\;2448\,k^5\,\Gamma(\mathrm{K}\,4\,n)\,\Gamma(\mathrm{K}\,3\,n+k)\,f(n)\,\Gamma(\mathrm{K}\,n+k)\,n^3$$
$$+ 85440\,k^4\,\Gamma(\mathrm{K}\,4\,n)\,\Gamma(\mathrm{K}\,3\,n+k)\,f(n)\,\Gamma(\mathrm{K}\,n+k)\,n^4\,\mathrm{K}\;1040640\,k^3\,\Gamma($$
$$\mathrm{K}\,4\,n)\,\Gamma(\mathrm{K}\,3\,n+k)\,f(n)\,\Gamma(\mathrm{K}\,n+k)\,n^5 + 5852160\,\Gamma(\mathrm{K}\,4\,n)\,\Gamma(\mathrm{K}\,3\,n$$
$$+ k)\,f(n)\,\Gamma(\mathrm{K}\,n+k)\,k^2\,n^6\,\mathrm{K}\;15605760\,k\,\Gamma(\mathrm{K}\,4\,n)\,\Gamma(\mathrm{K}\,3\,n+k)\,f(n)\,\Gamma($$
$$\mathrm{K}\,n+k)\,n^7 + 309\,k^5\,\Gamma(\mathrm{K}\,4\,n)\,\Gamma(\mathrm{K}\,3\,n+k)\,\Gamma(\mathrm{K}\,n+k)\,g(n+1)\,n^2$$
$$\mathrm{K}\;14550\,\Gamma(\mathrm{K}\,4\,n)\,\Gamma(\mathrm{K}\,n+k)\,\Gamma(\mathrm{K}\,3\,n+k)\,k^4\,g(n+1)\,n^3 + 215490\,\Gamma($$
$$\mathrm{K}\,4\,n)\,\Gamma(\mathrm{K}\,n+k)\,\Gamma(\mathrm{K}\,3\,n+k)\,k^3\,g(n+1)\,n^4\,\mathrm{K}\;1405065\,\Gamma(\mathrm{K}\,4\,n)\,\Gamma(\mathrm{K}\,n$$
$$+ k)\,\Gamma(\mathrm{K}\,3\,n+k)\,k^2\,g(n+1)\,n^5 + 4226280\,\Gamma(\mathrm{K}\,4\,n)\,\Gamma(\mathrm{K}\,n+k)\,\Gamma(\mathrm{K}\,3\,n$$
$$+ k)\,k\,g(n+1)\,n^6 + 282\,k^5\,\Gamma(\mathrm{K}\,4\,n)\,\Gamma(\mathrm{K}\,3\,n+k)\,\Gamma(\mathrm{K}\,n+k)\,f(n+1)\,n^2$$
$$\mathrm{K}\;13605\,\Gamma(\mathrm{K}\,4\,n)\,\Gamma(\mathrm{K}\,n+k)\,\Gamma(\mathrm{K}\,3\,n+k)\,k^4\,f(n+1)\,n^3 + 204690\,\Gamma($$
$$\mathrm{K}\,4\,n)\,\Gamma(\mathrm{K}\,n+k)\,\Gamma(\mathrm{K}\,3\,n+k)\,k^3\,f(n+1)\,n^4\,\mathrm{K}\;1348905\,\Gamma(\mathrm{K}\,4\,n)\,\Gamma(\mathrm{K}\,n$$
$$+ k)\,\Gamma(\mathrm{K}\,3\,n+k)\,k^2\,f(n+1)\,n^5 + 4088040\,\Gamma(\mathrm{K}\,4\,n)\,\Gamma(\mathrm{K}\,n+k)\,\Gamma(\mathrm{K}\,3\,n$$
$$+ k)\,k\,f(n+1)\,n^6\,\mathrm{K}\;1336\,\Gamma(\mathrm{K}\,4\,n)\,\Gamma(\mathrm{K}\,3\,n+k)\,\Gamma(\mathrm{K}\,n+k)\,g(n)\,k^5\,n^2$$
$$+ 67280\,k^4\,\Gamma(\mathrm{K}\,4\,n)\,\Gamma(\mathrm{K}\,3\,n+k)\,\Gamma(\mathrm{K}\,n+k)\,g(n)\,n^3\,\mathrm{K}\;1069440\,\Gamma($$
$$\mathrm{K}\,4\,n)\,\Gamma(\mathrm{K}\,3\,n+k)\,\Gamma(\mathrm{K}\,n+k)\,g(n)\,k^3\,n^4 + 7402240\,\Gamma(\mathrm{K}\,4\,n)\,\Gamma(\mathrm{K}\,3\,n$$
$$+ k)\,\Gamma(\mathrm{K}\,n+k)\,g(n)\,k^2\,n^5\,\mathrm{K}\;23398400\,k\,\Gamma(\mathrm{K}\,4\,n)\,\Gamma(\mathrm{K}\,3\,n+k)\,\Gamma(\mathrm{K}\,n$$
$$+ k)\,g(n)\,n^6\,\mathrm{K}\;952\,k^5\,\Gamma(\mathrm{K}\,4\,n)\,\Gamma(\mathrm{K}\,3\,n+k)\,f(n)\,\Gamma(\mathrm{K}\,n+k)\,n^2$$
$$+ 55760\,k^4\,\Gamma(\mathrm{K}\,4\,n)\,\Gamma(\mathrm{K}\,3\,n+k)\,f(n)\,\Gamma(\mathrm{K}\,n+k)\,n^3\,\mathrm{K}\;946560\,k^3\,\Gamma($$
$$\mathrm{K}\,4\,n)\,\Gamma(\mathrm{K}\,3\,n+k)\,f(n)\,\Gamma(\mathrm{K}\,n+k)\,n^4 + 6787840\,\Gamma(\mathrm{K}\,4\,n)\,\Gamma(\mathrm{K}\,3\,n$$
$$+ k)\,f(n)\,\Gamma(\mathrm{K}\,n+k)\,k^2\,n^5\,\mathrm{K}\;21923840\,\Gamma(\mathrm{K}\,4\,n)\,\Gamma(\mathrm{K}\,3\,n+k)\,f(n)\,\Gamma(\mathrm{K}\,n$$
$$+ k)\,k\,n^6 + 54\,k^5\,\Gamma(\mathrm{K}\,4\,n)\,\Gamma(\mathrm{K}\,3\,n+k)\,\Gamma(\mathrm{K}\,n+k)\,g(n+1)\,n$$
$$\mathrm{K}\;82944\,\Gamma(\mathrm{K}\,3\,n)\,\Gamma(\mathrm{K}\,n)\,\Gamma(\mathrm{K}\,4\,n+k)\,\Gamma(k)\,g(n+1)\,k^2\,n^9 + 82944\,\Gamma($$
$$\mathrm{K}\,3\,n)\,\Gamma(\mathrm{K}\,n)\,\Gamma(\mathrm{K}\,4\,n+k)\,\Gamma(k)\,g(n+1)\,k\,n^{10}\,\mathrm{K}\;768\,\Gamma(\mathrm{K}\,3\,n)\,\Gamma(\mathrm{K}\,n)\,\Gamma($$
$$\mathrm{K}\,4\,n+k)\,\Gamma(k)\,f(n+1)\,k^5\,n^6 + 21120\,\Gamma(\mathrm{K}\,3\,n)\,\Gamma(\mathrm{K}\,n)\,\Gamma(\mathrm{K}\,4\,n$$
$$+ k)\,\Gamma(k)\,f(n+1)\,k^4\,n^7\,\mathrm{K}\;162432\,k^3\,\Gamma(\mathrm{K}\,n)\,\Gamma(\mathrm{K}\,3\,n)\,f(n+1)\,n^8\,\Gamma(\mathrm{K}\,4\,n$$
$$+ k)\,\Gamma(k) + 487296\,k^2\,\Gamma(\mathrm{K}\,n)\,\Gamma(\mathrm{K}\,3\,n)\,f(n+1)\,n^9\,\Gamma(\mathrm{K}\,4\,n+k)\,\Gamma(k)$$
$$\mathrm{K}\;508032\,k\,\Gamma(\mathrm{K}\,n)\,\Gamma(\mathrm{K}\,3\,n)\,f(n+1)\,n^{10}\,\Gamma(\mathrm{K}\,4\,n+k)\,\Gamma(k) + 3072\,\Gamma($$
$$\mathrm{K}\,3\,n)\,\Gamma(\mathrm{K}\,n)\,\Gamma(\mathrm{K}\,4\,n+k)\,\Gamma(k)\,g(n)\,k^4\,n^7\,\mathrm{K}\;27648\,\Gamma(\mathrm{K}\,3\,n)\,\Gamma(\mathrm{K}\,n)\,\Gamma($$
$$\mathrm{K}\,4\,n+k)\,\Gamma(k)\,g(n)\,k^3\,n^8 + 82944\,\Gamma(\mathrm{K}\,3\,n)\,\Gamma(\mathrm{K}\,n)\,\Gamma(\mathrm{K}\,4\,n$$
$$+ k)\,\Gamma(k)\,g(n)\,k^2\,n^9\,\mathrm{K}\;82944\,\Gamma(\mathrm{K}\,3\,n)\,\Gamma(\mathrm{K}\,n)\,\Gamma(\mathrm{K}\,4\,n$$
$$+ k)\,\Gamma(k)\,g(n)\,k\,n^{10} + 768\,\Gamma(\mathrm{K}\,3\,n)\,\Gamma(\mathrm{K}\,n)\,\Gamma(\mathrm{K}\,4\,n+k)\,\Gamma(k)\,f(n)\,k^5\,n^6$$

