Integration in D-modules

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MathemAmplitudes, September 25, Mainz

Algebraic integration

- Can we compute integrals

$$\int_\Gamma f(x_1, \dots, x_n) \mathrm{d}x_1 \cdots \mathrm{d}x_n$$

  using algebraic relations?

- Yes!
  D-module theory provides an expressive and effective framework.

- Saito, M., Sturmfels, B., & Takayama, N. (2000). *Gröbner deformations of hypergeometric differential equations* (Vol. 6). Springer-Verlag

- How to compute them faster?

What does *computing* mean? I

✔ **Computing master integrals**

Given $f_1(\mathbf{x}), \ldots, f_r(\mathbf{x})$, find linear relations

$$a_1 \int f_1(\mathbf{x})\mathrm{d}\mathbf{x} + \cdots + a_r \int f_r(\mathbf{x})\mathrm{d}\mathbf{x} = 0.$$

What does *computing* mean? II

✔ **Picard–Fuchs equations**

Given $y(t) = \int f(t, \mathbf{x}) \mathrm{d}\mathbf{x}$, find a differential equation

$$a_r(t) y^{(r)}(t) + \cdots + a_1(t) y'(t) + a_0(t) y(t) = 0.$$

This reduces to finding a relation between several integrals:

- ∗ Use $\mathbb{C}(t)$ as the base field.
- ∗ Find a relation between the integrals

$$y(t) = \int f(t, \mathbf{x}) \mathrm{d}\mathbf{x},$$

$$y'(t) = \int \frac{\partial f(t, \mathbf{x})}{\partial t} \mathrm{d}\mathbf{x},$$

$$y''(t) = \int \frac{\partial^2 f(t, \mathbf{x})}{\partial t^2} \mathrm{d}\mathbf{x}, \ldots$$

What does *computing* not mean?

Numerical evaluation
Given $f(\mathbf{x})$, compute a numerical approximation of

$$\int f(\mathbf{x})\mathrm{d}\mathbf{x}.$$

Finding master integrals, or computing differential equations, *may* help, but it is only a step.

D-modules

Definition

A D-module M is a space of *functions* of $x_1, \ldots, x_n$ in which you can:

* multiply by polynomial functions in $\mathbf{x}$
* differentiate with respect to $\mathbf{x}$

Definition (alternative)

Let $D = \mathbb{C}[x_1, \ldots, x_n]\langle \partial_1, \ldots, \partial_n \rangle$ be the nth Weyl algebra:

* $[x_i, x_j] = [\partial_i, \partial_j] = [\partial_i, x_j] = 0$
* $[\partial_i, x_i] = 1$ (Leibniz' rule)

A D-module is a D-module.

Examples

* Polynomials in $\mathbf{x}$
* Holomorphic functions on an open subset of $\mathbb{C}^n$
* Schwartz distributions on $\mathbb{R}^n$

Holonomic D-modules

Definition

A D-module M is *holonomic* if:

1. it is finitely generated (as a module over D)
2. for any $f \in M$, as $s \to \infty$,

$$\dim_{\mathbb{C}} \operatorname{Vect} \left\{ \mathbf{x}^{\alpha} \partial^{\beta} \cdot f \;\middle|\; |\alpha| + |\beta| \leq s \right\} = O(s^n).$$

Examples

- ✔ $M = \mathbb{C}[x_1, \ldots, x_n]$
- ✘ $M = \mathbb{C}(x_1, \ldots, x_n)$
- ✔ $M = \mathbb{C}[x_1, \ldots, x_n, F^{-1}] = \left\{ aF^{-k} \;\middle|\; a \in \mathbb{C}[\mathbf{x}], k \geq 0 \right\}$
- ✔ $M = \mathbb{C}[x_1, \ldots, x_n, F^{-1}]F^{\varepsilon}$
- ✘ $M = D \cdot \frac{1}{1+\exp(x)}$

Computer representation of holonomic D-modules

$$M \simeq \frac{D^{m}}{Dg_1 + \cdots + Dg_s},$$

with:

- m: the number of generators of M (we can always assume $m = 1$, but this is not free)
- $Dg_1 + \cdots + Dg_s$: the module of relations between the generators

