Transcendental methods in numerical algebraic geometry

Pierre Lairez

MATHEXP, Université Paris–Saclay, Inria, France

includes joint work with E. Pichon-Pharabod, E. Sertöz, and P. Vanhove

July 16, 2024
Raleigh, NC, USA / ISSAC tutorial

High precision quadrature
uncovers fine invariants
of algebraic varieties.

High precision quadrature

uncovers fine invariants

of algebraic varieties.

today's menu: elliptic curves and quartic surfaces

High precision quadrature

uncovers fine invariants

of algebraic varieties.

invariants that characterize
nongeneric yet dense
behaviour

today's menu: elliptic curves
and quartic surfaces

High precision quadrature uncovers fine invariants of algebraic varieties.

typically 1000 decimal digits, uses symbolic-numeric algorithms

invariants that characterize nongeneric yet dense behaviour

today's menu: elliptic curves and quartic surfaces

Contents lists available at ScienceDirect

Journal of Symbolic Computation

www.elsevier.com/locate/jsc

Foreword

What is numerical algebraic geometry?

CrossMark

ARTICLE INFO

MSC:
65H10
68W30
14Q99

Keywords:
Witness set
Generic point
Homotopy continuation
Cascade homotopy
Irreducible component
Multiplicity
Numerical algebraic geometry
Polynomial system
Numerical irreducible decomposition
Primary decomposition
Algebraic set
Algebraic variety
Number field

ABSTRACT

The foundation of algebraic geometry is the solving of systems of polynomial equations. When the equations to be considered are defined over a subfield of the complex numbers, numerical methods can be used to perform algebraic geometric computations forming the area of numerical algebraic geometry. This article provides a short introduction to numerical algebraic geometry with the subsequent articles in this special issue considering three current research topics: solving structured systems, certifying the results of numerical computations, and performing algebraic computations numerically via Macaulay dual spaces.

© 2016 Elsevier Ltd. All rights reserved.

(Jonathan D. Hauenstein, Andrew J. Sommese)

A TRANSCENDENTAL METHOD IN ALGEBRAIC GEOMETRY

by PHILLIP A. GRIFFITHS

1. Introduction and an example from curves.

It is well known that the basic objects of algebraic geometry, the smooth projective varieties, depend continuously on parameters as well as having the usual discrete invariants such as homotopy and homology groups. What I shall attempt here is to outline a procedure for measuring this continuous variation of structure. This method uses the periods of suitably defined rational differential forms to construct an intrinsic " continuous " invariant of arbitrary smooth projective varieties. The original aim in defining this " period matrix " of an algebraic variety was to give, at least in some cases, a complete invariant (i. e. " moduli ") of the algebraic structure, as turns out to happen for curves. It is too soon to evaluate the success of this program, but a few interesting things have turned up, and there remain very many attractive unsolved problems. In presenting this talk, I shall not give references as these, together with a more detailed discussion of the material discussed, may be found in my survey paper which appeared in the March (1970) *Bulletin of the American Mathematical Society.*

A transcendental method in numerical AG?

A transcendental method in numerical AG?

The method = symbolic integration

compute a basis of indepentent integrals, compute differential equations for integrals with a parameter

A transcendental method in numerical AG?

The method = **symbolic integration**

compute a basis of indepentent integrals, compute differential equations for integrals with a parameter

\+ **seminumerical methods for solving linear ODEs**

high-precision numerical solving, higher-order methods required

A transcendental method in numerical AG?

The method = symbolic integration

compute a basis of indepentent integrals, compute differential equations for integrals with a parameter

\+ seminumerical methods for solving linear ODEs

high-precision numerical solving, higher-order methods required

\+ effective algebraic topology

to know where to integrate

A transcendental method in numerical AG?