$$\mathrm{K}\ 16512\,\Gamma(\mathrm{K}\ 3\,n)\,\Gamma(\mathrm{K}\ n)\,\Gamma(\mathrm{K}\ 4\,n+k)\,\Gamma(k)\,f(n)\,k^4\,n^7+120960\,k^3\,\Gamma($$
$$\mathrm{K}\ n)\,\Gamma(\mathrm{K}\ 3\,n)\,f(n)\,n^8\,\Gamma(\mathrm{K}\ 4\,n+k)\,\Gamma(k)\,\mathrm{K}\ 362880\,k^2\,\Gamma(\mathrm{K}\ n)\,\Gamma($$
$$\mathrm{K}\ 3\,n)\,f(n)\,n^9\,\Gamma(\mathrm{K}\ 4\,n+k)\,\Gamma(k)+383616\,k\,\Gamma(\mathrm{K}\ n)\,\Gamma(\mathrm{K}\ 3\,n)\,f(n)\,n^{10}\,\Gamma($$
$$\mathrm{K}\ 4\,n+k)\,\Gamma(k)+1024\,\Gamma(\mathrm{K}\ 3\,n)\,\Gamma(\mathrm{K}\ n)\,\Gamma(\mathrm{K}\ 4\,n+k)\,\Gamma(k)\,g(n+1)\,k^5\,n^5$$
$$\mathrm{K}\ 28928\,\Gamma(\mathrm{K}\ 3\,n)\,\Gamma(\mathrm{K}\ n)\,\Gamma(\mathrm{K}\ 4\,n+k)\,\Gamma(k)\,g(n+1)\,k^4\,n^6$$
$$+223488\,k^3\,\Gamma(\mathrm{K}\ n)\,\Gamma(\mathrm{K}\ 3\,n)\,g(n+1)\,n^7\,\Gamma(\mathrm{K}\ 4\,n+k)\,\Gamma(k)$$
$$\mathrm{K}\ 670464\,k^2\,\Gamma(\mathrm{K}\ n)\,\Gamma(\mathrm{K}\ 3\,n)\,g(n+1)\,n^8\,\Gamma(\mathrm{K}\ 4\,n+k)\,\Gamma(k)$$
$$+698112\,k\,\Gamma(\mathrm{K}\ n)\,\Gamma(\mathrm{K}\ 3\,n)\,g(n+1)\,n^9\,\Gamma(\mathrm{K}\ 4\,n+k)\,\Gamma(k)\,\mathrm{K}\ 1920\,\Gamma($$
$$\mathrm{K}\ 3\,n)\,\Gamma(\mathrm{K}\ n)\,\Gamma(\mathrm{K}\ 4\,n+k)\,\Gamma(k)\,f(n+1)\,k^5\,n^5+54960\,k^4\,\Gamma(\mathrm{K}\ n)\,\Gamma($$
$$\mathrm{K}\ 3\,n)\,f(n+1)\,n^6\,\Gamma(\mathrm{K}\ 4\,n+k)\,\Gamma(k)\,\mathrm{K}\ 462384\,k^3\,\Gamma(\mathrm{K}\ n)\,\Gamma(\mathrm{K}\ 3\,n)\,f(n$$
$$+1)\,n^7\,\Gamma(\mathrm{K}\ 4\,n+k)\,\Gamma(k)+1523088\,k^2\,\Gamma(\mathrm{K}\ n)\,\Gamma(\mathrm{K}\ 3\,n)\,f(n+1)\,n^8\,\Gamma($$
$$\mathrm{K}\ 4\,n+k)\,\Gamma(k)\,\mathrm{K}\ 1736208\,k\,\Gamma(\mathrm{K}\ n)\,\Gamma(\mathrm{K}\ 3\,n)\,f(n+1)\,n^9\,\Gamma(\mathrm{K}\ 4\,n$$
$$+k)\,\Gamma(k)\,\mathrm{K}\ 1024\,\Gamma(\mathrm{K}\ 3\,n)\,\Gamma(\mathrm{K}\ n)\,\Gamma(\mathrm{K}\ 4\,n+k)\,\Gamma(k)\,g(n)\,k^5\,n^5$$
$$+25856\,\Gamma(\mathrm{K}\ 3\,n)\,\Gamma(\mathrm{K}\ n)\,\Gamma(\mathrm{K}\ 4\,n+k)\,\Gamma(k)\,g(n)\,k^4\,n^6\,\mathrm{K}\ 195840\,\Gamma($$
$$\mathrm{K}\ 3\,n)\,\Gamma(\mathrm{K}\ n)\,\Gamma(\mathrm{K}\ 4\,n+k)\,\Gamma(k)\,g(n)\,k^3\,n^7+587520\,\Gamma(\mathrm{K}\ 3\,n)\,\Gamma(\mathrm{K}\ n)\,\Gamma($$
$$\mathrm{K}\ 4\,n+k)\,\Gamma(k)\,g(n)\,k^2\,n^8\,\mathrm{K}\ 615168\,\Gamma(\mathrm{K}\ 3\,n)\,\Gamma(\mathrm{K}\ n)\,\Gamma(\mathrm{K}\ 4\,n$$
$$+k)\,\Gamma(k)\,g(n)\,k\,n^9+384\,\Gamma(\mathrm{K}\ 3\,n)\,\Gamma(\mathrm{K}\ n)\,\Gamma(\mathrm{K}\ 4\,n+k)\,\Gamma(k)\,f(n)\,k^5\,n^5$$
$$\mathrm{K}\ 17328\,k^4\,\Gamma(\mathrm{K}\ n)\,\Gamma(\mathrm{K}\ 3\,n)\,f(n)\,n^6\,\Gamma(\mathrm{K}\ 4\,n+k)\,\Gamma(k)+178992\,k^3\,\Gamma($$
$$\mathrm{K}\ n)\,\Gamma(\mathrm{K}\ 3\,n)\,f(n)\,n^7\,\Gamma(\mathrm{K}\ 4\,n+k)\,\Gamma(k)\,\mathrm{K}\ 672912\,k^2\,\Gamma(\mathrm{K}\ n)\,\Gamma($$
$$\mathrm{K}\ 3\,n)\,f(n)\,n^8\,\Gamma(\mathrm{K}\ 4\,n+k)\,\Gamma(k)+844560\,k\,\Gamma(\mathrm{K}\ n)\,\Gamma(\mathrm{K}\ 3\,n)\,f(n)\,n^9\,\Gamma($$
$$\mathrm{K}\ 4\,n+k)\,\Gamma(k)+2816\,\Gamma(\mathrm{K}\ 3\,n)\,\Gamma(\mathrm{K}\ n)\,\Gamma(\mathrm{K}\ 4\,n+k)\,\Gamma(k)\,g(n+1)\,k^5\,n^4$$
$$\mathrm{K}\ 79552\,k^4\,\Gamma(\mathrm{K}\ n)\,\Gamma(\mathrm{K}\ 3\,n)\,g(n+1)\,n^5\,\Gamma(\mathrm{K}\ 4\,n+k)\,\Gamma(k)$$
$$+663744\,k^3\,\Gamma(\mathrm{K}\ n)\,\Gamma(\mathrm{K}\ 3\,n)\,g(n+1)\,n^6\,\Gamma(\mathrm{K}\ 4\,n+k)\,\Gamma(k)$$
$$\mathrm{K}\ 2172480\,k^2\,\Gamma(\mathrm{K}\ n)\,\Gamma(\mathrm{K}\ 3\,n)\,g(n+1)\,n^7\,\Gamma(\mathrm{K}\ 4\,n+k)\,\Gamma(k)$$
$$+2463552\,k\,\Gamma(\mathrm{K}\ n)\,\Gamma(\mathrm{K}\ 3\,n)\,g(n+1)\,n^8\,\Gamma(\mathrm{K}\ 4\,n+k)\,\Gamma(k)\,\mathrm{K}\ 1680\,\Gamma($$
$$\mathrm{K}\ 3\,n)\,\Gamma(\mathrm{K}\ n)\,\Gamma(\mathrm{K}\ 4\,n+k)\,\Gamma(k)\,f(n+1)\,k^5\,n^4+64200\,k^4\,\Gamma(\mathrm{K}\ n)\,\Gamma($$
$$\mathrm{K}\ 3\,n)\,f(n+1)\,n^5\,\Gamma(\mathrm{K}\ 4\,n+k)\,\Gamma(k)\,\mathrm{K}\ 659400\,k^3\,\Gamma(\mathrm{K}\ n)\,\Gamma(\mathrm{K}\ 3\,n)\,f(n$$
$$+1)\,n^6\,\Gamma(\mathrm{K}\ 4\,n+k)\,\Gamma(k)+2535384\,k^2\,\Gamma(\mathrm{K}\ n)\,\Gamma(\mathrm{K}\ 3\,n)\,f(n+1)\,n^7\,\Gamma($$
$$\mathrm{K}\ 4\,n+k)\,\Gamma(k)\,\mathrm{K}\ 3282696\,k\,\Gamma(\mathrm{K}\ n)\,\Gamma(\mathrm{K}\ 3\,n)\,f(n+1)\,n^8\,\Gamma(\mathrm{K}\ 4\,n$$
$$+k)\,\Gamma(k)\,\mathrm{K}\ 1792\,\Gamma(\mathrm{K}\ 3\,n)\,\Gamma(\mathrm{K}\ n)\,\Gamma(\mathrm{K}\ 4\,n+k)\,\Gamma(k)\,g(n)\,k^5\,n^4$$
$$+54464\,\Gamma(\mathrm{K}\ 3\,n)\,\Gamma(\mathrm{K}\ n)\,\Gamma(\mathrm{K}\ 4\,n+k)\,\Gamma(k)\,g(n)\,k^4\,n^5\,\mathrm{K}\ 474816\,k^3\,\Gamma($$
$$\mathrm{K}\ n)\,\Gamma(\mathrm{K}\ 3\,n)\,g(n)\,n^6\,\Gamma(\mathrm{K}\ 4\,n+k)\,\Gamma(k)+1605696\,k^2\,\Gamma(\mathrm{K}\ n)\,\Gamma($$
$$\mathrm{K}\ 3\,n)\,g(n)\,n^7\,\Gamma(\mathrm{K}\ 4\,n+k)\,\Gamma(k)\,\mathrm{K}\ 1869120\,k\,\Gamma(\mathrm{K}\ n)\,\Gamma(\mathrm{K}\ 3\,n)\,g(n)\,n^8\,\Gamma($$
$$\mathrm{K}\ 4\,n+k)\,\Gamma(k)\,\mathrm{K}\ 624\,\Gamma(\mathrm{K}\ 3\,n)\,\Gamma(\mathrm{K}\ n)\,\Gamma(\mathrm{K}\ 4\,n+k)\,\Gamma(k)\,f(n)\,k^5\,n^4$$

