Monodromy in computer algebra

Pierre Lairez
Université Paris–Saclay, Inria, France

XVII Encuentro Álgebra Computacional y Aplicaciones
Castellò de la Plana, Spain, 20 June 2022

Take home message

Monodromy computed numerically give access to
an exact geometric information,
even in situations not likely of approximation

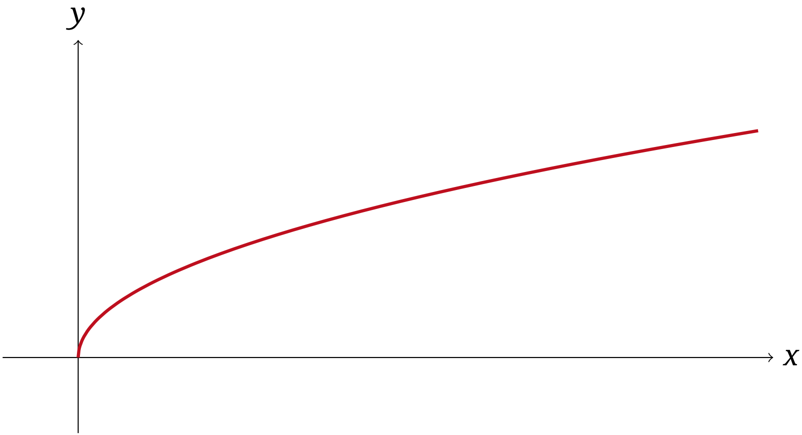
Overview

1. Algebraic functions
    1.1 Monodromy action
    1.2 Irreducible decomposition

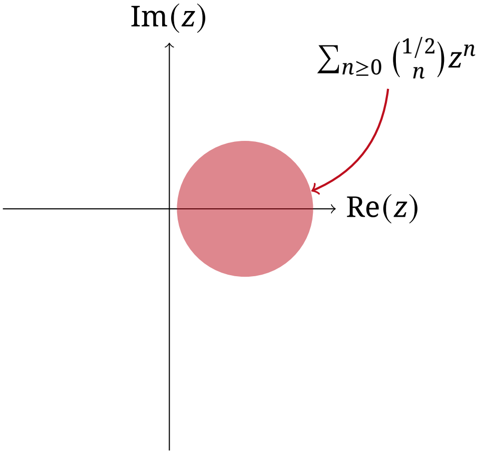
2. Holonomic functions
    2.1 Factorization of differential operator
    2.2 Testing algebraicity