Examples

- ✓ $\mathbb{C}[\mathbf{x}] \simeq D/(D\partial_1 + \cdots + D\partial_n)$
- ✓ $\mathbb{C}[\mathbf{x}]e^{f(\mathbf{x})} \simeq D/\left(\sum_i D\left(\partial_i - \frac{\partial f}{\partial x_i}\right)\right)$
- ! $D \cdot \frac{1}{x^2-y^3} \simeq D/\big(D(3x\partial_x + 2y\partial_y + 6) + D(3y^2\partial_x + 2x\partial_y)\big)$

Rational functions as D-modules

Let $f = 1 - (1-xy)z - xyz(1-x)(1-y)(1-z)$ (Beukers & Peters, 1984) and consider $M = \mathbb{C}[x,y,z,f^{-1}]$.

- ⚠ It is not finitely generated as a $\mathbb{C}[x,y,z]$-module.
- 💡 It is generated by f^{-2} as a D-module:

$$\mathbb{C}[x,y,z,f^{-1}] = D \cdot f^{-2} \simeq D/I$$

where I is the left ideal generated by

1. $x^2z^2\partial_x - y^2z^2\partial_y - 2xz^3\partial_z + 2yz^3\partial_z - 2x^2z\partial_x - xz^2\partial_x + 2y^2z\partial_y + yz^2\partial_y + 4xz^2\partial_z - 4yz^2\partial_z - 4xz^2 + 4yz^2 + x^2\partial_x + 3xz\partial_x - y^2\partial_y - 3yz\partial_y - 2xz\partial_z + 2yz\partial_z + 4xz - 4yz$
2. $-2y^2z^3\partial_z + 3xy^2z\partial_y - y^3z\partial_y - 2xyz^2\partial_z + 4y^2z^2\partial_z + 2yz^3\partial_z - 8y^2z^2 + 2x^2z\partial_x - 2xyz\partial_x - xy^2\partial_y + y^3\partial_y - 3xyz\partial_y + 5y^2z\partial_y + 2xyz\partial_z - 2y^2z\partial_z - xz^2\partial_z - 5yz^2\partial_z + 4xyz + 8y^2z + 8yz^2 - x^2\partial_x - xy\partial_x + 2xz\partial_x + xy\partial_y - y^2\partial_y - 4yz\partial_y + xz\partial_z + 3yz\partial_z - 2z^2\partial_z - 2xz - 6yz - 2x\partial_x - 2z\partial_x + 2y\partial_y + 2z\partial_z - 4z + 2\partial_x$
3. $-4xyz^3\partial_z - 5xy^2z\partial_y - 6y^2z^2\partial_y + 10xyz^2\partial_z + 12yz^3\partial_z - 16xyz^2 - 3x^2z\partial_x + 3xy^2\partial_y + 5xyz\partial_y + 12y^2z\partial_y + 6yz^2\partial_y - 6xyz\partial_z + xz^2\partial_z - 24yz^2\partial_z - 6z^3\partial_z + 12xyz + 24yz^2 + 2x^2\partial_x + 5xz\partial_x - 3xy\partial_y - 6y^2\partial_y - 16yz\partial_y - xz\partial_z + 12yz\partial_z + 9z^2\partial_z + 2xz - 24yz - 12z^2 + x\partial_x + y\partial_y - 3z\partial_z + 6z$
4. $-2x^2z^3\partial_z - x^3z\partial_x + 3xy^2z\partial_y + 4x^2z^2\partial_z - 2xyz^2\partial_z + 2xz^3\partial_z - 8x^2z^2 + x^3\partial_x + 7x^2z\partial_x - xy^2\partial_y - 5xyz\partial_y - 2x^2z\partial_z + 2xyz\partial_z - 6xz^2\partial_z + 8x^2z + 4xyz + 8xz^2 - 2x^2\partial_x - 6xz\partial_x + 4yz\partial_y + 4xz\partial_z - 2z^2\partial_z - 8xz + 2x\partial_x - 2y\partial_y - 2z\partial_y + 2z\partial_z - 4z + 2\partial_y$