$+ 8088\,\Gamma(\mathrm{K}\,3\,n)\,\Gamma(\mathrm{K}\,n)\,\Gamma(\mathrm{K}\,4\,n+k)\,\Gamma(k)\,f(n)\,k^{4}\,n^{5} + 18024\,k^{3}\,\Gamma($
$\mathrm{K}\,n)\,\Gamma(\mathrm{K}\,3\,n)\,f(n)\,n^{6}\,\Gamma(\mathrm{K}\,4\,n+k)\,\Gamma(k)\,\mathrm{K}\;339384\,k^{2}\,\Gamma(\mathrm{K}\,n)\,\Gamma($
$\mathrm{K}\,3\,n)\,f(n)\,n^{7}\,\Gamma(\mathrm{K}\,4\,n+k)\,\Gamma(k) + 701928\,k\,\Gamma(\mathrm{K}\,n)\,\Gamma(\mathrm{K}\,3\,n)\,f(n)\,n^{8}\,\Gamma($
$\mathrm{K}\,4\,n+k)\,\Gamma(k) + 2624\,\Gamma(\mathrm{K}\,3\,n)\,\Gamma(\mathrm{K}\,n)\,\Gamma(\mathrm{K}\,4\,n+k)\,\Gamma(k)\,g(n+1)\,k^{5}\,n^{3}$
$\mathrm{K}\;97648\,k^{4}\,\Gamma(\mathrm{K}\,n)\,\Gamma(\mathrm{K}\,3\,n)\,g(n+1)\,n^{4}\,\Gamma(\mathrm{K}\,4\,n+k)\,\Gamma(k)$
$+ 986096\,k^{3}\,\Gamma(\mathrm{K}\,n)\,\Gamma(\mathrm{K}\,3\,n)\,g(n+1)\,n^{5}\,\Gamma(\mathrm{K}\,4\,n+k)\,\Gamma(k)$
$\mathrm{K}\;3746512\,k^{2}\,\Gamma(\mathrm{K}\,n)\,\Gamma(\mathrm{K}\,3\,n)\,g(n+1)\,n^{6}\,\Gamma(\mathrm{K}\,4\,n+k)\,\Gamma(k)$
$+ 4807056\,k\,\Gamma(\mathrm{K}\,n)\,\Gamma(\mathrm{K}\,3\,n)\,g(n+1)\,n^{7}\,\Gamma(\mathrm{K}\,4\,n+k)\,\Gamma(k)\,\mathrm{K}\;600\,\Gamma($
$\mathrm{K}\,3\,n)\,\Gamma(\mathrm{K}\,n)\,\Gamma(\mathrm{K}\,4\,n+k)\,\Gamma(k)\,f(n+1)\,k^{5}\,n^{3} + 37416\,k^{4}\,\Gamma(\mathrm{K}\,n)\,\Gamma($
$\mathrm{K}\,3\,n)\,f(n+1)\,n^{4}\,\Gamma(\mathrm{K}\,4\,n+k)\,\Gamma(k)\,\mathrm{K}\;520584\,k^{3}\,\Gamma(\mathrm{K}\,n)\,\Gamma(\mathrm{K}\,3\,n)\,f(n$
$+ 1)\,n^{5}\,\Gamma(\mathrm{K}\,4\,n+k)\,\Gamma(k) + 2483112\,k^{2}\,\Gamma(\mathrm{K}\,n)\,\Gamma(\mathrm{K}\,3\,n)\,f(n+1)\,n^{6}\,\Gamma($
$\mathrm{K}\,4\,n+k)\,\Gamma(k) + 2304\,\Gamma(\mathrm{K}\,3\,n)\,\Gamma(\mathrm{K}\,n)\,\Gamma(\mathrm{K}\,4\,n+k)\,\Gamma(k)\,f(n+1)\,k^{4}\,n^{8}$
$\mathrm{K}\;20736\,\Gamma(\mathrm{K}\,3\,n)\,\Gamma(\mathrm{K}\,n)\,\Gamma(\mathrm{K}\,4\,n+k)\,\Gamma(k)\,f(n+1)\,k^{3}\,n^{9} + 62208\,\Gamma($
$\mathrm{K}\,3\,n)\,\Gamma(\mathrm{K}\,n)\,\Gamma(\mathrm{K}\,4\,n+k)\,\Gamma(k)\,f(n+1)\,k^{2}\,n^{10}\,\mathrm{K}\;62208\,\Gamma(\mathrm{K}\,3\,n)\,\Gamma($
$\mathrm{K}\,n)\,\Gamma(\mathrm{K}\,4\,n+k)\,\Gamma(k)\,f(n+1)\,k\,n^{11}\,\mathrm{K}\;2304\,\Gamma(\mathrm{K}\,3\,n)\,\Gamma(\mathrm{K}\,n)\,\Gamma(\mathrm{K}\,4\,n$
$+ k)\,\Gamma(k)\,f(n)\,k^{4}\,n^{8} + 20736\,\Gamma(\mathrm{K}\,3\,n)\,\Gamma(\mathrm{K}\,n)\,\Gamma(\mathrm{K}\,4\,n+k)\,\Gamma(k)\,f(n)\,k^{3}\,n^{9}$
$\mathrm{K}\;62208\,\Gamma(\mathrm{K}\,3\,n)\,\Gamma(\mathrm{K}\,n)\,\Gamma(\mathrm{K}\,4\,n+k)\,\Gamma(k)\,f(n)\,k^{2}\,n^{10} + 62208\,\Gamma($