3. Homology of complex varieties

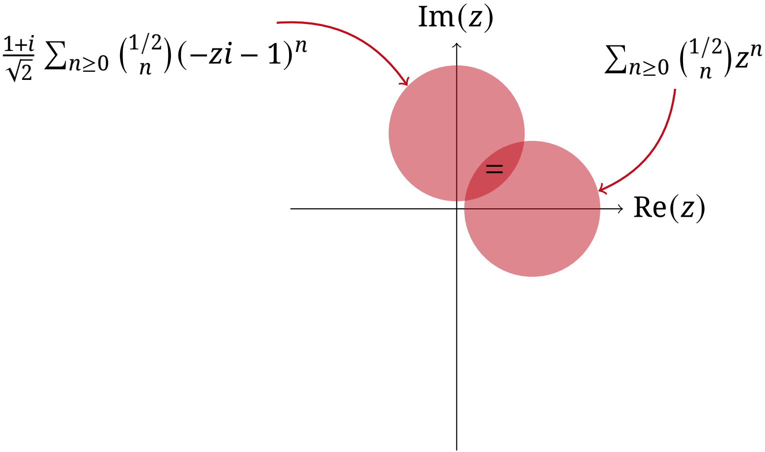
The square root function



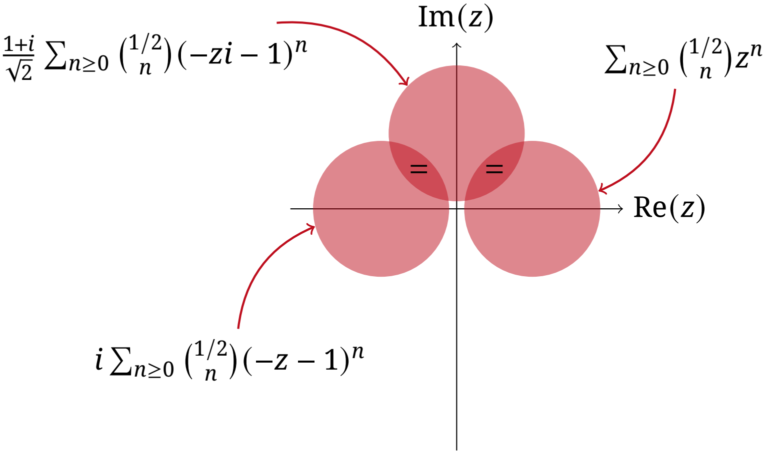
Square root, in the complex plane



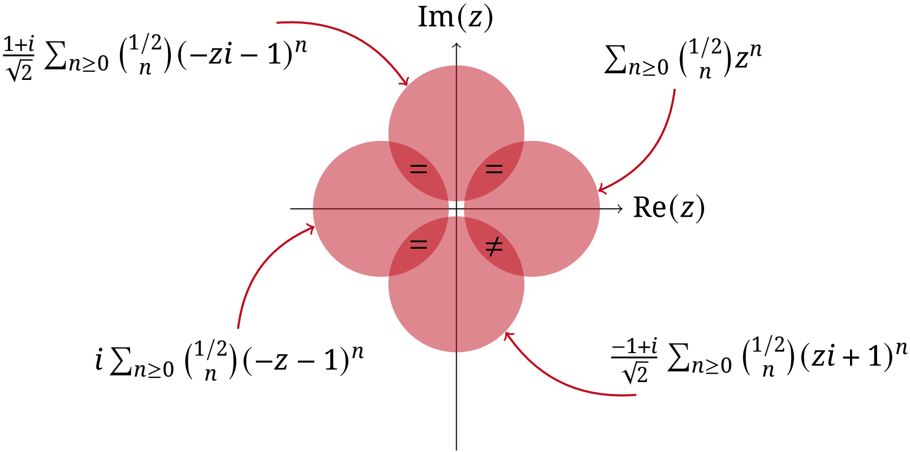
Square root, in the complex plane



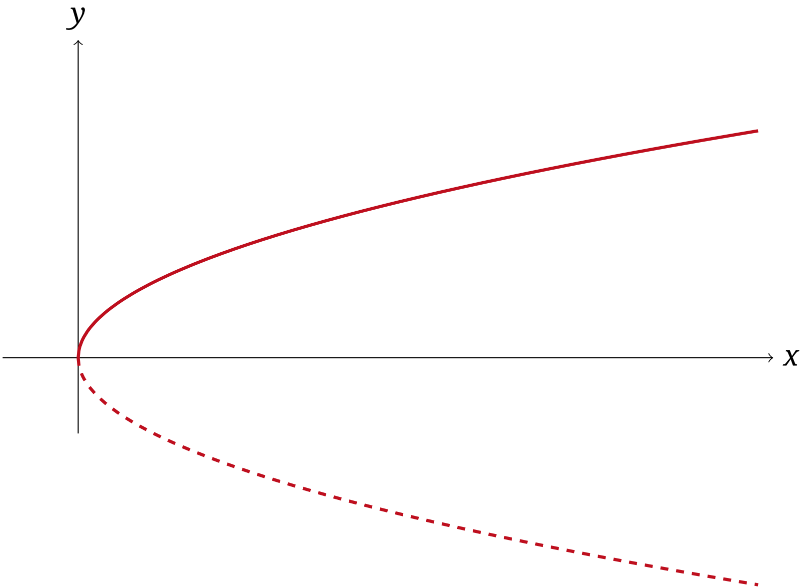
Square root, in the complex plane



Square root, in the complex plane



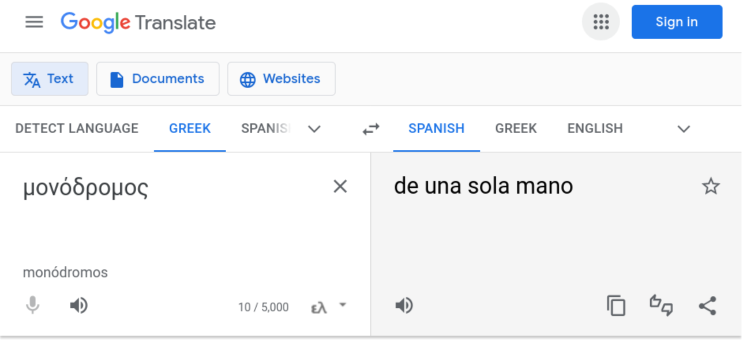
The square root has two determinations...