5. $-y^3z^2\partial_y + 3xy^2z\partial_y + y^3z\partial_y + 2y^2z^2\partial_y - 2xyz^2\partial_z - yz^3\partial_z - 4y^2z^2 + 2x^2z\partial_x - xyz\partial_x - xy^2\partial_y - 3xyz\partial_y - yz^2\partial_y + 2xyz\partial_z - xz^2\partial_z + yz^2\partial_z + z^3\partial_z + 4xyz + 4y^2z + 2yz^2 - x^2\partial_x + xz\partial_x + xy\partial_y - yz\partial_y + xz\partial_z - 3z^2\partial_z - 2xz + 2z^2 - 2x\partial_x - 2z\partial_x + 2y\partial_y + 2z\partial_z - 4z + 2\partial_x$
6. $-8xy^2z^2\partial_z - 4xy^3\partial_y - 25xy^2z\partial_y - 15y^3z\partial_y + 8xy^2z\partial_z + 26xyz^2\partial_z + 18y^2z^2\partial_z - 32xy^2z - 16x^2z\partial_x + 15xy^2\partial_y + 9y^3\partial_y + 25xyz\partial_y + 23y^2z\partial_y - 26xyz\partial_z - 18y^2z\partial_z + 7xz^2\partial_z - 19yz^2\partial_z + 4xyz + 12y^2z + 9x^2\partial_x - 5xy\partial_x + 16xz\partial_x - 11xy\partial_y - y^2\partial_y - 8yz\partial_y - 7xz\partial_z + 7yz\partial_z - 8z^2\partial_z + 14xz - 14yz + 8x\partial_x - 24y\partial_y + 8z\partial_z - 16z - 12\partial_x$
7. $-2x^2yz^2\partial_z - 4x^3z\partial_x - x^2y^2\partial_y - 6xy^2z\partial_y + 2x^2yz\partial_z + 5x^2z^2\partial_z + 6xyz^2\partial_z - 8x^2yz + 2x^3\partial_x + 2x^2z\partial_x + x^2y\partial_y + 3xy^2\partial_y + 6xyz\partial_y - 5x^2z\partial_z - 6xyz\partial_z - 3xz^2\partial_z + 4x^2z + 2x^2\partial_x + 2xz\partial_x - 4xy\partial_y - 2z^2\partial_z - 6x\partial_x + 2y\partial_y + 2z\partial_z - 4z - 3\partial_y$
8. $-4xy^3z\partial_y - 7xy^2z\partial_y - 5y^3z\partial_y + 10xyz^2\partial_z + 6y^2z^2\partial_z - 16xy^2z - 7x^2z\partial_x + 5xy^2\partial_y + 3y^3\partial_y + 11xyz\partial_y + 8y^2z\partial_y - 10xyz\partial_z - 6y^2z\partial_z + 3xz^2\partial_z - 7yz^2\partial_z + 4xyz + 4y^2z + 4x^2\partial_x - 3xy\partial_x + 7xz\partial_x - 5xy\partial_y - 3yz\partial_y - 3xz\partial_z + 3yz\partial_z - 4z^2\partial_z + 6xz - 6yz + 4x\partial_x - 12y\partial_y + 4z\partial_z - 8z - 4\partial_x$
9. $-x^2y^2z\partial_y - 2x^3z\partial_x + x^2yz\partial_y - 2xy^2z\partial_y + 2x^2z^2\partial_z + 2xyz^2\partial_z - 4x^2yz + x^3\partial_x + x^2z\partial_x + xy^2\partial_y + 2xyz\partial_y - 2x^2z\partial_z - 2xyz\partial_z - xz^2\partial_z + 2x^2z + x^2\partial_x + xz\partial_x - 2xy\partial_y - z^2\partial_z - 3x\partial_x + y\partial_y + z\partial_z - 2z - \partial_y$
10. $4x^3z\partial_x^2 + 6xy^2z\partial_x\partial_y - 6xy^2z\partial_y^2 - 4y^3z\partial_y^2 - 4xyz^2\partial_x\partial_z + 4xyz^2\partial_y\partial_z - 8xz^3\partial_z^2 + 8yz^3\partial_z^2 - 2x^3\partial_x^2 - 2x^2z\partial_x^2 - 2xy^2\partial_x\partial_y - 4x^2z\partial_x\partial_y - 6xyz\partial_x\partial_y + 2xy^2\partial_y^2 + 2y^3\partial_y^2 + 6xyz\partial_y^2 + 6y^2z\partial_y^2 - 4x^2z\partial_x\partial_z + 4xyz\partial_x\partial_z - 2xz^2\partial_x\partial_z - 4xyz\partial_y\partial_z + 4y^2z\partial_y\partial_z + 2xz^2\partial_y\partial_z + 16xz^2\partial_z^2 - 16yz^2\partial_z^2 + 21x^2z\partial_x + 8xyz\partial_x - 2x^2\partial_x^2 - 2xz\partial_x^2 - 20xyz\partial_y - 15y^2z\partial_y + 2x^2\partial_x\partial_y + 2xy\partial_x\partial_y + 4xz\partial_x\partial_y - 2xy\partial_y^2 - 2yz\partial_y^2 - 46xz^2\partial_z + 46yz^2\partial_z + 4x^2\partial_x\partial_z + 12xz\partial_x\partial_z + 2z^2\partial_x\partial_z - 4y^2\partial_y\partial_z - 2xz\partial_y\partial_z - 10yz\partial_y\partial_z - 2z^2\partial_y\partial_z - 8xz\partial_z^2 + 8yz\partial_z^2 - 15x^2\partial_x - 17xz\partial_x + 6x\partial_x^2 + 4xy\partial_y + 13y^2\partial_y + 10xz\partial_y + 15yz\partial_y + 2x\partial_x\partial_y - 2y\partial_x\partial_y - 6y\partial_y^2 + 54xz\partial_z - 54yz\partial_z - 2z^2\partial_z - 2z\partial_x\partial_z + 2z\partial_y\partial_z - 44xz + 44yz + 6x\partial_x + 2z\partial_x - 2x\partial_y - 8y\partial_y - 6z\partial_y - 8x\partial_z + 8y\partial_z + 2z\partial_z + 16x - 16y - 4z + 6\partial_x - 6\partial_y$
11. $-18xy^2z\partial_x\partial_y + 48y^4\partial_y^2 - 306xy^2z\partial_y^2 + 228y^3z\partial_y^2 + 60xyz^2\partial_x\partial_z + 288xy^2z\partial_y\partial_z + 204xyz^2\partial_y\partial_z - 216y^2z^2\partial_y\partial_z - 192xyz^2\partial_z^2 + 56xz^3\partial_z^2 - 280yz^3\partial_z^2 - 64z^4\partial_z^2 + 14x^3\partial_x^2 + 182x^2z\partial_x^2 - 584xy^2z\partial_y - 168y^3z\partial_y - 192y^2z^2\partial_y - 90xy^2\partial_x\partial_y - 204x^2z\partial_x\partial_y + 210xyz\partial_x\partial_y + 102xy^2\partial_y^2 - 58y^3\partial_y^2 + 306xyz\partial_y^2 - 430y^2z\partial_y^2 + 464xyz^2\partial_z + 144y^2z^2\partial_z + 384yz^3\partial_z + 220x^2z\partial_x\partial_z - 12xyz\partial_x\partial_z - 702xz^2\partial_x\partial_z - 96xy^2\partial_y\partial_z - 492xyz\partial_y\partial_z - 116y^2z\partial_y\partial_z + 102xz^2\partial_y\partial_z + 72yz^2\partial_y\partial_z + 192xyz\partial_z^2 - 208xz^2\partial_z^2 + 656yz^2\partial_z^2 + 928z^3\partial_z^2 - 289x^2z\partial_x + 168xyz\partial_x - 70x^2\partial_x^2 - 182xz\partial_x^2 + 520xy^2\partial_y + 408y^3\partial_y - 436xyz\partial_y + 1123y^2z\partial_y + 192yz^2\partial_y + 102x^2\partial_x\partial_y - 538xy\partial_x\partial_y + 76xz\partial_x\partial_y - 128yz\partial_x\partial_y - 102xy\partial_y^2 + 512y^2\partial_y^2 + 202yz\partial_y^2 - 464xyz\partial_z - 144y^2z\partial_z + 646xz^2\partial_z - 3070yz^2\partial_z - 704z^3\partial_z - 124x^2\partial_x\partial_z + 636xz\partial_x\partial_z + 438z^2\partial_x\partial_z + 96xy\partial_y\partial_z + 188y^2\partial_y\partial_z - 102xz\partial_y\partial_z + 330yz\partial_y\partial_z - 150z^2\partial_y\partial_z + 152xz\partial_z^2 - 376yz\partial_z^2 - 1536z^2\partial_z^2 - 480xyz - 96y^2z + 768yz^2 + 459x^2\partial_x - 312xy\partial_x - 431xz\partial_x + 98x\partial_x^2 - 316xy\partial_y - 681y^2\partial_y + 510xz\partial_y - 1291yz\partial_y + 278x\partial_x\partial_y + 266y\partial_x\partial_y + 64z\partial_x\partial_y - 706y\partial_y^2 + 192xy\partial_z - 1086xz\partial_z + 2966yz\partial_z + 4634z^2\partial_z - 352x\partial_x\partial_z - 630z\partial_x\partial_z + 32y\partial_y\partial_z + 150z\partial_y\partial_z + 672z\partial_z^2 + 384xy + 384y^2 + 252xz - 2860yz - 1152z^2 + 58x\partial_x + 694z\partial_x - 102x\partial_y + 928y\partial_y - 98z\partial_y - 176\partial_x\partial_y + 152x\partial_z - 376y\partial_z - 4330z\partial_z + 192\partial_x\partial_z - 304x + 752y + 3284z - 478\partial_x - 706\partial_y + 672\partial_z - 1344$
12. $-2x^2y^3\partial_y - x^3z\partial_x + 3x^2y^2\partial_y + 2xy^3\partial_y + 7xy^2z\partial_y + x^2z^2\partial_z - 6xyz^2\partial_z - 8x^2y^2 + x^3\partial_x + 7x^2z\partial_x - x^2y\partial_y - 2xy^2\partial_y - 7xyz\partial_y - x^2z\partial_z + 4xyz\partial_z - 5xz^2\partial_z + 8x^2y + 8xy^2 + 4xyz - 2x^2\partial_x - 6xz\partial_x + 4xz\partial_z + 6z^2\partial_z - 8xz + 2x\partial_x + 4y\partial_y - 10z\partial_z + 12z - \partial_y + 4\partial_z - 8$
13. $4xy^3\partial_y^2 + 25xy^2z\partial_y^2 + 15y^3z\partial_y^2 - 12xy^2z\partial_y\partial_z - 18xyz^2\partial_y\partial_z - 34y^2z^2\partial_y\partial_z + 8xyz^2\partial_z^2 - 8xz^3\partial_z^2 + 32yz^3\partial_z^2 + 16xy^2z\partial_y + 16x^2z\partial_x\partial_y - 15xy^2\partial_y^2 - 9y^3\partial_y^2 - 25xyz\partial_y^2 - 23y^2z\partial_y^2 - 16xyz^2\partial_z - 12x^2z\partial_x\partial_z + 4xy^2\partial_y\partial_z + 30xyz\partial_y\partial_z + 50y^2z\partial_y\partial_z - 7xz^2\partial_y\partial_z + 35yz^2\partial_y\partial_z - 8xyz\partial_z^2 + 20xz^2\partial_z^2 - 64yz^2\partial_z^2 - 16z^3\partial_z^2 + 8xy^2\partial_y + 62xyz\partial_y + y^2z\partial_y - 9x^2\partial_x\partial_y + 5xy\partial_x\partial_y - 16xz\partial_x\partial_y + 11xy\partial_y^2 + y^2\partial_y^2 + 8yz\partial_y^2 + 16xyz\partial_z - 66xz^2\partial_z + 124yz^2\partial_z + 8x^2\partial_x\partial_z + 20xz\partial_x\partial_z - 4xy\partial_y\partial_z - 16y^2\partial_y\partial_z + 7xz\partial_y\partial_z - 55yz\partial_y\partial_z + 8z^2\partial_y\partial_z - 12xz\partial_z^2 + 32yz\partial_z^2 + 28z^2\partial_z^2 - 12x^2\partial_x - 26xy\partial_y + 5y^2\partial_y - 39xz\partial_y - 8x\partial_x\partial_y + 24y\partial_y^2 - 8xy\partial_z + 90xz\partial_z - 156yz\partial_z - 61z^2\partial_z + 4x\partial_x\partial_z + 4y\partial_y\partial_z - 8z\partial_y\partial_z - 12z\partial_z^2 + 16xy - 36xz + 104yz + 25x\partial_x + 11x\partial_y - 46y\partial_y + 24z\partial_y + 12\partial_x\partial_y - 12x\partial_z + 32y\partial_z + 73z\partial_z + 24x - 64y - 50z + 24\partial_y - 12\partial_z + 24$