$\mathrm{K}\,3\,n)\,\Gamma(\mathrm{K}\,n)\,\Gamma(\mathrm{K}\,4\,n+k)\,\Gamma(k)\,f(n)\,k\,n^{11}\,\mathrm{K}\;3072\,\Gamma(\mathrm{K}\,3\,n)\,\Gamma(\mathrm{K}\,n)\,\Gamma(\mathrm{K}\,4\,n$
$+ k)\,\Gamma(k)\,g(n+1)\,k^{4}\,n^{7} + 27648\,\Gamma(\mathrm{K}\,3\,n)\,\Gamma(\mathrm{K}\,n)\,\Gamma(\mathrm{K}\,4\,n+k)\,\Gamma(k)\,g(n$
$+ 1)\,k^{3}\,n^{8} + 2880\,\Gamma(\mathrm{K}\,4\,n)\,\Gamma(\mathrm{K}\,3\,n+k)\,g(n)\,\Gamma(\mathrm{K}\,n+k)\,\mathrm{K}\;51840\,k^{2}\,\Gamma($
$\mathrm{K}\,4\,n)\,\Gamma(\mathrm{K}\,3\,n+k)\,\Gamma(\mathrm{K}\,n+k)\,g(n+1)\,n^{8} + 103680\,k\,\Gamma(\mathrm{K}\,4\,n)\,\Gamma(\mathrm{K}\,3\,n$
$+ k)\,\Gamma(\mathrm{K}\,n+k)\,g(n+1)\,n^{9} + 81\,k^{5}\,\Gamma(\mathrm{K}\,4\,n)\,\Gamma(\mathrm{K}\,3\,n+k)\,\Gamma(\mathrm{K}\,n+k)\,f(n$
$+ 1)\,n^{5}\,\mathrm{K}\;1620\,k^{4}\,\Gamma(\mathrm{K}\,4\,n)\,\Gamma(\mathrm{K}\,3\,n+k)\,\Gamma(\mathrm{K}\,n+k)\,f(n+1)\,n^{6}$
$+ 12960\,k^{3}\,\Gamma(\mathrm{K}\,4\,n)\,\Gamma(\mathrm{K}\,3\,n+k)\,\Gamma(\mathrm{K}\,n+k)\,f(n+1)\,n^{7}\,\mathrm{K}\;51840\,k^{2}\,\Gamma($
$\mathrm{K}\,4\,n)\,\Gamma(\mathrm{K}\,3\,n+k)\,\Gamma(\mathrm{K}\,n+k)\,f(n+1)\,n^{8} + 103680\,k\,\Gamma(\mathrm{K}\,4\,n)\,\Gamma(\mathrm{K}\,3\,n$
$+ k)\,\Gamma(\mathrm{K}\,n+k)\,f(n+1)\,n^{9}\,\mathrm{K}\;768\,\Gamma(\mathrm{K}\,4\,n)\,\Gamma(\mathrm{K}\,3\,n+k)\,\Gamma(\mathrm{K}\,n$
$+ k)\,g(n)\,k^{5}\,n^{5} + 15360\,\Gamma(\mathrm{K}\,4\,n)\,\Gamma(\mathrm{K}\,3\,n+k)\,\Gamma(\mathrm{K}\,n+k)\,g(n)\,k^{4}\,n^{6}$
$\mathrm{K}\;122880\,\Gamma(\mathrm{K}\,4\,n)\,\Gamma(\mathrm{K}\,3\,n+k)\,\Gamma(\mathrm{K}\,n+k)\,g(n)\,k^{3}\,n^{7} + 491520\,\Gamma($
$\mathrm{K}\,4\,n)\,\Gamma(\mathrm{K}\,3\,n+k)\,\Gamma(\mathrm{K}\,n+k)\,g(n)\,k^{2}\,n^{8}\,\mathrm{K}\;983040\,k\,\Gamma(\mathrm{K}\,4\,n)\,\Gamma(\mathrm{K}\,3\,n$
$+ k)\,\Gamma(\mathrm{K}\,n+k)\,g(n)\,n^{9}\,\mathrm{K}\;768\,\Gamma(\mathrm{K}\,4\,n)\,\Gamma(\mathrm{K}\,3\,n+k)\,f(n)\,\Gamma(\mathrm{K}\,n+k)\,k^{5}\,n^{5}$
$+ 15360\,\Gamma(\mathrm{K}\,4\,n)\,\Gamma(\mathrm{K}\,3\,n+k)\,f(n)\,\Gamma(\mathrm{K}\,n+k)\,k^{4}\,n^{6}\,\mathrm{K}\;122880\,\Gamma($
$\mathrm{K}\,4\,n)\,\Gamma(\mathrm{K}\,3\,n+k)\,f(n)\,\Gamma(\mathrm{K}\,n+k)\,k^{3}\,n^{7} + 491520\,\Gamma(\mathrm{K}\,4\,n)\,\Gamma(\mathrm{K}\,3\,n$
$+ k)\,f(n)\,\Gamma(\mathrm{K}\,n+k)\,k^{2}\,n^{8}\,\mathrm{K}\;983040\,k\,\Gamma(\mathrm{K}\,4\,n)\,\Gamma(\mathrm{K}\,3\,n+k)\,f(n)\,\Gamma(\mathrm{K}\,n$
$+ k)\,n^{9} + 378\,k^{5}\,\Gamma(\mathrm{K}\,4\,n)\,\Gamma(\mathrm{K}\,3\,n+k)\,\Gamma(\mathrm{K}\,n+k)\,g(n+1)\,n^{4}$

