Generalized Hermite reduction, Creative telescoping, and Definite integration of differentially finite functions

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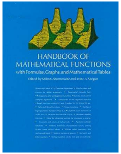
Automatic computation of sums and integrals

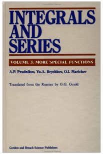
$$\sum_{i=0}^{n}\sum_{j=0}^{n}\binom{i+j}{i}^2\binom{4n-2i-2j}{2n-2i} = (2n+1)\binom{2n}{n}^2 \qquad \text{(Blodgelt, 1990)}$$

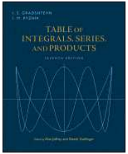
$$\int_0^{+\infty} x\, J_1(ax)\, I_1(ax)\, Y_0(x)\, K_0(x)\, dx = -\frac{\ln(1-a^4)}{2\pi a^2} \qquad \text{(Glasser, Montaldi, 1994)}$$

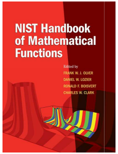
$$\int_{-1}^{1} \frac{e^{-px}\, T_n(x)}{\sqrt{1-x^2}}\, dx = (-1)^n \pi\, I_n(p)$$

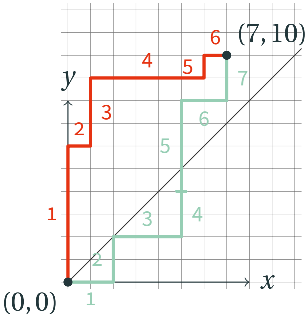
$$\sum_{j=0}^{n}\sum_{i=0}^{n-j} \frac{q^{(i+j)^2+j^2}}{(q;q)_{n-i-j}(q;q)_i(q;q)_j} = \sum_{k=-n}^{n} \frac{(-1)^k q^{7/2k^2+1/2k}}{(q;q)_{n+k}(q;q)_{n-k}} \qquad \text{(Paule, 1985)}$$











$$u_n = \#\left\{\text{rook paths from } (0,\ldots,0) \text{ to } (n,\ldots,n) \text{ in } \mathbb{N}^d\right\}$$

- dimension 2

$$9nu_n + (-14-10n)u_{n+1} + (2+n)u_{n+2} = 0$$

- dimension 3

$$\begin{aligned}&-192n^2(1+n)(88+35n)u_n\\&+(1+n)(54864+100586n+59889n^2+11305n^3)u_{n+1}\\&-(2+n)(43362+63493n+30114n^2+4655n^3)u_{n+2}\\&+2(2+n)(3+n)^2(53+35n)u_{n+3} = 0\end{aligned}$$

- dimension 4

$$\begin{aligned}&5000n^3(1+n)^2(2705080+3705334n+1884813n^2+421590n^3+34983n^4)u_n\\&-(1+n)^2(80002536960+282970075928n+\cdots+6386508141n^6+393838614n^7)u_{n+1}\\&+2(2+n)(143370725280+500351938492n+\cdots+2636030943n^7+131501097n^8)u_{n+2}\\&-(3+n)^2(26836974336+80191745800n+100381179794n^2+\cdots+44148546n^7)u_{n+3}\\&+2(3+n)^2(4+n)^3(497952+1060546n+829941n^2+281658n^3+34983n^4)u_{n+4} = 0\end{aligned}$$

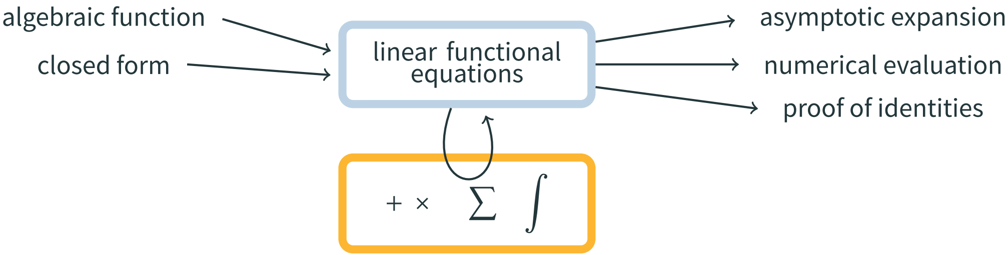
The problem of definite integration

input $F(t_1, \ldots, t_n, x)$

output $G(t_1, \ldots, t_n) = \int_D F(t_1, \ldots, t_n, x) \mathrm{d}x$

assumption $\displaystyle\int_D \frac{\partial}{\partial x}(\ldots)\mathrm{d}x = 0$