The method = **symbolic integration**
compute a basis of indepentent integrals, compute differential equations for integrals with a parameter

\+ **seminumerical methods for solving linear ODEs**
high-precision numerical solving, higher-order methods required

\+ **effective algebraic topology**
to know where to integrate

\+ **integer relation algorithm (LLL, PSLQ, HJLS)**
to recover exact information from numerical data

Today's goal

Explain on two examples:

* how to compute periods with high precision,
* how to solve a concrete algebraic problem with them.

Periods

X complex algebraic variety manifold of dimension n

$$\int_{\gamma} \omega$$

algebraic differential n-form (ω)

n-cycle (γ)

$*$ boils down to a n-fold integral of an algebraic function

Periods

X complex algebraic variety manifold of dimension n

$$\int_{\gamma} \omega$$

ω: algebraic differential n-form

γ: n-cycle

∗ boils down to a n-fold integral of an algebraic function

💡 contains information about the geometry of X

Periods

X complex algebraic variety manifold of dimension n

$$\int_{\gamma} \omega$$

algebraic differential n-form

n-cycle

* boils down to a n-fold integral of an algebraic function

- contains information about the geometry of X
- often not computable exactly, need hundreds or thousands of digits

Periods

X complex algebraic variety manifold of dimension n

$$\int_\gamma \omega$$

algebraic differential n-form

n-cycle

* boils down to a n-fold integral of an algebraic function

- contains information about the geometry of X
- often not computable exactly, need hundreds or thousands of digits
- in this regime, direct numerical recipes do not work well

Why periods are called periods?

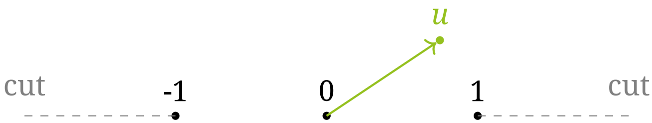
$$X = \{(t,s) \in \mathbb{C}^2 \mid t^2 + s^2 = 1\}, \quad s = \pm\sqrt{1-t^2}$$

$$\sin\left(\int_0^u \frac{\mathrm{d}t}{\sqrt{1-t^2}}\right) = u$$

Why periods are called periods?

$$X = \{(t, s) \in \mathbb{C}^2 \mid t^2 + s^2 = 1\}, \quad s = \pm\sqrt{1 - t^2}$$

$$\sin\left(\int_0^u \frac{\mathrm{d}t}{\sqrt{1-t^2}}\right) = u$$



Why periods are called periods?

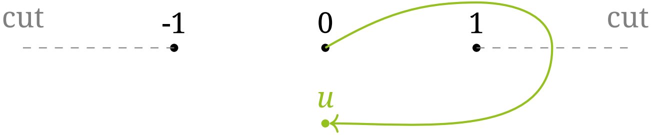
$$X = \left\{ (t,s) \in \mathbb{C}^2 \;\middle|\; t^2 + s^2 = 1 \right\}, \quad s = \pm\sqrt{1-t^2}$$

$$\sin\left(\int_0^u \frac{\mathrm{d}t}{\sqrt{1-t^2}} \right) = u$$

Why periods are called periods?

$$X = \{(t,s) \in \mathbb{C}^2 \mid t^2 + s^2 = 1\}, \quad s = \pm\sqrt{1-t^2}$$

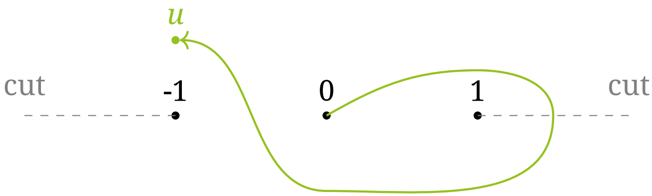
$$\sin\left(\int_0^u \frac{\mathrm{d}t}{\sqrt{1-t^2}}\right) = u$$



Why periods are called periods?

$$X = \left\{(t,s) \in \mathbb{C}^2 \;\middle|\; t^2 + s^2 = 1\right\}, \quad s = \pm\sqrt{1-t^2}$$