14. $12x^4\partial_x^2 - 162xy^2z\partial_x\partial_y + 81xy^2z\partial_y^2 + 57y^3z\partial_y^2 + 108xyz^2\partial_x\partial_z + 72xy^2z\partial_y\partial_z - 42xyz^2\partial_y\partial_z - 54y^2z^2\partial_y\partial_z - 48xyz^2\partial_z^2 - 64xz^3\partial_z^2 + 8yz^3\partial_z^2 - 16z^4\partial_z^2 - 42x^3z\partial_x + 14x^3\partial_x^2 - 130x^2z\partial_x^2 - 146xy^2z\partial_y - 48y^2z^2\partial_y + 54xy^2\partial_x\partial_y + 48x^2z\partial_x\partial_y + 210xyz\partial_x\partial_y - 51xy^2\partial_y^2 - 25y^3\partial_y^2 - 81xyz\partial_y^2 - 31y^2z\partial_y^2 + 36x^2z^2\partial_z + 116xyz^2\partial_z + 96yz^3\partial_z - 32x^2z\partial_x\partial_z - 108xyz\partial_x\partial_z + 18xz^2\partial_x\partial_z - 24xy^2\partial_y\partial_z - 18xyz\partial_y\partial_z + 58y^2z\partial_y\partial_z - 15xz^2\partial_y\partial_z - 135yz^2\partial_y\partial_z + 48xyz\partial_z^2 + 128xz^2\partial_z^2 - 16yz^2\partial_z^2 + 232z^3\partial_z^2 + 102x^3\partial_x - 97x^2z\partial_x - 216xyz\partial_x + 182x^2\partial_x^2 + 130xz\partial_x^2 + 130xy^2\partial_y + 464xyz\partial_y + 148y^2z\partial_y + 48yz^2\partial_y - 33x^2\partial_x\partial_y - 211xy\partial_x\partial_y - 80xz\partial_x\partial_y - 32yz\partial_x\partial_y + 51xy\partial_y^2 - 13y^2\partial_y^2 - 26yz\partial_y^2 - 36x^2z\partial_z - 116xyz\partial_z - 416xz^2\partial_z - 190yz^2\partial_z - 176z^3\partial_z + 20x^2\partial_x\partial_z - 66z^2\partial_x\partial_z + 24xy\partial_y\partial_z - 4y^2\partial_y\partial_z + 15xz\partial_y\partial_z + 201yz\partial_y\partial_z + 138z^2\partial_y\partial_z - 64xz\partial_z^2 + 8yz\partial_z^2 - 384z^2\partial_z^2 - 24x^2z - 120xyz + 192yz^2 + 105x^2\partial_x - 5xz\partial_x - 262x\partial_x^2 - 310xy\partial_y - 84y^2\partial_y - 111xz\partial_y - 385yz\partial_y + 38x\partial_x\partial_y + 98y\partial_x\partial_y + 16z\partial_x\partial_y + 110y\partial_y^2 + 48xy\partial_z + 384xz\partial_z + 86yz\partial_z + 1199z^2\partial_z + 8x\partial_x\partial_z + 66z\partial_x\partial_z - 88y\partial_y\partial_z - 186z\partial_y\partial_z + 168z\partial_z^2 + 96x^2 + 96xy - 240xz - 412yz - 288z^2 + 289x\partial_x - 2z\partial_x + 51x\partial_y - 2y\partial_y + 250z\partial_y - 44\partial_x\partial_y - 64x\partial_z + 8y\partial_z - 1123z\partial_z + 48\partial_y\partial_z + 128x - 16y + 902z - 262\partial_x - 34\partial_y + 168\partial_z - 336$