$$
\begin{aligned}
&\mathrm{K}\ 8775\,k^4\,\Gamma(\mathrm{K}\ 4\,n)\,\Gamma(\mathrm{K}\ 3\,n+k)\,\Gamma(\mathrm{K}\ n+k)\,g(n+1)\,n^5+79920\,k^3\,\Gamma(\\
&\mathrm{K}\ 4\,n)\,\Gamma(\mathrm{K}\ 3\,n+k)\,\Gamma(\mathrm{K}\ n+k)\,g(n+1)\,n^6\,\mathrm{K}\ 358560\,k^2\,\Gamma(\mathrm{K}\ 4\,n)\,\Gamma(\mathrm{K}\ 3\,n\\
&+k)\,\Gamma(\mathrm{K}\ n+k)\,g(n+1)\,n^7+794880\,k\,\Gamma(\mathrm{K}\ 4\,n)\,\Gamma(\mathrm{K}\ 3\,n+k)\,\Gamma(\mathrm{K}\ n\\
&+k)\,g(n+1)\,n^8+378\,k^5\,\Gamma(\mathrm{K}\ 4\,n)\,\Gamma(\mathrm{K}\ 3\,n+k)\,\Gamma(\mathrm{K}\ n+k)\,f(n+1)\,n^4\\
&\mathrm{K}\ 8775\,k^4\,\Gamma(\mathrm{K}\ 4\,n)\,\Gamma(\mathrm{K}\ 3\,n+k)\,\Gamma(\mathrm{K}\ n+k)\,f(n+1)\,n^5+79920\,k^3\,\Gamma(\\
&\mathrm{K}\ 4\,n)\,\Gamma(\mathrm{K}\ 3\,n+k)\,\Gamma(\mathrm{K}\ n+k)\,f(n+1)\,n^6\,\mathrm{K}\ 358560\,k^2\,\Gamma(\mathrm{K}\ 4\,n)\,\Gamma(\mathrm{K}\ 3\,n\\
&+k)\,\Gamma(\mathrm{K}\ n+k)\,f(n+1)\,n^7+794880\,k\,\Gamma(\mathrm{K}\ 4\,n)\,\Gamma(\mathrm{K}\ 3\,n+k)\,\Gamma(\mathrm{K}\ n\\
&+k)\,f(n+1)\,n^8\,\mathrm{K}\ 2432\,k^5\,\Gamma(\mathrm{K}\ 4\,n)\,\Gamma(\mathrm{K}\ 3\,n+k)\,\Gamma(\mathrm{K}\ n+k)\,g(n)\,n^4\\
&+60160\,k^4\,\Gamma(\mathrm{K}\ 4\,n)\,\Gamma(\mathrm{K}\ 3\,n+k)\,\Gamma(\mathrm{K}\ n+k)\,g(n)\,n^5\,\mathrm{K}\ 573440\,k^3\,\Gamma(\\
&\mathrm{K}\ 4\,n)\,\Gamma(\mathrm{K}\ 3\,n+k)\,\Gamma(\mathrm{K}\ n+k)\,g(n)\,n^6+2662400\,k^2\,\Gamma(\mathrm{K}\ 4\,n)\,\Gamma(\mathrm{K}\ 3\,n\\
&+k)\,\Gamma(\mathrm{K}\ n+k)\,g(n)\,n^7\,\mathrm{K}\ 6062080\,k\,\Gamma(\mathrm{K}\ 4\,n)\,\Gamma(\mathrm{K}\ 3\,n+k)\,\Gamma(\mathrm{K}\ n\\
&+k)\,g(n)\,n^8\,\mathrm{K}\ 2432\,k^5\,\Gamma(\mathrm{K}\ 4\,n)\,\Gamma(\mathrm{K}\ 3\,n+k)\,f(n)\,\Gamma(\mathrm{K}\ n+k)\,n^4\\
&+60160\,k^4\,\Gamma(\mathrm{K}\ 4\,n)\,\Gamma(\mathrm{K}\ 3\,n+k)\,f(n)\,\Gamma(\mathrm{K}\ n+k)\,n^5\,\mathrm{K}\ 573440\,k^3\,\Gamma(\\
&\mathrm{K}\ 4\,n)\,\Gamma(\mathrm{K}\ 3\,n+k)\,f(n)\,\Gamma(\mathrm{K}\ n+k)\,n^6+2662400\,k^2\,\Gamma(\mathrm{K}\ 4\,n)\,\Gamma(\mathrm{K}\ 3\,n\\
&+k)\,f(n)\,\Gamma(\mathrm{K}\ n+k)\,n^7\,\mathrm{K}\ 6062080\,k\,\Gamma(\mathrm{K}\ 4\,n)\,\Gamma(\mathrm{K}\ 3\,n+k)\,f(n)\,\Gamma(\mathrm{K}\ n\\
&+k)\,n^8+558\,k^5\,\Gamma(\mathrm{K}\ 4\,n)\,\Gamma(\mathrm{K}\ 3\,n+k)\,\Gamma(\mathrm{K}\ n+k)\,g(n+1)\,n^3\\
&\mathrm{K}\ 16830\,\Gamma(\mathrm{K}\ 4\,n)\,\Gamma(\mathrm{K}\ n+k)\,\Gamma(\mathrm{K}\ 3\,n+k)\,k^4\,g(n+1)\,n^4\,\mathrm{K}\ 3797136\,k\,\Gamma(\\