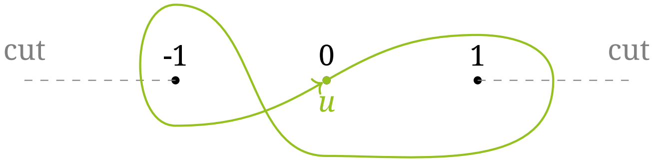
$$\sin\left(\int_0^u \frac{\mathrm{d}t}{\sqrt{1-t^2}}\right) = u$$



Why periods are called periods?

$$X = \left\{(t,s) \in \mathbb{C}^2 \;\middle|\; t^2 + s^2 = 1\right\}, \quad s = \pm\sqrt{1-t^2}$$

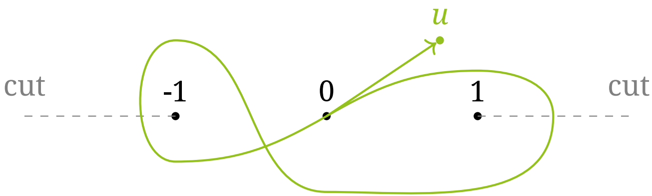
$$\sin\left(\int_0^u \frac{\mathrm{d}t}{\sqrt{1-t^2}}\right) = u$$



Why periods are called periods?

$$X = \{(t,s) \in \mathbb{C}^2 \mid t^2 + s^2 = 1\}, \quad s = \pm\sqrt{1-t^2}$$

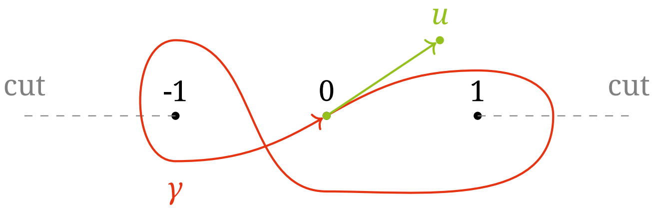
$$\sin\left(\int_0^u \frac{\mathrm{d}t}{\sqrt{1-t^2}}\right) = u$$



Why periods are called periods?

$$X = \left\{ (t,s) \in \mathbb{C}^2 \;\middle|\; t^2 + s^2 = 1 \right\}, \quad s = \pm\sqrt{1-t^2}$$

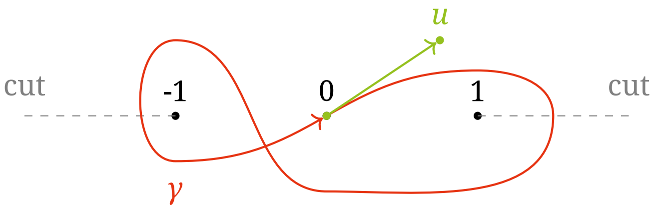
$$\sin\left(\int_\gamma \frac{\mathrm{d}t}{\sqrt{1-t^2}} + \int_0^u \frac{\mathrm{d}t}{\sqrt{1-t^2}} \right) = u$$



Why periods are called periods?

$$X = \left\{(t,s) \in \mathbb{C}^2 \;\middle|\; t^2 + s^2 = 1\right\}, \quad s = \pm\sqrt{1-t^2}$$

$$\sin\Bigg(\underbrace{\int_\gamma \frac{\mathrm{d}t}{\sqrt{1-t^2}}}_{\text{period!}} + \int_0^u \frac{\mathrm{d}t}{\sqrt{1-t^2}}\Bigg) = u$$



SUR LES RÉSIDUS DES INTÉGRALES DOUBLES

PAR

H. POINCARÉ

À PARIS.