data structure linear functional equations



Previous works

input $F(t,x) \in \mathbb{Q}(t,x)$

output A differential equation for $G(t) = \oint F(t,x)\mathrm{d}x$

references Ostrogradsky (1845), Hermite (1872), Bostan, Chen, Chyzak, Li (2010a)

$$\begin{aligned}
F &= \frac{A_0}{B} + \frac{\partial}{\partial x} H_0 \\
\frac{\partial}{\partial t} F &= \frac{A_1}{B} + \frac{\partial}{\partial x} H_1 \\
\frac{\partial^2}{\partial t^2} F &= \frac{A_2}{B} + \frac{\partial}{\partial x} H_2 \\
&\ \ \vdots \\
\frac{\partial^r}{\partial t^r} F &= \frac{A_r}{B} + \frac{\partial}{\partial x} H_r
\end{aligned}$$

(simple poles)

$\xRightarrow[\text{finite dimension}]{\text{confinement in}}$

$$\sum_{k=0}^{r} a_k(t) \frac{\partial^k}{\partial t^k} F = 0 + \frac{\partial}{\partial x} H$$

$$\rightsquigarrow \sum_{k=0}^{r} a_k(t) G^{(k)} = 0 \quad ✓$$

theorem (Bostan, Chen, Chyzak, Li 2010a)
On input of degree d, one can compute the output in $\mathcal{O}(d^{\omega+4})$ arithmetic operations.

input $F(t, x_1, \dots, x_n) \in \mathbb{Q}(t, x_1, \dots, x_n)$

output A differential equation for $G(t) = \oint F(t, x_1, \dots, x_n) \mathrm{d}x_1 \cdots \mathrm{d}x_n$

references Dwork (1962), Griffiths (1969), Bostan, Lairez, Salvy (2013), and Lairez (2016)

Compute $a_0(t), \dots, a_r(t) \in \mathbb{Q}(t)$ such that

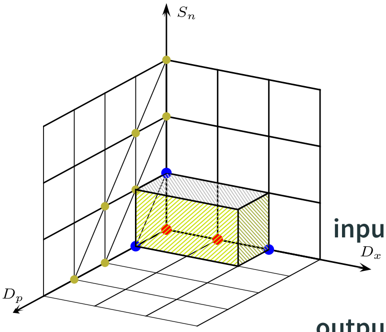
$$\sum_{k=0}^{r} a_k(t) \frac{\partial^k}{\partial t^k} F = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \text{(some rational function)}$$

theorem (Bostan, Lairez, Salvy 2013)

One input of degree d, one can compute the output in $d^{8n+\mathcal{O}(1)}$ arithmetic operations.

Generically, the certificate has size $> d^{n^2/2}$.

A differentially finite example



$$\int_{-1}^{1} \underbrace{\frac{e^{-px}T_n(x)}{\sqrt{1-x^2}}}_{=F_n(p,x)}\,dx = (-1)^n \pi\, I_n(p)$$

input $\frac{\partial}{\partial p}F_n = -xF_n, \quad nF_{n+1} = \frac{\partial}{\partial x}\big((x^2-1)F_n\big) + (px^2 + (n-1)x - p)F_n,$

$(1-x^2)\frac{\partial^2}{\partial x^2}F_n = (2px^2+3x-2p)\frac{\partial}{\partial x}F_n + (p^2x^2+3px-n^2-p^2+1)F_n$

output $p^2\frac{\partial^2}{\partial p^2}G_n + p\frac{\partial}{\partial p}G_n - (n^2+p^2)G_n = 0$

$G_{n+1} + \frac{\partial}{\partial p}G_n - \frac{n}{p}G_n = 0$

differential finiteness For all $i, j, k \geqslant 0$, there are a_{ijk} and $b_{ijk} \in \mathbb{Q}(n,p,x)$ such that

$$\frac{\partial^i}{\partial x^i}\frac{\partial^j}{\partial p^j}F_{n+k}(p,x) = a_{ijk}(n,p,x)F_n(p,x) + b_{ijk}(n,p,x)\frac{\partial}{\partial x}F_n(p,x).$$