&\mathrm{K}\ n)\,\Gamma(\mathrm{K}\ 3\,n)\,f(n+1)\,n^7\,\Gamma(\mathrm{K}\ 4\,n+k)\,\Gamma(k)\,\mathrm{K}\ 1088\,\Gamma(\mathrm{K}\ 3\,n)\,\Gamma(\mathrm{K}\ n)\,\Gamma(\\
&\mathrm{K}\ 4\,n+k)\,\Gamma(k)\,g(n)\,k^5\,n^3+49456\,k^4\,\Gamma(\mathrm{K}\ n)\,\Gamma(\mathrm{K}\ 3\,n)\,g(n)\,n^4\,\Gamma(\mathrm{K}\ 4\,n\\
&+k)\,\Gamma(k)\,\mathrm{K}\ 558512\,k^3\,\Gamma(\mathrm{K}\ n)\,\Gamma(\mathrm{K}\ 3\,n)\,g(n)\,n^5\,\Gamma(\mathrm{K}\ 4\,n+k)\,\Gamma(k)\\
&+2282512\,k^2\,\Gamma(\mathrm{K}\ n)\,\Gamma(\mathrm{K}\ 3\,n)\,g(n)\,n^6\,\Gamma(\mathrm{K}\ 4\,n+k)\,\Gamma(k)\,\mathrm{K}\ 3086544\,k\,\Gamma(\\
&\mathrm{K}\ n)\,\Gamma(\mathrm{K}\ 3\,n)\,g(n)\,n^7\,\Gamma(\mathrm{K}\ 4\,n+k)\,\Gamma(k)\,\mathrm{K}\ 456\,\Gamma(\mathrm{K}\ 3\,n)\,\Gamma(\mathrm{K}\ n)\,\Gamma(\mathrm{K}\ 4\,n\\
&+k)\,\Gamma(k)\,f(n)\,k^5\,n^3+18696\,k^4\,\Gamma(\mathrm{K}\ n)\,\Gamma(\mathrm{K}\ 3\,n)\,f(n)\,n^4\,\Gamma(\mathrm{K}\ 4\,n+k)\,\Gamma(k)\\
&\mathrm{K}\ 156840\,k^3\,\Gamma(\mathrm{K}\ n)\,\Gamma(\mathrm{K}\ 3\,n)\,f(n)\,n^5\,\Gamma(\mathrm{K}\ 4\,n+k)\,\Gamma(k)+391656\,k^2\,\Gamma(\\
&\mathrm{K}\ n)\,\Gamma(\mathrm{K}\ 3\,n)\,f(n)\,n^6\,\Gamma(\mathrm{K}\ 4\,n+k)\,\Gamma(k)\,\mathrm{K}\ 187488\,k\,\Gamma(\mathrm{K}\ n)\,\Gamma(\\
&\mathrm{K}\ 3\,n)\,f(n)\,n^7\,\Gamma(\mathrm{K}\ 4\,n+k)\,\Gamma(k)+976\,\Gamma(\mathrm{K}\ 3\,n)\,\Gamma(\mathrm{K}\ n)\,\Gamma(\mathrm{K}\ 4\,n\\
&+k)\,\Gamma(k)\,g(n+1)\,k^5\,n^2\,\mathrm{K}\ 59240\,k^4\,\Gamma(\mathrm{K}\ n)\,\Gamma(\mathrm{K}\ 3\,n)\,g(n+1)\,n^3\,\Gamma(\mathrm{K}\ 4\,n\\
&+k)\,\Gamma(k)+807016\,k^3\,\Gamma(\mathrm{K}\ n)\,\Gamma(\mathrm{K}\ 3\,n)\,g(n+1)\,n^4\,\Gamma(\mathrm{K}\ 4\,n+k)\,\Gamma(k)\\
&\mathrm{K}\ 3789944\,k^2\,\Gamma(\mathrm{K}\ n)\,\Gamma(\mathrm{K}\ 3\,n)\,g(n+1)\,n^5\,\Gamma(\mathrm{K}\ 4\,n+k)\,\Gamma(k)\\
&+5726952\,k\,\Gamma(\mathrm{K}\ n)\,\Gamma(\mathrm{K}\ 3\,n)\,g(n+1)\,n^6\,\Gamma(\mathrm{K}\ 4\,n+k)\,\Gamma(k)\,\mathrm{K}\ 72\,\Gamma(\\
&\mathrm{K}\ 3\,n)\,\Gamma(\mathrm{K}\ n)\,\Gamma(\mathrm{K}\ 4\,n+k)\,\Gamma(k)\,f(n+1)\,k^5\,n^2+10440\,k^4\,\Gamma(\mathrm{K}\ n)\,\Gamma(\\
&\mathrm{K}\ 3\,n)\,f(n+1)\,n^3\,\Gamma(\mathrm{K}\ 4\,n+k)\,\Gamma(k)\,\mathrm{K}\ 229368\,k^3\,\Gamma(\mathrm{K}\ n)\,\Gamma(\mathrm{K}\ 3\,n)\,f(n\\
&+1)\,n^4\,\Gamma(\mathrm{K}\ 4\,n+k)\,\Gamma(k)+1471344\,k^2\,\Gamma(\mathrm{K}\ n)\,\Gamma(\mathrm{K}\ 3\,n)\,f(n+1)\,n^5\,\Gamma(\\
&\mathrm{K}\ 4\,n+k)\,\Gamma(k)\,\mathrm{K}\ 2789640\,k\,\Gamma(\mathrm{K}\ n)\,\Gamma(\mathrm{K}\ 3\,n)\,f(n+1)\,n^6\,\Gamma(\mathrm{K}\ 4\,n
\end{aligned}
$$