L'intégrale envisagée par M. Picard est alors:

$$\int_{u_0}^{u_1} du \int_{v_0}^{v_1} dv\, \Phi(u,\ v)\left(\frac{d\varphi}{du}\frac{d\psi}{dv} - \frac{d\varphi}{dv}\frac{d\psi}{du}\right).$$

M. Picard a donné à ces intégrales le nom de périodes; je ne saurais l'en blâmer puisque cette dénomination lui a permis d'exprimer dans un langage plus concis les intéressants résultats auxquels il est parvenu. Mais je crois qu'il serait fâcheux qu'elle s'introduisît définitivement dans la science et qu'elle serait propre à engendrer de nombreuses confusions.

"M. Picard gave these integrals the name of periods; I cannot blame him since this name allowed him to express in more concise language the interesting results he achieved. But I believe that it would be unfortunate if it were definitively introduced into science and that it would be likely to generate numerous confusions."

Periods

X_t a family of complex algebraic variety manifold of dimension n

$$\alpha(t) = \int_{\gamma_t} \omega_t$$

algebraic differential n-form, rational in t

continuously varying n-cycle

Periods

X_t a family of complex algebraic variety manifold of dimension n

$$\alpha(t) = \int_{\gamma_t} \omega_t$$

ω_t: algebraic differential n-form, rational in t

γ_t: continuously varying n-cycle

- contains information about the geometry of X_t

Periods

X_t a family of complex algebraic variety manifold of dimension n

$$\alpha(t) = \int_{\gamma_t} \omega_t$$

ω_t: algebraic differential n-form, rational in t

γ_t: continuously varying n-cycle

- contains information about the geometry of X_t
- computable exactly, up to finitely many constants

Periods

X_t a family of complex algebraic variety manifold of dimension n

$$\alpha(t) = \int_{\gamma_t} \omega_t$$

ω_t: algebraic differential n-form, rational in t

γ_t: continuously varying n-cycle

- contains information about the geometry of X_t
- computable exactly, up to finitely many constants
- symbolic integration

Periods

X_t a family of complex algebraic variety manifold of dimension n

$$\alpha(t) = \int_{\gamma_t} \omega_t$$

ω_t: algebraic differential n-form, rational in t

γ_t: continuously varying n-cycle

- contains information about the geometry of X_t
- computable exactly, up to finitely many constants
- symbolic integration

Picard–Fuchs equations

There are polynomials $p_0(t), \ldots, p_r(t) \neq 0$ such that

$$p_r(t)\alpha^{(r)}(t) + \cdots + p_1(t)\alpha'(t) + p_0(t)\alpha(t) = 0.$$

High precision numerical integration of linear ODEs

We have a differential equation

$$p_r(t)y^{(r)}(t) + \cdots + p_1(t)y'(t) + p_0(t)y(t) = 0,$$

and initial conditions $y(0), \ldots, y^{(r-1)}(0)$.

ϟ We want $y(1)$ with high precision.

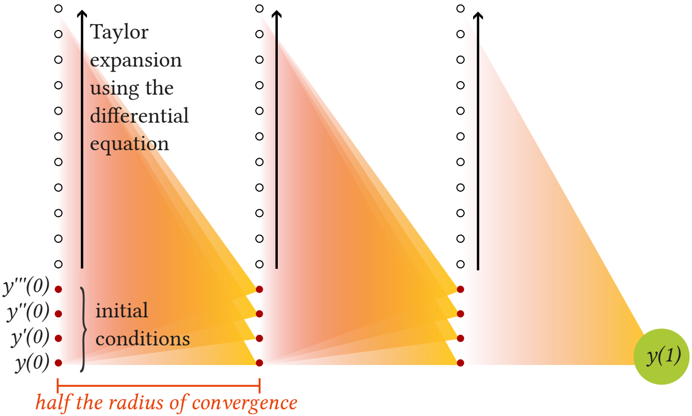
High precision numerical integration of linear ODEs

We have a differential equation

$$p_r(t)y^{(r)}(t) + \cdots + p_1(t)y'(t) + p_0(t)y(t) = 0,$$

and initial conditions $y(0), \ldots, y^{(r-1)}(0)$.