... so there is a monodromy phenomenon

- It is possible to extend the square root function holomorphically at any point in $\mathbb{C}^{\times}$...
- ... but not in a consistent way.
- As we go around 0, $\sqrt{z}$ becomes $-\sqrt{z}$.
- This phenomenon is called *monodromy*.

μονόδρομος?



- coined by Cauchy with the meaning of “in a single way”
- now refers to the presence of multiple determinations

Analytic continuation of algebraic functions

a polynomial equation $P_z(T) \in \mathbb{C}[z][T]$

a base point $b \in \mathbb{C}$ such that $\mathrm{disc}(P_b) \neq 0$

an initial value $y_b \in \mathbb{C}$ such that $P_b(y_b) = 0$

a open set $U \subseteq \mathbb{C} \setminus \{z \in \mathbb{C} \mid \mathrm{disc}(P_z) = 0\}$ simply connected

Analytic continuation of algebraic functions

a polynomial equation	$P_z(T) \in \mathbb{C}[z][T]$
a base point	$b \in \mathbb{C}$ such that $\operatorname{disc}(P_b) \neq 0$
an initial value	$y_b \in \mathbb{C}$ such that $P_b(y_b) = 0$
a open set	$U \subseteq \mathbb{C} \setminus \{z \in \mathbb{C} \mid \operatorname{disc}(P_z) = 0\}$ simply connected
theorem	there exists a unique holomorphic function $Y : U \longrightarrow \mathbb{C}$ such that $P(Y) = 0$ and $Y(b) = y_b$.

Analytic continuation of algebraic functions

a polynomial equation $P_z(T) \in \mathbb{C}[z][T]$

a base point $b \in \mathbb{C}$ such that $\operatorname{disc}(P_b) \neq 0$

an initial value $y_b \in \mathbb{C}$ such that $P_b(y_b) = 0$

a open set $U \subseteq \mathbb{C} \setminus \{z \in \mathbb{C} \mid \operatorname{disc}(P_z) = 0\}$ simply connected

theorem there exists a unique holomorphic function $Y : U \to \mathbb{C}$ such that $P(Y) = 0$ and $Y(b) = y_b$.

proof Apply the global Picard-Lindelöf theorem to

$$Y'(z) = \left(\frac{\partial P}{\partial T}\right)(Y)^{-1} \cdot \frac{\partial P}{\partial z}(Y)$$

Analytic continuation of algebraic functions: algorithm

input $P \in \mathbb{C}[z][T]$, base point b, initial value y_b, target point c

output $Y(c)$ where Y is the analytic continuation as above along the line segment $[b, c]$.

$t \leftarrow 0$
$y \leftarrow y_b$
while $t < 1$ **do**
    $t \leftarrow t + \delta t$ (*many different ways to choose* δt)
    $y \leftarrow y - \left(\frac{\partial P}{\partial z}\right)(y)^{-1} \cdot \frac{\partial P}{\partial T}(y)|_{z \leftarrow (1-t)b+tc}$
end
return y

Monodromy action

polynomial equation $P \in \mathbb{C}[z][T]$, squarefree

critical values $\Sigma = \{z \in \mathbb{C} \mid \operatorname{disc}(P) = 0\}$

base point $b \in \mathbb{C} \setminus \Sigma$

monodromy action Continuation along a path induces the morphism

$$\phi : \pi_1 (\mathbb{C} \setminus \Sigma, b) \longrightarrow \operatorname{Bij} (\{y \in \mathbb{C} \mid P_b(y) = 0\}) .$$

monodromy group $M = \operatorname{im} \phi$

Theorem

- *The orbits of this action are in one-to-one correspondance with the irreducible factors of P in* $\mathbb{C}(z)[T]$.
- *If P is irreducible, the monodromy group is isomorphic to the Galois group of P over the field* $\mathbb{C}(z)$.

Counting irreducible factors

Given $P \in \mathbb{C}[z][T]$, how many irreducible factors does it have?
Easy reduction to the following case:

- the coefficients of P (as a polynomial in T) do not have common factors;
- P does not have a multiple factor.