$+ k) \Gamma(k) \mathrm{K}\ 272 \Gamma(\mathrm{K}\ 3 n) \Gamma(\mathrm{K}\ n) \Gamma(\mathrm{K}\ 4 n + k) \Gamma(k) g(n) k^5 n^2$

$+ 21832 k^4 \Gamma(\mathrm{K}\ n) \Gamma(\mathrm{K}\ 3 n) g(n) n^3 \Gamma(\mathrm{K}\ 4 n + k) \Gamma(k) \mathrm{K}\ 355400 k^3 \Gamma($

$\mathrm{K}\ n) \Gamma(\mathrm{K}\ 3 n) g(n) n^4 \Gamma(\mathrm{K}\ 4 n + k) \Gamma(k) + 1873432 k^2 \Gamma(\mathrm{K}\ n) \Gamma($

$\mathrm{K}\ 3 n) g(n) n^5 \Gamma(\mathrm{K}\ 4 n + k) \Gamma(k) \mathrm{K}\ 3070536 k \Gamma(\mathrm{K}\ n) \Gamma(\mathrm{K}\ 3 n) g(n) n^6 \Gamma($

$\mathrm{K}\ 4 n + k) \Gamma(k) \mathrm{K}\ 72 \Gamma(\mathrm{K}\ 3 n) \Gamma(\mathrm{K}\ n) \Gamma(\mathrm{K}\ 4 n + k) \Gamma(k) f(n) k^5 n^2$

$+ 8280 k^4 \Gamma(\mathrm{K}\ n) \Gamma(\mathrm{K}\ 3 n) f(n) n^3 \Gamma(\mathrm{K}\ 4 n + k) \Gamma(k) \mathrm{K}\ 134376 k^3 \Gamma($

$\mathrm{K}\ n) \Gamma(\mathrm{K}\ 3 n) f(n) n^4 \Gamma(\mathrm{K}\ 4 n + k) \Gamma(k) + 620112 k^2 \Gamma(\mathrm{K}\ n) \Gamma($

$\mathrm{K}\ 3 n) f(n) n^5 \Gamma(\mathrm{K}\ 4 n + k) \Gamma(k) \mathrm{K}\ 820008 k \Gamma(\mathrm{K}\ n) \Gamma(\mathrm{K}\ 3 n) f(n) n^6 \Gamma($

$\mathrm{K}\ 4 n + k) \Gamma(k) + 120 \Gamma(\mathrm{K}\ 3 n) \Gamma(\mathrm{K}\ n) \Gamma(\mathrm{K}\ 4 n + k) \Gamma(k) g(n + 1) k^5 n$

$\mathrm{K}\ 17040 k^4 \Gamma(\mathrm{K}\ n) \Gamma(\mathrm{K}\ 3 n) g(n + 1) \Gamma(\mathrm{K}\ 4 n + k) \Gamma(k) n^2$

$+ 366448 k^3 \Gamma(\mathrm{K}\ n) \Gamma(\mathrm{K}\ 3 n) g(n + 1) n^3 \Gamma(\mathrm{K}\ 4 n + k) \Gamma(k)$

$\mathrm{K}\ 2310368 k^2 \Gamma(\mathrm{K}\ n) \Gamma(\mathrm{K}\ 3 n) g(n + 1) n^4 \Gamma(\mathrm{K}\ 4 n + k) \Gamma(k)$

$+ 4321128 k \Gamma(\mathrm{K}\ n) \Gamma(\mathrm{K}\ 3 n) g(n + 1) n^5 \Gamma(\mathrm{K}\ 4 n + k) \Gamma(k) + 1080 k^4 \Gamma($

$\mathrm{K}\ n) \Gamma(\mathrm{K}\ 3 n) f(n + 1) n^2 \Gamma(\mathrm{K}\ 4 n + k) \Gamma(k) \mathrm{K}\ 52176 k^3 \Gamma(\mathrm{K}\ n) \Gamma($

$\mathrm{K}\ 3 n) f(n + 1) n^3 \Gamma(\mathrm{K}\ 4 n + k) \Gamma(k) + 515472 k^2 \Gamma(\mathrm{K}\ n) \Gamma(\mathrm{K}\ 3 n) f(n$

$+ 1) n^4 \Gamma(\mathrm{K}\ 4 n + k) \Gamma(k) \mathrm{K}\ 1301112 k \Gamma(\mathrm{K}\ n) \Gamma(\mathrm{K}\ 3 n) f(n + 1) n^5 \Gamma($

$\mathrm{K}\ 4 n + k) \Gamma(k) \mathrm{K}\ 24 \Gamma(\mathrm{K}\ 3 n) \Gamma(\mathrm{K}\ n) \Gamma(\mathrm{K}\ 4 n + k) \Gamma(k) g(n) k^5 n$

$+ 4560 k^4 \Gamma(\mathrm{K}\ n) \Gamma(\mathrm{K}\ 3 n) g(n) n^2 \Gamma(\mathrm{K}\ 4 n + k) \Gamma(k) \mathrm{K}\ 123952 k^3 \Gamma($

$\mathrm{K}\ n) \Gamma(\mathrm{K}\ 3 n) g(n) n^3 \Gamma(\mathrm{K}\ 4 n + k) \Gamma(k) + 916064 k^2 \Gamma(\mathrm{K}\ n) \Gamma($

$\mathrm{K}\ 3 n) g(n) n^4 \Gamma(\mathrm{K}\ 4 n + k) \Gamma(k) \mathrm{K}\ 1914696 k \Gamma(\mathrm{K}\ n) \Gamma(\mathrm{K}\ 3 n) g(n) n^5 \Gamma($

$\mathrm{K}\ 4 n + k) \Gamma(k) + 1080 k^4 \Gamma(\mathrm{K}\ n) \Gamma(\mathrm{K}\ 3 n) f(n) n^2 \Gamma(\mathrm{K}\ 4 n + k) \Gamma(k)$

$\mathrm{K}\ 42816 k^3 \Gamma(\mathrm{K}\ n) \Gamma(\mathrm{K}\ 3 n) f(n) n^3 \Gamma(\mathrm{K}\ 4 n + k) \Gamma(k) + 335760 k^2 \Gamma($

$\mathrm{K}\ n) \Gamma(\mathrm{K}\ 3 n) f(n) n^4 \Gamma(\mathrm{K}\ 4 n + k) \Gamma(k) \mathrm{K}\ 668520 k \Gamma(\mathrm{K}\ n) \Gamma($

$\mathrm{K}\ 3 n) f(n) n^5 \Gamma(\mathrm{K}\ 4 n + k) \Gamma(k) \mathrm{K}\ 1800 k^4 \Gamma(\mathrm{K}\ n) \Gamma(\mathrm{K}\ 3 n) g(n + 1) \Gamma($

$\mathrm{K}\ 4 n + k) \Gamma(k) n + 85400 k^3 \Gamma(\mathrm{K}\ n) \Gamma(\mathrm{K}\ 3 n) g(n + 1) \Gamma(\mathrm{K}\ 4 n$

$+ k) \Gamma(k) n^2 \mathrm{K}\ 829168 k^2 \Gamma(\mathrm{K}\ n) \Gamma(\mathrm{K}\ 3 n) g(n + 1) n^3 \Gamma(\mathrm{K}\ 4 n + k) \Gamma(k)$

$+ 2063088 k \Gamma(\mathrm{K}\ n) \Gamma(\mathrm{K}\ 3 n) g(n + 1) n^4 \Gamma(\mathrm{K}\ 4 n + k) \Gamma(k) \mathrm{K}\ 4680 k^3 \Gamma($

$\mathrm{K}\ n) \Gamma(\mathrm{K}\ 3 n) f(n + 1) n^2 \Gamma(\mathrm{K}\ 4 n + k) \Gamma(k) + 97416 k^2 \Gamma(\mathrm{K}\ n) \Gamma($

$\mathrm{K}\ 3 n) f(n + 1) n^3 \Gamma(\mathrm{K}\ 4 n + k) \Gamma(k) \mathrm{K}\ 370944 k \Gamma(\mathrm{K}\ n) \Gamma(\mathrm{K}\ 3 n) f(n$