We want $y(1)$ with high precision.



Complexity of numerical LODE solving

Theorem (Chudnovsky and Chudnovsky, 1990; van der Hoeven, 1999; Mezzarobba, 2010)

Consider

- *a linear ODE* $(*)\ p_r(t)y^{(r)}(t) + \cdots + p_1(t)y'(t) + p_0(t)y(t) = 0$
- *a path* $\gamma : [0,1] \to \mathbb{C} \setminus \mathit{zeros}(p_r)$
- *initial condition* $u_0, \ldots, u_{r-1} \in \mathbb{C}$

Then we can compute $y(\gamma_1)$*, up to precision* 2^{-p}*, where* y *is the unique solution of* $(*)$ *such that* $y^{(i)}(\gamma_0) = u_i$ $(0 \leq i < r)$*, analytically continued along* γ*.*

Moreover:

- *The error bound is explicit.*
- *As* $p \to \infty$ *(everything else is fixed), the algorithm runs in time* $\tilde{O}(p)$*.*

High precision numerical integration (variant)

Corollary

In the same context, we can compute $\int_\gamma y(z)\mathrm{d}z$, up to precision 2^{-p}.

Moreover:

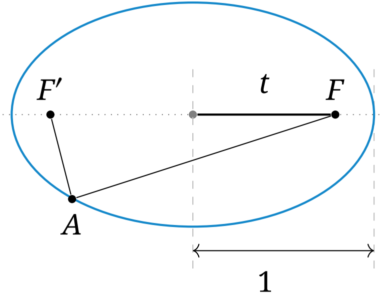
- The error bound is explicit
- As $p \to \infty$ (everything else is fixed), the algorithm runs in time $\tilde{O}(p)$.

Proof. Apply the theorem to the differential equation

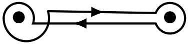
$$p_r(t)I^{(r+1)}(t) + \cdots + p_1(t)I''(t) + p_0(t)I'(t) = 0$$

of which $I(t) = \int_{\gamma_0}^{t} y(z)\mathrm{d}z$ is solution.

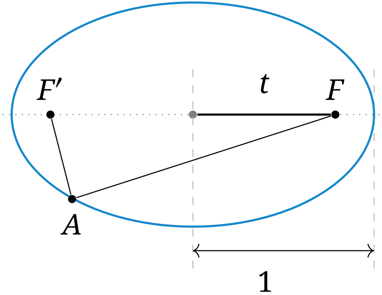
Perimeter of an ellipse



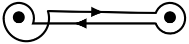
$$E(t) = 2\int_{-1}^{1}\sqrt{1+y'(x)^2}\,\mathrm{d}x$$
$$= 2\int_{-1}^{1}\sqrt{\frac{1-t^2x^2}{1-x^2}}\,\mathrm{d}x$$
$$= \int_{\gamma}\sqrt{\frac{1-t^2x^2}{1-x^2}}\,\mathrm{d}x$$

Where $\gamma =$  .

Perimeter of an ellipse



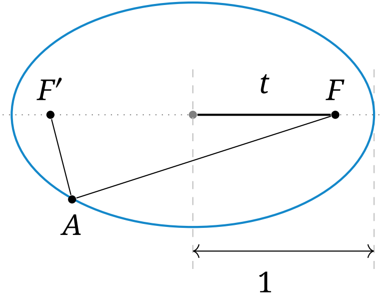
$$E(t) = 2\int_{-1}^{1} \sqrt{1 + y'(x)^2}\,\mathrm{d}x$$
$$= 2\int_{-1}^{1} \sqrt{\frac{1 - t^2x^2}{1 - x^2}}\,\mathrm{d}x$$
$$= \int_{\gamma} \sqrt{\frac{1 - t^2x^2}{1 - x^2}}\,\mathrm{d}x$$

Where $\gamma =$ .

