Rigid continuation paths I
Quasilinear average complexity for solving polynomial systems

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Numerical continuation

F_t a polynomial system depending continuously on $t \in [0, 1]$

z_0 a zero of F_0

function NumericalContinuation(F_t, z_0)
  $t \leftarrow 0$
  $z \leftarrow z_0$
  repeat
    $t \leftarrow t + \Delta t$
    $z \leftarrow \mathrm{Newton}(F_t, z)$
  until $t \geqslant 1$
  return z
end function

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- How to set the step size Δt ?

Input size and Bézout bound

	n complex variables n equations of degree d
Bézout bound	d^n
input size	$N = n\binom{n+d}{n} \geqslant c^{\min(n,d)}$
quadratic case	$d = 2$ $N \sim \frac{1}{2}n^3 \ll 2^d$
medium degree case	$d = n$ $N \sim \frac{1}{\sqrt{\pi}} n^{\frac{1}{2}} 4^n \ll n^n$
large degree case	$d \gg n$ $N \sim \frac{1}{(n-1)!} d^n$

Renegar (1987)

n complex variables
n random equations of degree d
input size N

# of steps	$\mathrm{poly}(d^n)$, with high probability
starting system	$x_1^d = 1, \ldots, x_n^d = 1$
continuation path	$(1-t)F_0 + tF_1$
previous best	$\varnothing$

Shub, Smale (1994)

n complex variables
n random equations of degree d
input size N

# of steps	$\text{poly}(N)$, with high probability
starting system	not constructive
continuation path	$(1-t)F_0 + tF_1$
previous best	$\text{poly}(d^n)$

Beltrán, Pardo (2009)

n complex variables
n random equations of degree d
input size N

# of steps	$O(nd^{3/2}N)$, on average
starting system	random system, sampled directly with a zero
continuation path	$(1-t)F_0 + tF_1$
previous best	$\mathrm{poly}(d^n) \to \mathrm{poly}(N)$

Armentano, Beltrán, Bürgisser, Cucker, Shub (2016)

n complex variables
n random equations of degree d
input size N

# of steps	$O(nd^{3/2}N^{1/2})$, on average
starting system	idem Beltrán-Pardo
continuation path	$(1-t)F_0 + tF_1$
previous best	$\text{poly}(d^n) \to \text{poly}(N) \to O(nd^{3/2}N)$

This talk (Lairez 2020)

n complex variables
n random equations of degree d
input size N

# of steps	$O(n^3 d^2)$, on average
starting system	an analogue of Beltrán-Pardo
continuation path	$(f_1 \circ u_1^{1-t}, \ldots, f_n \circ u_n^{1-t})$, with $u_i \in U(n+1)$ (rigid motion of each equations)
previous best	$\mathrm{poly}(d^n) \to \mathrm{poly}(N) \to O(nd^{3/2}N) \to O(nd^{3/2}N^{1/2})$

How to improve the complexity?

By making **bigger steps!**

z = the current zero
$\rho(F,z)$ = inverse of the radius of the bassin of attraction of z
$\mu(F,z) = \sup\,[\text{over } F' \sim F \text{ and } F'(z') = 0]\ \frac{\operatorname{dist}(z,z')}{\|F-F'\|}$

step size heuristic + μ-estimate

$$\frac{1}{\Delta t} \approx \rho(F,z) \cdot \frac{\Delta z}{\Delta t} \lessapprox \underbrace{\mu(F,z)}_{\text{loose}} \cdot \underbrace{\mu(F,z)}_{\text{sharp}} .$$

(loose on average)

Each factor μ contributes $O(N^{1/2})$ in the average # of steps.
To go down to $\operatorname{poly}(n,d)$, we must improve both.

Conditioning and dimension

Rule of thumb

$$\mu(F, z) \underset{\text{on average}}{\approx} \left(\text{dimension of ambient space}\right)^{c}$$

What is the ambient space?
Is it the space of all polynomials systems?