$b \leftarrow$ generic point in $\mathbb{C}$
$y_1, \dots, y_r \leftarrow$ roots of $P_b(T)$
$G \leftarrow$ graph with r nodes and no edge
repeat (*how many times?*)
    $u, v \leftarrow$ random points in $\mathbb{C}$
    for i from 1 to r **do**
        $y_j \leftarrow$ continuation of y_i along the loop $[b, u, v, b]$
        insert an edge (i, j) in G

return the number of connected components of G

Halting condition: the trace test

(Sommese, Verschelde, & Wampler, 2002)
Assume generic coordinates

input $S \subseteq \{y \in \mathbb{C} \mid P_b(y) = 0\}$

problem Is S closed under the monodromy action?

$u, v \leftarrow$ random points in $\mathbb{C}$

Halting condition: the trace test

(Sommese, Verschelde, & Wampler, 2002)
Assume generic coordinates

input $S \subseteq \{y \in \mathbb{C} \mid P_b(y) = 0\}$
problem Is S closed under the monodromy action?

$u, v \leftarrow$ random points in $\mathbb{C}$
$\sigma_b \leftarrow \sum_{y \in S} y$

Halting condition: the trace test

(Sommese, Verschelde, & Wampler, 2002)
Assume generic coordinates

input $S \subseteq \{y \in \mathbb{C} \mid P_b(y) = 0\}$
problem Is S closed under the monodromy action?

$u, v \leftarrow$ random points in $\mathbb{C}$
$\sigma_b \leftarrow \sum_{y \in S} y$
$\sigma_u \leftarrow \sum_{y \in S}$ continuation of y along $[b, u]$
$\sigma_v \leftarrow \sum_{y \in S}$ continuation of y along $[b, v]$

Halting condition: the trace test

(Sommese, Verschelde, & Wampler, 2002)
Assume generic coordinates

input $S \subseteq \{y \in \mathbb{C} \mid P_b(y) = 0\}$

problem Is S closed under the monodromy action?

$u, v \leftarrow$ random points in $\mathbb{C}$
$\sigma_b \leftarrow \sum_{y \in S} y$
$\sigma_u \leftarrow \sum_{y \in S}$ continuation of y along $[b, u]$
$\sigma_v \leftarrow \sum_{y \in S}$ continuation of y along $[b, v]$

return $(b - u)(\sigma_b - \sigma_v) == (b - v)(\sigma_b - \sigma_u)$

Halting condition: the trace test

(Sommese, Verschelde, & Wampler, 2002)
Assume generic coordinates

input $S \subseteq \{y \in \mathbb{C} \mid P_b(y) = 0\}$

problem Is S closed under the monodromy action?

$u, v \leftarrow$ random points in $\mathbb{C}$
$\sigma_b \leftarrow \sum_{y \in S} y$
$\sigma_u \leftarrow \sum_{y \in S}$ continuation of y along $[b, u]$
$\sigma_v \leftarrow \sum_{y \in S}$ continuation of y along $[b, v]$

return $(b - u)(\sigma_b - \sigma_v) == (b - v)(\sigma_b - \sigma_u)$

in words Check that $\sigma_u - \sigma_b$ depends linearly on u.

proof If it does, then it has no monodromy, so S is closed.
For the converse: the sum of roots of a monic polynomial P is minus the coefficient of T^{d-1}.

Overview

1. Algebraic functions
    1.1 Monodromy action
    1.2 Irreducible decomposition

2. Holonomic functions
    2.1 Factorization of differential operator
    2.2 Testing algebraicity

3. Homology of complex varieties

Linear differential operators

open set	$U \subseteq \mathbb{C}$
function space	$\mathcal{O}(U)$, holomorphic functions on U
differential ops	$\mathbb{C}(z)\langle\partial\rangle$ is the subalgebra of $\mathrm{End}_{\mathbb{C}}(\mathcal{O}(U))$ generated by multiplications by rational functions and $\partial = \frac{\mathrm{d}}{\mathrm{dz}}$.

For $L \in \mathbb{C}[z]\langle\partial\rangle$ nonzero, we can always write

$$L = a_r(z)\partial^r + a_{r-1}(z)\partial^{r-1} + \cdots + a_1(z)\partial + a_0(z),$$

for some $r \geq 0$ and $a_r \neq 0$.