$+ 1) n^4 \Gamma(\mathrm{K}\ 4 n + k) \Gamma(k) + 360 k^4 \Gamma(\mathrm{K}\ n) \Gamma(\mathrm{K}\ 3 n) g(n) n \Gamma(\mathrm{K}\ 4 n$

$+ k) \Gamma(k) \mathrm{K}\ 22072 k^3 \Gamma(\mathrm{K}\ n) \Gamma(\mathrm{K}\ 3 n) g(n) n^2 \Gamma(\mathrm{K}\ 4 n + k) \Gamma(k)$

$+ 261680 k^2 \Gamma(\mathrm{K}\ n) \Gamma(\mathrm{K}\ 3 n) g(n) n^3 \Gamma(\mathrm{K}\ 4 n + k) \Gamma(k) \mathrm{K}\ 750000 k \Gamma($

$\mathrm{K}\ n) \Gamma(\mathrm{K}\ 3 n) g(n) n^4 \Gamma(\mathrm{K}\ 4 n + k) \Gamma(k) \mathrm{K}\ 4680 k^3 \Gamma(\mathrm{K}\ n) \Gamma($

$\mathrm{K}\ 3 n) f(n) n^2 \Gamma(\mathrm{K}\ 4 n + k) \Gamma(k) + 82296 k^2 \Gamma(\mathrm{K}\ n) \Gamma(\mathrm{K}\ 3 n) f(n) n^3 \Gamma($

$$\mathrm{K}\ 4\,n+k)\ \Gamma(k)\ \mathrm{K}\ 261648\,k\,\Gamma(\mathrm{K}\ n)\ \Gamma(\mathrm{K}\ 3\,n)\ f(n)\ n^4\,\Gamma(\mathrm{K}\ 4\,n+k)\ \Gamma(k)$$
$$+\,7800\,k^3\,\Gamma(\mathrm{K}\ n)\ \Gamma(\mathrm{K}\ 3\,n)\ g(n+1)\ \Gamma(\mathrm{K}\ 4\,n+k)\ \Gamma(k)\ n\mathrm{K}\ 159840\,k^2\,\Gamma($$
$$\mathrm{K}\ n)\ \Gamma(\mathrm{K}\ 3\,n)\ g(n+1)\ \Gamma(\mathrm{K}\ 4\,n+k)\ \Gamma(k)\ n^2+600024\,k\,\Gamma(\mathrm{K}\ n)\ \Gamma($$
$$\mathrm{K}\ 3\,n)\ g(n+1)\ n^3\,\Gamma(\mathrm{K}\ 4\,n+k)\ \Gamma(k)\ +7560\,k^2\,\Gamma(\mathrm{K}\ n)\ \Gamma(\mathrm{K}\ 3\,n)\ f(n$$
$$+\,1)\ n^2\,\Gamma(\mathrm{K}\ 4\,n+k)\ \Gamma(k)\ \mathrm{K}\ 58536\,f(n+1)\ \Gamma(\mathrm{K}\ 3\,n)\ \Gamma(\mathrm{K}\ n)\ n^3\,k\,\Gamma(\mathrm{K}\ 4\,n$$
$$+\,k)\ \Gamma(k)\ \mathrm{K}\ 1560\,k^3\,\Gamma(\mathrm{K}\ n)\ \Gamma(\mathrm{K}\ 3\,n)\ g(n)\ n\,\Gamma(\mathrm{K}\ 4\,n+k)\ \Gamma(k)$$
$$+\,40032\,k^2\,\Gamma(\mathrm{K}\ n)\ \Gamma(\mathrm{K}\ 3\,n)\ g(n)\ n^2\,\Gamma(\mathrm{K}\ 4\,n+k)\ \Gamma(k)\ \mathrm{K}\ 178296\,k\,\Gamma($$
$$\mathrm{K}\ n)\ \Gamma(\mathrm{K}\ 3\,n)\ g(n)\ n^3\,\Gamma(\mathrm{K}\ 4\,n+k)\ \Gamma(k)\ +7560\,k^2\,\Gamma(\mathrm{K}\ n)\ \Gamma($$
$$\mathrm{K}\ 3\,n)\ f(n)\ n^2\,\Gamma(\mathrm{K}\ 4\,n+k)\ \Gamma(k)\ \mathrm{K}\ 50760\,f(n)\ \Gamma(\mathrm{K}\ n)\ n^3\,\Gamma(\mathrm{K}\ 3\,n)\ k\,\Gamma($$
$$\mathrm{K}\ 4\,n+k)\ \Gamma(k)\ \mathrm{K}\ 12600\,k^2\,\Gamma(\mathrm{K}\ n)\ \Gamma(\mathrm{K}\ 3\,n)\ g(n+1)\ \Gamma(\mathrm{K}\ 4\,n+k)\ \Gamma(k)\ n$$
$$+\,96264\,g(n+1)\ \Gamma(\mathrm{K}\ n)\ \Gamma(\mathrm{K}\ 3\,n)\ k\,\Gamma(\mathrm{K}\ 4\,n+k)\ \Gamma(k)\ n^2\mathrm{K}\ 3888\,f(n$$
$$+\,1)\ \Gamma(\mathrm{K}\ 3\,n)\ \Gamma(\mathrm{K}\ n)\ n^2\,k\,\Gamma(\mathrm{K}\ 4\,n+k)\ \Gamma(k)\ +2520\,k^2\,\Gamma(\mathrm{K}\ n)\ \Gamma($$
$$\mathrm{K}\ 3\,n)\ g(n)\ n\,\Gamma(\mathrm{K}\ 4\,n+k)\ \Gamma(k)\ \mathrm{K}\ 23400\,k\,\Gamma(\mathrm{K}\ n)\ \Gamma(\mathrm{K}\ 3\,n)\ g(n)\ n^2\,\Gamma($$
$$\mathrm{K}\ 4\,n+k)\ \Gamma(k)\ \mathrm{K}\ 3888\,f(n)\ \Gamma(\mathrm{K}\ n)\ n^2\,\Gamma(\mathrm{K}\ 3\,n)\ k\,\Gamma(\mathrm{K}\ 4\,n+k)\ \Gamma(k)$$
$$+\,6480\,g(n+1)\ \Gamma(\mathrm{K}\ n)\ \Gamma(\mathrm{K}\ 3\,n)\ k\,\Gamma(\mathrm{K}\ 4\,n+k)\ \Gamma(k)\ n\mathrm{K}\ 1296\,k\,\Gamma($$
$$\mathrm{K}\ n)\ \Gamma(\mathrm{K}\ 3\,n)\ g(n)\ n\,\Gamma(\mathrm{K}\ 4\,n+k)\ \Gamma(k)\big)\ (\mathrm{K}\ 1)^k\big)\Big/\big(8\,n\,(n+1)\ \Gamma(k)\ \Gamma($$
$$\mathrm{K}\ 4\,n+k)\ (\mathrm{K}\ 3\,n+k\mathrm{K}\ 1)\ \Gamma(\mathrm{K}\ 3\,n)\ \Gamma(\mathrm{K}\ n)\ (4\,n+1)\ (2\,n+1)\ (4\,n+3)\big)$$

At this point, this seems hopeless...