Theorem (Euler, 1733)

$$(t - t^3)E'' + (1 - t^2)E' + tE = 0$$

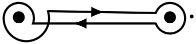
Perimeter of an ellipse



$$E(t) = 2\int_{-1}^{1} \sqrt{1 + y'(x)^2}\,\mathrm{d}x$$

$$= 2\int_{-1}^{1} \sqrt{\frac{1 - t^2x^2}{1 - x^2}}\,\mathrm{d}x$$

$$= \int_{\gamma} \sqrt{\frac{1 - t^2x^2}{1 - x^2}}\,\mathrm{d}x$$

Where $\gamma =$ .

Theorem (Euler, 1733)

$(t - t^3)E'' + (1 - t^2)E' + tE = 0$

Theorem (Liouville, 1834)

*$E(t)$ is **transcendental**.*
It is not even expressible in terms of elementary functions.

Proof of Euler's theorem

Let $F(t,x) = \sqrt{\frac{1-t^2x^2}{1-x^2}}$, so that $E(t) = \int_\gamma F(t,x)\mathrm{d}x$.

Proof of Euler's theorem

Let $F(t,x) = \sqrt{\frac{1-t^2x^2}{1-x^2}}$, so that $E(t) = \int_\gamma F(t,x)\mathrm{d}x$.

$$(t-t^3)\frac{\partial^2 F}{\partial t^2} + (1-t^2)\frac{\partial F}{\partial t} + tF = \frac{\partial}{\partial x}\left(\frac{tx(1-x^2)}{1-t^2x^2}F\right) \quad (*)$$

Proof of Euler's theorem

Let $F(t,x) = \sqrt{\frac{1-t^2x^2}{1-x^2}}$, so that $E(t) = \int_\gamma F(t,x)\mathrm{d}x$.

$$(t - t^3)\frac{\partial^2 F}{\partial t^2} + (1-t^2)\frac{\partial F}{\partial t} + tF = \frac{\partial}{\partial x}\left(\frac{tx(1-x^2)}{1-t^2x^2}F\right) \quad (*)$$

$$\rightsquigarrow (t - t^3)\frac{\partial^2}{\partial t^2}\int_\gamma F\mathrm{d}x + (1-t^2)\frac{\partial}{\partial t}\int_\gamma F\mathrm{d}x + t\int_\gamma F\mathrm{d}x = \int_\gamma \frac{\partial}{\partial x}\left(\frac{tx(1-x^2)}{1-t^2x^2}F\right)\mathrm{d}x$$

$$\rightsquigarrow (t - t^3)E'' + (1-t^2)E' + tE = 0.$$

Proof of Euler's theorem

Let $F(t,x) = \sqrt{\frac{1-t^2x^2}{1-x^2}}$, so that $E(t) = \int_\gamma F(t,x)\mathrm{d}x$.

$$(t-t^3)\frac{\partial^2 F}{\partial t^2} + (1-t^2)\frac{\partial F}{\partial t} + tF = \frac{\partial}{\partial x}\left(\frac{tx(1-x^2)}{1-t^2x^2}F\right) \qquad (*)$$

$$\rightsquigarrow (t-t^3)\frac{\partial^2}{\partial t^2}\int_\gamma F\mathrm{d}x + (1-t^2)\frac{\partial}{\partial t}\int_\gamma F\mathrm{d}x + t\int_\gamma F\mathrm{d}x = \int_\gamma \frac{\partial}{\partial x}\left(\frac{tx(1-x^2)}{1-t^2x^2}F\right)\mathrm{d}x$$

$$\rightsquigarrow (t-t^3)E'' + (1-t^2)E' + tE = 0.$$

- $*$ Symbolic integration provides algorithms for finding the magical relation $(*)$. Keywords: creative telescoping, D-module integration. (Chyzak, 2000; Koutschan, 2010; Oaku & Takayama, 2001; Lairez, 2016; Chen, van Hoeij, Kauers, & Koutschan, 2018; Bostan, Chyzak, Lairez, & Salvy, 2018)