- Singular (Andres et al., 2010) or Macaulay2 (Leykin, 2002).
- ? Do we deal correctly with poles?

Integrals

Definition
An *integral* $\int$ on a D-module M is a linear form $\int : M \to \mathbb{C}$ such that for any $f_1, \dots, f_n \in M$,

$$\int (\partial_1 f_1 + \dots + \partial_n f_n)\, \mathrm{d}\mathbf{x} = 0.$$

Definition
The *integral of a D-module M* is the $\mathbb{C}$-linear space

$$\int M \triangleq \frac{M}{\partial_1 M + \dots + \partial_n M}.$$

(So an integral on M is a linear form $\int M \to \mathbb{C}$.)

Theorem (Kashiwara)
If M is holonomic, $\int M$ is finite dimensional.

Integrals with boundaries

In general

$$\int_\Gamma (\partial_1 f_1 + \cdots + \partial_n f_n)\, \mathrm{d}\mathbf{x} = \int_{\partial\Gamma} [\ldots] \neq 0.$$

* In the D-module approach, we need $\partial\Gamma = \emptyset$.
* Even if $\partial\Gamma \neq \emptyset$, you still learn something useful from your integral by studying the consequences of

  $$\int (\partial_1 f_1 + \cdots + \partial_n f_n)\, \mathrm{d}\mathbf{x} = 0.$$

* The approach of Oaku (2013) may apply:

  $$\int_\Gamma f \mathrm{d}\mathbf{x} = \int_{\mathbb{R}^n} \mathbb{1}_\Gamma f \mathrm{d}\mathbf{x},$$

  if you can work with Schwartz distributions.

Takayama's algorithm for integration

Let $M = D/I$ a holonomic D-module with $I \subseteq D$ a left ideal.

$$\int M = \frac{M}{\partial_1 M + \cdots + \partial_n M} \simeq \frac{D}{I + \partial_1 D + \cdots + \partial_n D}.$$

Algorithm (Takayama, 1990)

1. Pick some r and some s large enough.
2. Compute the finite dimensional vector space

   $$V_{r+s} = I \cap D_{r+s} + \partial_1 D_{r+s-1} + \cdots + \partial_n D_{r+s-1},$$

   where $D_k = \mathrm{Vect}\left\{ \mathbf{x}^\alpha \partial^\beta \,\middle|\, |\alpha| + |\beta| \leq k \right\}$.
3. Return $D_r/(V_{r+s} \cap D_r)$.

Correctness. There is a canonical map $D_r/(V_{r+s} \cap D_r) \to \int M$.
It is surjective if $r \gg 0$ and injective if $s \gg 0$.

Issues with Takayama's algorithm

? How to choose r and s?

Important theoretical question, but in practice:

- Choose r large enough so that D_r contains what you want
- Increase s until you discover no new relations

⌛ Linear algebra in large dimension

A priori bounds

- Let $w(\mathbf{x}^\alpha \partial^\beta) = |\alpha| - |\beta|$ be the *weight* of a monomial.
- Let $w(g) = \max \{w(m) \mid m \text{ is a monomial in } g\}$
- Let $\theta = \sum_i x_i \partial_i$.

NB $\theta \cdot \mathbf{x}^\alpha = |\alpha| \mathbf{x}^\alpha$ and $\theta = -n + \sum_i \partial_i x_i$.

[b-function theory] There is some $g \in I$ and $b \in \mathbb{C}[s]$ such that

$$g = b(\theta) + \text{terms of negative weights.}$$

For any $\mathbf{x}^\alpha$,

$$\mathbf{x}^\alpha g = b(-|\alpha| - n)\mathbf{x}^\alpha + \text{lower order terms} + \sum_i \partial_i (\ldots)$$

In Takayama's algorithm,
if $r \geq \max \{k \in \mathbb{N} \mid b(-k-n) = 0\}$
then $D_r / (V_{r+s} \cap D_r) \to \int M$ surjective.
(It is also injective, but this is more subtle.)