$L(y) = 0$ is the linear differential equation

$$a_r(z)y^{(r)} + a_{r-1}(z)y^{(r-1)} + \cdots + a_1 y' + a_0 y = 0.$$

Two problems for Fuchsian operators

Fuchsian operator $L \in \mathbb{C}(z)\langle\partial\rangle$ is Fuchsian if the solutions grow at most polynomially near singularities (including near ∞)
Naturally happens in many contexts

Two problems for Fuchsian operators

Fuchsian operator $L \in \mathbb{C}(z)\langle\partial\rangle$ is Fuchsian if the solutions grow at most polynomially near singularities (including near ∞)
Naturally happens in many contexts

Problem #1 Given L, find a nontrivial fractorization $L = AB$, or prove that there is none.

Two problems for Fuchsian operators

Fuchsian operator $L \in \mathbb{C}(z)\langle\partial\rangle$ is Fuchsian if the solutions grow at most polynomially near singularities (including near ∞)
Naturally happens in many contexts

Problem #1 Given L, find a nontrivial fractorization $L = AB$, or prove that there is none.

Problem #2 Given L, prove or disprove that all solutions of L are algebraic.

Analytic continution of holonomic functions

a differential operator $L \in \mathbb{C}[z]\langle\partial\rangle$ or order r

a base point $b \in \mathbb{C}$ such that $\mathrm{lc}(L)|_{z=b} \neq 0$

initial conditions $y_0, \ldots, y_{r-1} \in \mathbb{C}$

a open set $U \subseteq \mathbb{C} \setminus \{\mathrm{lc}(L) = 0\}$ simply connected

Analytic continution of holonomic functions

a differential operator	$L \in \mathbb{C}[z]\langle\partial\rangle$ or order r
a base point	$b \in \mathbb{C}$ such that $\mathrm{lc}(L)\|_{z=b} \neq 0$
initial conditions	$y_0, \ldots, y_{r-1} \in \mathbb{C}$
a open set	$U \subseteq \mathbb{C} \setminus \{\mathrm{lc}(L) = 0\}$ simply connected

theorem there exists a unique holomorphic function $Y : U \longrightarrow \mathbb{C}$ such that $L(Y) = 0$ and $Y(b) = y_0$, $Y'(b) = y_1$, ..., $Y^{(r-1)}(b) = y_{r-1}$.

Analytic continution of holonomic functions

a differential operator $L \in \mathbb{C}[z]\langle\partial\rangle$ or order r

a base point $b \in \mathbb{C}$ such that $\mathrm{lc}(L)|_{z=b} \neq 0$

initial conditions $y_0, \ldots, y_{r-1} \in \mathbb{C}$

a open set $U \subseteq \mathbb{C} \setminus \{\mathrm{lc}(L) = 0\}$ simply connected

theorem there exists a unique holomorphic function $Y : U \longrightarrow \mathbb{C}$ such that $L(Y) = 0$ and $Y(b) = y_0$, $Y'(b) = y_1$, ..., $Y^{(r-1)}(b) = y_{r-1}$.

proof Apply the global Picard-Lindelöf theorem

Analytic continution of holonomic functions

a differential operator $L \in \mathbb{C}[z]\langle\partial\rangle$ or order r

a base point $b \in \mathbb{C}$ such that $\mathrm{lc}(L)|_{z=b} \neq 0$

initial conditions $y_0, \dots, y_{r-1} \in \mathbb{C}$

a open set $U \subseteq \mathbb{C} \setminus \{\mathrm{lc}(L) = 0\}$ simply connected

theorem there exists a unique holomorphic function $Y : U \to \mathbb{C}$ such that $L(Y) = 0$ and $Y(b) = y_0$, $Y'(b) = y_1$, ..., $Y^{(r-1)}(b) = y_{r-1}$.

proof Apply the global Picard-Lindelöf theorem

$$\{y \in \mathcal{O}(U) \mid L(y) = 0\} \xrightarrow{\sim} \mathbb{C}^{r-1}$$

$$y \mapsto \left(y(b), y'(b), \dots, y^{(r-1)}(b)\right)$$

Monodromy action

differential op $L \in\in \mathbb{C}(z)\langle\partial\rangle$ Fuchsian

singular points $\Sigma = \{z \in \mathbb{C} \mid \mathrm{lc}(L) = 0\}$

base point $b \in \mathbb{C} \setminus \Sigma$

local solutions $V_b = \{y \in \mathcal{O}(D(b, \epsilon)) \mid L(y) = 0\}$

monodromy action Continuation along a path induces the morphism

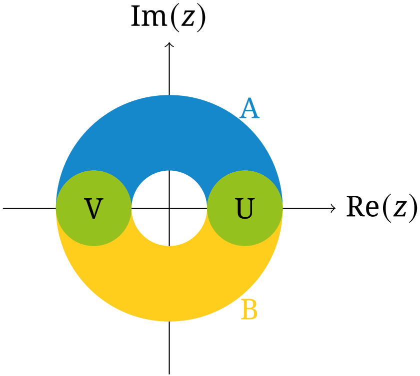
$$\phi : \pi_1(\mathbb{C} \setminus \Sigma, b) \longrightarrow \mathrm{Aut}_{\mathbb{C}}(V_b).$$