Proof of Euler's theorem

Let $F(t,x) = \sqrt{\frac{1-t^2x^2}{1-x^2}}$, so that $E(t) = \int_\gamma F(t,x)\mathrm{d}x$.

$$(t-t^3)\frac{\partial^2 F}{\partial t^2} + (1-t^2)\frac{\partial F}{\partial t} + tF = \frac{\partial}{\partial x}\left(\frac{tx(1-x^2)}{1-t^2x^2}F\right) \qquad (*)$$

$$\rightsquigarrow (t-t^3)\frac{\partial^2}{\partial t^2}\int_\gamma F\mathrm{d}x + (1-t^2)\frac{\partial}{\partial t}\int_\gamma F\mathrm{d}x + t\int_\gamma F\mathrm{d}x = \int_\gamma \frac{\partial}{\partial x}\left(\frac{tx(1-x^2)}{1-t^2x^2}F\right)\mathrm{d}x$$

$$\rightsquigarrow (t-t^3)E'' + (1-t^2)E' + tE = 0.$$

- Symbolic integration provides algorithms for finding the magical relation $(*)$. Keywords: creative telescoping, D-module integration. (Chyzak, 2000; Koutschan, 2010; Oaku & Takayama, 2001; Lairez, 2016; Chen, van Hoeij, Kauers, & Koutschan, 2018; Bostan, Chyzak, Lairez, & Salvy, 2018)
- Many implementations

Computing the perimeter, 1st method

Gauss quadrature

Let f be a multivalued analytic function on the complex plane.

$$\int_\gamma f(x)\mathrm{d}x = \sum_{i=1}^{N} w_i f(x_i) + O(C^{-N}),$$

for a suitable choice of w_i and $x_i \in (-1, 1)$.

Computing the perimeter, 1st method

Gauss quadrature

Let f be a multivalued analytic function on the complex plane.

$$\int_\gamma f(x)\mathrm{d}x = \sum_{i=1}^{N} w_i f(x_i) + O(C^{-N}),$$

for a suitable choice of w_i and $x_i \in (-1, 1)$.

* Effective error bounds

Computing the perimeter, 1st method

Gauss quadrature

Let f be a multivalued analytic function on the complex plane.

$$\int_\gamma f(x)\mathrm{d}x = \sum_{i=1}^{N} w_i f(x_i) + O(C^{-N}),$$

for a suitable choice of w_i and $x_i \in (-1,1)$.

- Effective error bounds
- Complexity $\tilde{O}(N^2)$ for computing the w_i and the x_i

Computing the perimeter, 1st method

Gauss quadrature

Let f be a multivalued analytic function on the complex plane.

$$\int_\gamma f(x)\mathrm{d}x = \sum_{i=1}^{N} w_i f(x_i) + O(C^{-N}),$$

for a suitable choice of w_i and $x_i \in (-1, 1)$.

- Effective error bounds
- Complexity $\tilde{O}(N^2)$ for computing the w_i and the x_i
- Needs evaluation of f at precision C^{-N} at N points

Computing the perimeter, 1st method

Gauss quadrature

Let f be a multivalued analytic function on the complex plane.

$$\int_\gamma f(x)\mathrm{d}x = \sum_{i=1}^{N} w_i f(x_i) + O(C^{-N}),$$

for a suitable choice of w_i and $x_i \in (-1, 1)$.

- ∗ Effective error bounds
- ∗ Complexity $\tilde{O}(N^2)$ for computing the w_i and the x_i
- ∗ Needs evaluation of f at precision C^{-N} at N points
- $\rightsquigarrow$ For k-fold integrals, this leads to a $\tilde{O}(N^{k+1})$ total complexity for computing N digits.