Challenges

minimality We want to find *all* relations satified by the integrals

bounds We want to understand and control:

- the size of the output,
- the computational complexity of the algorithm.

certificateless We want to avoiding computing the certificate (otherwise, the complexity gets out of control):

- the certificate is much bigger than the output
- not possible to compute it with good complexity
- it is often useless

We give up:

- simple certification of the output
- case where $\int_D \frac{\partial}{\partial x}(\ldots)\mathrm{d}x \neq 0$

principle Find all relations

$$\begin{aligned}\sum_{(j,k)\in B} c_{j,k}(n,p)\frac{\partial}{\partial p^j}F_{n+k}(p,x) &= \frac{\partial}{\partial x}\left(u(n,p,x)F_n(p,x) + v(n,p,x)\frac{\partial}{\partial x}F_n(p,x)\right)\\ \rightsquigarrow \sum_{(j,k)\in B} c_{j,k}(n,p)\frac{\partial}{\partial p^j}G_{n+k}(p) &= 0.\end{aligned}$$

equivalently Find all $B \subset \mathbb{N}^2$ and $\left(c_{jk}\right) \in \mathbb{Q}(n,p)^B$ such that

$$\begin{cases} \dfrac{\partial}{\partial x} u = - a_{200} v + \displaystyle\sum_{(i,j)\in B} c_{jk} a_{0jk} \\ \dfrac{\partial}{\partial x} v = -u - b_{200} v + \displaystyle\sum_{(i,j)\in B} c_{jk} b_{0jk}, \end{cases}$$

has a rational solution $u, v \in \mathbb{Q}(n,p,x)$.

problem Find all $B \subset \mathbb{N}^2$ and $\left(c_{jk}\right) \in \mathbb{Q}(n,p)^B$ s.t. $\exists u, v \in \mathbb{Q}(n,p,x)$

$$\frac{\partial}{\partial x} u = -a_{200} v + \sum_{(i,j)\in B} c_{jk} a_{0jk} \quad \text{and} \quad \frac{\partial}{\partial x} v = -u - b_{200} v + \sum_{(i,j)\in B} c_{jk} b_{0jk}.$$

elimination Only look for solutions with $u, v \in \mathbb{Q}(n,p)$.

Akin to polynomial elimination.

☞ Fasenmyer (1949); see also Takayama (1990), Galligo (1985), etc.

minimality	bounds	certificateless
✖	✖	✖

problem Find all $B \subset \mathbb{N}^2$ and $\left(c_{jk}\right) \in \mathbb{Q}(n,p)^B$ s.t. $\exists u, v \in \mathbb{Q}(n,p,x)$

$$\frac{\partial}{\partial x} u = -a_{200} v + \sum_{(i,j)\in B} c_{jk} a_{0jk} \quad \text{and} \quad \frac{\partial}{\partial x} v = -u - b_{200} v + \sum_{(i,j)\in B} c_{jk} b_{0jk}.$$

rational solutions Iteratively solve the differential system (Abramov 1989; Barkatou 1999) with increasing support B (FGLM-like).

☞ Chyzak (2000) ; see also Picard (1906), Zeilberger (1990)

minimality	bounds	certificateless
✔	✖	✖

problem Find all $B \subset \mathbb{N}^2$ and $\left(c_{jk}\right) \in \mathbb{Q}(n,p)^B$ s.t. $\exists u, v \in \mathbb{Q}(n,p,x)$

$$\frac{\partial}{\partial x} u = -a_{200} v + \sum_{(i,j)\in B} c_{jk} a_{0jk} \quad \text{and} \quad \frac{\partial}{\partial x} v = -u - b_{200} v + \sum_{(i,j)\in B} c_{jk} b_{0jk}.$$

linear algebra Predict the denominator of solutions $u, v \in \mathbb{Q}(n,p,x)$, reduce to linear algebra over $\mathbb{Q}(n,p)$.