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Rigid paths

Fix polynomials $f_1,\ldots,f_n$, then

$$\text{ambient space} = \left\{ (f_1 \circ u_1, \ldots, f_n \circ u_n) \;\middle|\; u_i \in U(n+1) \right\}.$$

The average conditioning is poly(n)!

Construction of a start system (Beltrán–Pardo analogue)

input $f_1, \dots, f_n$, homogeneous polynomials of degree d in $x_0, \dots, x_n$

1 Sample $H_1, \dots, H_n$ hyperplanes in $\mathbb{P}^n(\mathbb{C})$, uniformly

2 Compute $x \in H_1 \cap \cdots \cap H_n$ (unique)

3 For $1 \leqslant i \leqslant n$,

- **a** sample points $p_i \in \mathbb{P}^n(\mathbb{C})$ such that $f_i(p_i) = 0$, uniformly
- **b** sample $u_i \in U(n+1)$ such that $u_i(x) = p_i$ and $u_i(H_i) = T_{p_i} V(f_i)$, uniformly

output x and $f_1 \circ u_1, \dots, f_n \circ u_n$

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Theorem

The output $u_1, \dots, u_n$ is uniformly distributed in $U(n+1)^n$ (Haar measure), and the zero x is uniformly distributed in $V(f_1 \circ u_1, \dots, f_n \circ u_n)$.

Rigid continuation algorithm

input $f_1, \dots, f_n$, homogeneous polynomials of degree d in $x_0, \dots, x_n$

1. Compute a start system $f_1 \circ u_1, \dots, f_n \circ u_n$ with a zero x
2. Perform the numerical continuation with
$$F_t = \left(f_1 \circ u_1^{1-t}, \dots, f_n \circ u_n^{1-t}\right).$$

big win the parameter space has $O(n^3)$ dimensions, the conditioning is $\mathrm{poly}(n)$ on average

total complexity $O(n^6 d^4 N) = N^{1+o(1)}$ operation on average, quasilinear

The step size

For a n-variable polynomial f and a point $z \in \mathbb{C}$,

$$\gamma(f, z) = \sup_{k \geqslant 2} \left(\| \mathrm{d}_z f \|^{-1} \| \mathrm{d}_z^k f \| \right)^{\frac{1}{k-1}}$$

$$\Delta t \approx \frac{1}{\mu_{\text{rigid}}(F, z)^2 \sum_i \gamma(f_i, z)}$$

- Computational difficulties in computing γ.
  $N^{1+o(1)}$ complexity but not by evaluation.
- Uses the γ-number of a single polynomial, not a system.

Complexity

average gamma $\Gamma(f)^2 = \mathbb{E}_{z\in V(f)}\left[\gamma(f,z)^2\right]$

input distribution $f_1,\ldots,f_n$ independant random polynomials with unitary invariant distribution

Theorem

On input $f_1,\ldots,f_n$, rigid continuation terminates after

$$\text{poly}(n)\sum_{i=1}^{n}\mathbb{E}\left[\Gamma(f_i)\right]$$

continuation steps, on average.

- Only requires independance of equations and unitary invariance
- To be useful, need to estimate $\mathbb{E}[\Gamma(f_i)]$

Average gamma for Gaussian polynomials

Theorem

For $f \in \mathbb{C}[x_0, \ldots, x_n]$ a homogeneous Gaussian random polynomial (a.k.a. Kostlan polynomial) of degree d,

$$\mathbb{E}\left[\Gamma(f)\right] \leqslant \tfrac{1}{4} d^3 (d+n).$$

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Corollary

On input $f_1, \dots, f_n$ Kostlan polynomials of degree d, the rigid continuation algorithm outputs an approximate zero after $\operatorname{poly}(n, d)$ steps on average, and $N^{1+o(1)}$ operations on average.

Low complexity polynomials (teaser for next talk)

fact The set of polynomials with low evaluation complexity is unitary invariant.

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Thank you!