monodromy group $M = \mathrm{im}\,\phi$

Theorem

- *The right-factors of L are in one-to-one correspondance with the stable subspaces of V_b under the monodromy action.*
- *A solution of L is rational if and only if monodromy acts trivially.*
- *A solution of L is algebraic if and only if it has a finite orbit under monodromy.*

Monodromy of the logarithm



$L = \partial z \partial = z\partial^2 - 1$

Basis of solutions on A:

$1, \quad \mathrm{Log}_A(z) = \log|z| + \arg_A(z)i,$ with $\arg_A(z) \in [-\frac{\pi}{2}, \frac{3\pi}{2})$

Basis of solutions on B:

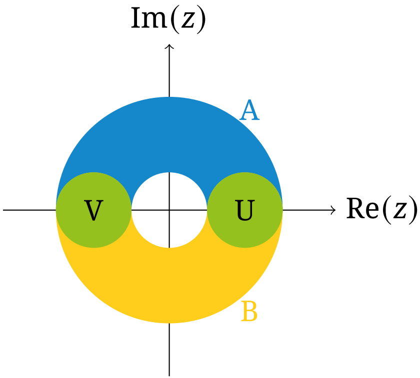
$1, \quad \mathrm{Log}_B(z) = \log|z| + \arg_B(z)i,$ with $\arg_B(z) \in [-\frac{3\pi}{2}, \frac{\pi}{2})$

On U: $\mathrm{Log}_A(z) = \mathrm{Log}_B(z)$

On V: $\mathrm{Log}_A(z) = \mathrm{Log}_B(z) + 2\pi i$

monodromy around 0: $\begin{pmatrix} 1 & 2\pi i \\ 0 & 1 \end{pmatrix}$

Monodromy of a power



$L = z\partial - \lambda, \quad \lambda \in \mathbb{C}$

Basis of solutions on A:
$1, \quad z^\lambda = \exp(\lambda \operatorname{Log}_A(z))$

Basis of solutions on B:
$1, \quad \widetilde{z^\lambda} = \exp(\lambda \operatorname{Log}_B(z))$

On U: $z^\lambda = \widetilde{z^\lambda}$
On V: $z^\lambda = \widetilde{z^\lambda} \cdot \exp(2\pi\lambda i)$

monodromy around 0: $\left(\exp(2\pi\lambda i)\right)$
(this is a 1×1 matrix)

Fuchsian holonomic functions with trivial or finite orbits

Let f be a Fuchsian holonomic function, such that monodromy acts trivially.

- Locally, we can expand f in $\mathbb{C}((z))[z^\lambda, \log z]$ for some $\lambda \in \mathbb{C}$.
- No monodromy, so f must be in $\mathbb{C}((z))$, so it is a meromorphic function on $\mathbb{P}^1$. To it is rational.

Fuchsian holonomic functions with trivial or finite orbits

Let f be a Fuchsian holonomic function, such that monodromy acts trivially.

- Locally, we can expand f in $\mathbb{C}((z))[z^\lambda, \log z]$ for some $\lambda \in \mathbb{C}$.
- No monodromy, so f must be in $\mathbb{C}((z))$, so it is a meromorphic function on $\mathbb{P}^1$. To it is rational.

Let f be a Fuchsian holonomic function, with a finite orbit $\{f_1, \ldots, f_n\}$ under monodromy.

- Form the polynomial $P(T) = \prod_i (T - f_i)$. Note that $P(f) = 0$.
- The coefficients of P have no monodromy, so they are rational functions.
- So f is algebraic.

Stable subspaces under monodromy

Let L be a Fuchsian differential operator.

- If $L = AB$, then $\{y \in V_b \mid B(y) = 0\}$ is a subspace of V_b stable under monodromy action.