☞ Lipshitz (1988), Apagodu, Zeilberger (2006), Koutschan (2010)

minimality	bounds	certificateless
✖	✔	✖

problem Find all $B \subset \mathbb{N}^2$ and $(c_{jk}) \in \mathbb{Q}(n,p)^B$ s.t. $\exists u, v \in \mathbb{Q}(n,p,x)$

$$\frac{\partial}{\partial x} u = -a_{200} v + \sum_{(i,j)\in B} c_{jk} a_{0jk} \quad \text{and} \quad \frac{\partial}{\partial x} v = -u - b_{200} v + \sum_{(i,j)\in B} c_{jk} b_{0jk}.$$

reduction of pole order Generalization of Hermite's reduction

☞ Bostan, Chen, Chyzak, Li (2010b), Chen, Kauers, Singer (2012) and Chen, Kauers, Koutschan (2016), Bostan, Chen, Chyzak, Li, Xin (2013), Chen, Huang, Kauers, Li (2015) and Huang (2016), Bostan, Dumont, Salvy (2016), Chen, Hoeij, Kauers, Koutschan (2018), Hoeven (2017)

minimality	bounds	certificateless
✔	✔	✔

New algorithm

Obstructions to integrability

problem Find all $B \subset \mathbb{N}^2$ and $\left(c_{jk}\right) \in \mathbb{Q}(n,p)^B$ s.t. $\exists u, v \in \mathbb{Q}(n,p,x)$

$$(*) \quad \frac{\partial}{\partial x} u = -a_{200} v + \sum_{(i,j)\in B} c_{jk} a_{0jk} \quad \text{and} \quad \frac{\partial}{\partial x} v = -u - b_{200} v + \sum_{(i,j)\in B} c_{jk} b_{0jk}.$$

2G/4G hybrid algorithm For all $(j,k) \in \mathbb{N}^2$, produce an *obstruction* λ_{jk} such that

$$\lambda_{jk} = 0 \quad \Leftrightarrow \quad \begin{cases} \frac{\partial}{\partial x} u = & -a_{200} v + a_{0jk} \\ \frac{\partial}{\partial x} v = -u & -b_{200} v + b_{0jk} \end{cases} \quad \text{has a solution.}$$

By linearity, $(*)$ has a solution if and only if

$$\sum_{(j,k)\in B} c_{jk} \lambda_{jk} = 0.$$

Lagrange identity

differential operator $L : f(x) \mapsto \sum_i a_i(x) \frac{\mathrm{d}^i}{\mathrm{d}x^i} f(x)$

adjoint operator $L^* : f(x) \mapsto \sum_i (-1)^i \frac{\mathrm{d}^i}{\mathrm{d}x^i} \big(a_i(x) f(x) \big)$

Lagrange's identity $uL(f) = L^*(u) f + \frac{\mathrm{d}}{\mathrm{d}x} \big(\dots \big).$

corollary 1 $M(f) = M^*(1) f + \frac{\mathrm{d}}{\mathrm{d}x} \big(\dots \big)$, for any diff. op. M.

corollary 2 If $L(f) = 0$ then $L^*(u) f = \frac{\mathrm{d}}{\mathrm{d}x} \big(\dots \big)$ for any $u(x)$

corollary 3 If L is the minimal annihilating operator of f,
then for any differential operator M,
$M(f)$ "is a derivative" $\quad \Leftrightarrow \quad \exists y \in K(x), M^*(1) = L^*(y)$.

Generalized Hermite reduction

	Hermite reduction	Generalized Hermite red.
input	$u \in K(x)$	$u \in K(x)$ and $M \in K[x]\langle \frac{d}{dx} \rangle$
output	$v \in K(x)$	$v \in K(x)$
prop. 1	$u - v \in \frac{d}{dx} K(x)$	$u - v \in M(K(x))$
prop. 2	$u = \frac{d}{dx}(\dots) \Rightarrow v = 0$	$u = M(\dots) \Rightarrow v = 0$

Testing integrability with GHR

input $\gamma(x)$ a "function"

L, the minimal annihilating operator of γ

$f \in K(x)\langle \frac{\mathrm{d}}{\mathrm{d}x} \rangle \cdot \gamma$, the function space generated by γ

output $\exists g \in K(x)\langle \frac{\mathrm{d}}{\mathrm{d}x} \rangle \cdot \gamma, f = \frac{\mathrm{d}}{\mathrm{d}x}\gamma$

algorithm

write $f = u(x)\gamma + \frac{\mathrm{d}}{\mathrm{d}x}(\dots)$ ▷ corollary 1

$v(x) \leftarrow \mathrm{GHR}(u, L^*)$

return $v = 0$ ▷ corollary 3

GHR powered variant of Chyzak's algorithm

input $\mathscr{I}$ a D-finite ideal and $f \in \mathbb{A}/\mathscr{I}$

output generators of the telescoping ideal $\mathscr{T}_f$ w.r.t. $\frac{\partial}{\partial x}$

algorithm

$\gamma \leftarrow$ a cyclic vector of $\mathbb{A}/\mathscr{I}$ with respect to $\frac{\partial}{\partial x}$