Integration of rational functions

Let $a, f \in \mathbb{C}[\mathbf{x}]$ homogeneous, $k > 0$ with $\deg a + n = k \deg f$.

The following questions are equivalent:

* Is $\int \frac{a}{f^k} \mathrm{d}\mathbf{x} = 0$?
* Is $\int a\, e^f \mathrm{d}\mathbf{x} = 0$?
* Does $a\, e^f = \sum_i \frac{\partial}{\partial x_i} \left(u_i\, e^f\right)$ for some polynomials u_i?
* Does $a \in I + \partial_1 D + \cdots + \partial_n D$ where $I = \sum_i D\left(\partial_i - \frac{\partial f}{\partial x_i}\right)$?

Griffiths–Dwork reduction

- The reduction step modulo $I + \partial_1 D + \cdots + \partial_n D$:

$$\begin{aligned} \sum_i b_i \frac{\partial f}{\partial x_i} &\equiv \sum_i b_i \partial_i \pmod{I} \\ &= \sum_i \partial_i b_i - \frac{\partial b_i}{\partial x_i} \quad \text{(commutation rule in } D\text{)} \\ &\equiv -\sum_i \frac{\partial b_i}{\partial x_i} \pmod{\partial_1 D + \cdots + \partial_n D} \end{aligned}$$

def $GD(a)$**:**

 while True:

  $r + \sum_i b_i \frac{\partial f}{\partial x_i} \leftarrow a$ [multivariate polynomial division]

  if $a = r$**:**

   return a

  $a \leftarrow r - \sum_i \frac{\partial b_i}{\partial x_i}$

Griffiths–Dwork reduction

Theorem (Dwork, 1962, 1964; Griffiths, 1969)
If $\{f = 0\}$ is *smooth* in $\mathbb{P}^{n-1}$, then $GD(a) = 0$ if and only if $\int a\, e^f = 0$.

More often than not, $\{f = 0\}$ is not smooth...

Syzygies give more relations! (Lairez, 2016)

$$\sum_i b_i \frac{\partial f}{\partial x_i} = 0 \Rightarrow \sum_i \frac{\partial b_i}{\partial x_i} \in I + \sum_i \partial_i D.$$

And only *nontrivial* syzygies may give new relations.

A Griffiths–Dwork reduction for holonomic ideal?

Let $M = D/I$ be a holonomic D-module. We want to compute in

$$\int M \simeq \frac{D}{\underbrace{I}_{\text{left ideal}} + \underbrace{\partial_1 D + \cdots + \partial_n D}_{\text{right ideal}}}.$$

def $\mathit{GenGD}(a)$**:** [Brochet, Chyzak, and Lairez, 2025]

  while a **is reducible:**

    $a \leftarrow \mathit{LeftRem}(a, I)$

    $a \leftarrow \mathit{RightRem}(a, \partial_1 D + \cdots + \partial_n D)$

  return a

Fixing *GenGD*

- An element $a \in D$ is *irreducible* if $GenGD(a) = a$.
  Irreducible elements forms a linear subspace $E \subseteq D$.

- We may have missed relations!
  They form the subspace $E \cap (I + \sum_i \partial_i D)$.

- To fix *GenGD*, we need to compute (or rather enumerate) a generating set of $E \cap (I + \sum_i \partial_i D)$.
- The missing relations come from monomials $m \in D$ that are reducible both by I and by $\sum_i \partial_i D$ (critical pairs)
- Among this critical pairs, many can be eliminated *a priori*.

Fixing *GenGD*

- ✔ *GenGD* Coincides with Griffiths–Dwork reduction when $M = \mathbb{C}[\mathbf{x}]e^f$.
- ! Does not reduce every derivatives to zero, but this can be fixed.
- ✔ After taking into accounts these critical pairs, we obtain all the relations.
- https://github.com/HBrochet/MultivariateCreativeTelescoping.jl

Is it fast?

- ✘ Sometimes not.
  For example: $I = D\partial_1 + \cdots + D\partial_n$, so that $D/I = \mathbb{C}[\mathbf{x}]$. We just want to integrate polynomials. This should be trivial, but *GenGD* is just the identity map...
- ✔ Sometimes yes.
  For example: computation of the generating series of the number of 8-regular graphs on k vertices.

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