Stable subspaces under monodromy

Let L be a Fuchsian differential operator.

- If $L = AB$, then $\{y \in V_b \mid B(y) = 0\}$ is a subspace of V_b stable under monodromy action.
- Conversely, let $S \subseteq V_b$ be subspace stable under the monodromy action. Pick a basis $y_1, \ldots, y_r$ of S and let

$$B = \begin{vmatrix} y_1 & \cdots & y_r \\ y_1' & \cdots & y_r' \\ \vdots & & \vdots \\ y_1^{(r-1)} & \cdots & y_r^{(r-1)} \end{vmatrix}^{-1} \begin{vmatrix} y_1 & \cdots & y_r & \partial \\ y_1' & \cdots & y_r' & \partial \\ \vdots & & \vdots & \vdots \\ y_1^{(r)} & \cdots & y_r^{(r)} & \partial^r \end{vmatrix} \in \mathbb{C}(z)\langle\partial\rangle$$

  The coefficients of this operator are monodromy-invariant, so rational.

  Every solution of B is a solution of L, so B right-divides L.

Factorization of Fuchsian differential operators

(van der Hoeven, 2007; Chyzak, Goyer, & Mezzarobba, 2022)

input $L \in \mathbb{C}(z)\langle\partial\rangle$ Fuchsian

output A right factor of L, or nothing if L is irreducible

$b \leftarrow$ a random point in $\mathbb{C}$
numerically compute generators $M_1, \ldots, M_s$
of the monodromy group, with base point b
find a nontrivial stable space $\mathbb{C}y_1 + \cdots + \mathbb{C}y_r \subseteq V_b$
if impossible **then return** $\varnothing$

return $$\begin{vmatrix} y_1 & \cdots & y_r \\ y_1' & \cdots & y_r' \\ \vdots & & \vdots \\ y_1^{(r-1)} & \cdots & y_r^{(r-1)} \end{vmatrix}^{-1} \begin{vmatrix} y_1 & \cdots & y_r & \partial \\ y_1' & \cdots & y_r' & \partial \\ \vdots & & \vdots & \vdots \\ y_1^{(r)} & \cdots & y_r^{(r)} & \partial^r \end{vmatrix} \in \mathbb{C}(z)\langle\partial\rangle$$

(*reconstruct the coefficients by evaluation-interpolation*)

Factorization of Fuchsian differential operators: comments

- Implemented in Sagemath (by Goyer)
- Relies on very high precision evaluation of the monodromy matrices (typically 1000 decimal digits)
- This is possible with quasilinear complexity! (algorithms and implementation by Mezzarobba)
- Performs very well

A famous hypergeometric function

$$\phi(z) = \sum_{n\geq 0} \frac{(30n)!n!}{(15n)!(10n)!(6n)!} z^n \in \mathbb{Z}[[z]]$$

Theorem (Beukers and Heckman, 1989; Rodriguez-Villegas, 2005)

There is a polynomial $P \in \mathbb{C}[z][T]$ of degree 483,840 such that $P(\phi(z)) = 0$.

- Follows from a result of Beukers and Heckman (1989) on the monodromy of generalized hypergeometric functions.
- Relies on an enormous classification work in finite group theory, especially (Shephard & Todd, 1954).
- Can we confirm this result computationally?
  Can we check that the orbit of ϕ under the monodromy action is finite?
  DEMO

Overview

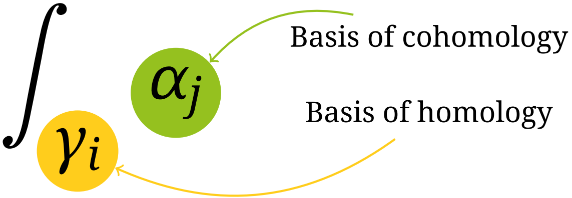
1. Algebraic functions
   1.1 Monodromy action
   1.2 Irreducible decomposition

2. Holonomic functions
   2.1 Factorization of differential operator
   2.2 Testing algebraicity

3. Homology of complex varieties

The matrix of periods

X smooth compact complex algebraic manifold of dimension n



Matrix of periods = matrix of the pairing $H_n(X, \mathbb{C}) \times H^n_{\mathrm{DR}}(X, \mathbb{C}) \longrightarrow \mathbb{C}$.

- describe fine algebraic invariants of X, related to the Hodge structure
- How to compute it?