$L \leftarrow$ the minimal operator annihilating γ

$\mathscr{L} \leftarrow [1]; G \leftarrow \{\}; Q \leftarrow \{\}$

while $\mu \leftarrow \mathrm{pop}(\mathscr{L})$ **do**

  if μ is a not multiple of the leading term of an element of G **then**

    write $\mu \cdot f = u_\mu(x)\gamma + \frac{\partial}{\partial x}(\dots)$

    $\lambda_\mu \leftarrow \mathrm{GHR}(u_\mu, L^*)$

    if $\exists$ a K-linear rel. between λ_μ and $\{\lambda_\nu \mid \nu \in Q\}$ **then**

      $(a_\nu)_{\nu \in Q} \leftarrow$ coeff. of the relation $\lambda_\mu u = \sum_{\nu \in Q} a_\nu \lambda_\nu$

      Add $\mu - \sum_{\nu \in Q} a_\nu \nu$ to G

    else

      add μ to Q; enqueue $\delta_1 \mu, \dots, \delta_e \mu$ in $\mathscr{L}$.

return G

$$\int \frac{2J_{m+n}(2tx)\,T_{m-n}(x)}{\sqrt{1-x^2}}\,\mathrm{d}x \quad \text{[diff. } t\text{, shift } n \text{ and } m\text{]} \tag{1}$$

$$\int_0^1 C_n^{(\lambda)}(x)C_m^{(\lambda)}(x)C_\ell^{(\lambda)}(x)(1-x^2)^{\lambda-\frac{1}{2}}\,\mathrm{d}x \quad \text{[shift } n, m, \ell\text{]} \tag{2}$$

$$\int_0^\infty xJ_1(ax)I_1(ax)Y_0(x)K_0(x)\,\mathrm{d}x \quad \text{[diff. } a\text{]} \tag{3}$$

$$\int \frac{n^2+x+1}{n^2+1}\left(\frac{(x+1)^2}{(x-4)(x-3)^2(x^2-5)^3}\right)^n\sqrt{x^2-5}\,e^{\frac{x^3+1}{x(x-3)(x-4)^2}}\,\mathrm{d}x \text{ [shift } n\text{]} \tag{4}$$

$$\int C_m^{(\mu)}(x)C_n^{(\nu)}(x)(1-x^2)^{\nu-1/2}\,\mathrm{d}x \quad \text{[shift } n, m, \mu, \nu\text{]} \tag{5}$$

$$\int x^\ell C_m^{(\mu)}(x)C_n^{(\nu)}(x)(1-x^2)^{\nu-1/2}\,\mathrm{d}x \quad \text{[shift } \ell, m, n, \mu, \nu\text{]} \tag{6}$$

$$\int (x+a)^{\gamma+\lambda-1}(a-x)^{\beta-1}C_m^{(\gamma)}(x/a)C_n^{(\lambda)}(x/a)\,\mathrm{d}x, \quad \text{[diff. } a\text{, shift } n, m, \beta, \gamma, \lambda\text{]} \tag{7}$$

Timings

Integral	(1)	(2)	(3)	(4)	(5)	(6)	(7)
New algorithm (Maple)	13 s	> 1h	> 1h	1.5 s	1.5 s	165 s	53 s
Chyzak[K]	19 s	253 s	45 s	232 s	516 s	>1h	>1h
Koutschan[K]	1.9 s*	2.3 s	5.3 s	>1h	2.3 s*	5.4 s	2.2 s*

* Non minimal output.

[K] Uses Koutschan's *HolonomicFunctions* (Mathematica package).

conclusion It really works! New algorithm for D-finite integration

New proof of the D-finiteness of the telescoping ideal of a D-finite function

2G/4G unification

future work Better understanding of the practical performance

Generalization to discrete sums

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