The long road to periods

(joint work with Eric Pichon-Pharabod and Pierre Vanhove)

- How to compute $\int_\gamma \alpha$ given γ and α?
  - what does it mean to give γ?
  - the description of α seems less of an issue
  - this is a problem of numerical integration

- How to compute a basis of De Rham cohomology?
  - For smooth hypersurfaces in a projective space: Griffiths–Dwork reduction

- How to compute a basis of the singular homology?
  - By Lefschetz, reduction to the case of a one-parameter family $(X_t)_{t\in\mathbb{P}^1}$ We need:
    - the homology of one fiber X_t (induction on dimension)
    - the monodromy action

Monodromy acting on homology

a family	$(X_t)_{t\in\mathbb{C}}$, such that X_t is compact and smooth for generic t
critical values	$\Sigma = \{t \in \mathbb{C} \mid X_t \text{ singular}\}$
base point	$b \in \mathbb{C} \setminus \Sigma$
a loop	$\gamma : [0,1] \to \mathbb{C} \setminus \Sigma$, $\gamma(0) = \gamma(1) = b$

- By Ehresmann's theorem, $X_{\gamma(u)}$ deforms continously as u goes from 0 to 1
- Induces diffeomorphism $X_{\gamma(0)} \simeq X_{\gamma(1)}$, determined up to homotopy.
- Induces $X_b \simeq X_b$ and in particular, an automorphism of $H_*(X_b, \mathbb{Z})$

monodromy action This induces

$$\phi : \pi_1(\mathbb{C} \setminus \Sigma, b) \longrightarrow \mathrm{Aut}_{\mathbb{Z}}(H_*(X_b, \mathbb{Z})).$$

How to compute it?

A family of elliptic curves

$$X_t = \left\{ [x : y : z] \in \mathbb{P}^2 \;\middle|\; (x+y)(y+z)(z+x) + txyz = 0 \right\}$$

- Given a basis γ_1, γ_2 of $H_2(X_b)$, there is a unique way to extend it continously to a basis $\gamma_1(t), \gamma_2(t)$ of $H_2(X_t)$.
- We want to compute the monodromy of this basis.
- Fix a basis $\alpha(t), \overline{\alpha}(t)$ of $H^2_{\mathrm{DR}}(X_t)$, where α depends *rationally on t*.
- $\omega_1(t) = \int_{\gamma_1(t)} \alpha(t)$ and $\omega_2(t) = \int_{\gamma_2(t)} \alpha(t)$ are a basis of solution of the *Picard-Fuchs differential equation*

$$t(t+8)(t-1)y'' + (3t^2 + 14t - 8)y' + (t+2)y = 0$$

Monodromy

Consider the continuation along a loop η in $\mathbb{C}$.

on the one hand $\eta\omega_i(t) = a_{i1}\omega_i(t) + a_{i2}\omega_2(t)$, as the monodromy acts on the solution space of the Picard-Fuchs equation.

on the other hand $\alpha(t)$ has no monodromy, so

$$\eta\omega_i(t) = \int_{\eta\gamma_i(t)} \alpha(t).$$

conclusion The monodromy on $H_2(X, \mathbb{Z})$ is given that of the PF equation:

$$\eta\gamma_i(t)(b) = a_{i1}\gamma_1(b) + a_{i2}\gamma_2(b)$$

DEMO

Shephard, G. C., & Todd, J. A. (1954). Finite Unitary Reflection Groups. *Can. J. Math.*, *6*, 274–304. https://doi.org/10/c4dbth

Beukers, F., & Heckman, G. (1989). Monodromy for the hypergeometric function ${}_nF_{n-1}$. *Invent Math*, *95*(2), 325–354. https://doi.org/10/d96v7p

Sommese, A. J., Verschelde, J., & Wampler, C. W. (2002). Symmetric Functions Applied to Decomposing Solution Sets of Polynomial Systems. *SIAM J. Numer. Anal.*, *40*(6), 2026–2046. https://doi.org/10/fcqgq8

Rodriguez-Villegas, F. (2005). *Integral ratios of factorials and algebraic hypergeometric functions*.

van der Hoeven, J. (2007). Around the numeric–symbolic computation of differential Galois groups. *J. Symb. Comput.*, *42*(1), 236–264. https://doi.org/10/cjxwkq

Chyzak, F., Goyer, A., & Mezzarobba, M. (2022). *Symbolic-numeric factorization of differential operators*.