Computing the perimeter, 2nd method

Goal: Compute $E(\frac{1}{2})$

Transcendental continuation, **outer** variant

Computing the perimeter, 2nd method

Goal: Compute $E(\frac{1}{2})$

Transcendental continuation, **outer** variant

1. We know the differential equation $(t - t^3)E'' + (1 - t^2)E' + tE = 0$.

Computing the perimeter, 2nd method

Goal: Compute $E(\frac{1}{2})$

Transcendental continuation, **outer** variant

1. We know the differential equation $(t - t^3)E'' + (1 - t^2)E' + tE = 0.$
2. We compute easily that $E(t) = 2\pi - \frac{\pi}{2}t^2 + O(t^4).$

Computing the perimeter, 2nd method

Goal: Compute $E(\frac{1}{2})$

Transcendental continuation, outer variant

1. We know the differential equation $(t - t^3)E'' + (1 - t^2)E' + tE = 0$.
2. We compute easily that $E(t) = 2\pi - \frac{\pi}{2}t^2 + O(t^4)$.
3. Apply the continuation algorithm to compute $E(\frac{1}{2})$.

Computing the perimeter, 2nd method

Goal: Compute $E(\frac{1}{2})$

Transcendental continuation, outer variant

1. We know the differential equation $(t - t^3)E'' + (1 - t^2)E' + tE = 0$.
2. We compute easily that $E(t) = 2\pi - \frac{\pi}{2}t^2 + O(t^4)$.
3. Apply the continuation algorithm to compute $E(\frac{1}{2})$.

* This is the "outer" method because to compute $E(\frac{1}{2})$, we embed it into the larger family $E(t)$.

Computing the perimeter, 2nd method

Goal: Compute $E(\frac{1}{2})$

Transcendental continuation, **outer** variant

1. We know the differential equation $(t - t^3)E'' + (1 - t^2)E' + tE = 0$.
2. We compute easily that $E(t) = 2\pi - \frac{\pi}{2}t^2 + O(t^4)$.
3. Apply the continuation algorithm to compute $E(\frac{1}{2})$.

* This is the “outer” method because to compute $E(\frac{1}{2})$, we embed it into the larger family $E(t)$.

⚠ Need to find a good starting point.

Computing the perimeter, 2nd method

Goal: Compute $E(\frac{1}{2})$

Transcendental continuation, outer variant

1. We know the differential equation $(t - t^3)E'' + (1 - t^2)E' + tE = 0$.
2. We compute easily that $E(t) = 2\pi - \frac{\pi}{2}t^2 + O(t^4)$.
3. Apply the continuation algorithm to compute $E(\frac{1}{2})$.

- ∗ This is the "outer" method because to compute $E(\frac{1}{2})$, we embed it into the larger family $E(t)$.
- ⚠ Need to find a good starting point.
- 💡 Little geometry involved.

Computing the perimeter, 2nd method

Goal: Compute $E(\frac{1}{2})$

Transcendental continuation, outer variant

1. We know the differential equation $(t - t^3)E'' + (1 - t^2)E' + tE = 0$.
2. We compute easily that $E(t) = 2\pi - \frac{\pi}{2}t^2 + O(t^4)$.
3. Apply the continuation algorithm to compute $E(\frac{1}{2})$.

- ∗ This is the "outer" method because to compute $E(\frac{1}{2})$, we embed it into the larger family $E(t)$.
- ⚠ Need to find a good starting point.
- 💡 Little geometry involved.
- 💡 Quasi-linear complexity with respect to precision.

Computing the perimeter, 3rd method

Goal: Compute $E(\frac{1}{2})$

Transcendental continuation, inner variant

Let $R(x) = \sqrt{\frac{1-\frac{1}{4}x^2}{1-x^2}}$, so that $E(\frac{1}{2}) = \int_\gamma R(x)\mathrm{d}x$.

Computing the perimeter, 3rd method

Goal: Compute $E(\frac{1}{2})$

Transcendental continuation, **inner** variant

Let $R(x) = \sqrt{\frac{1-\frac{1}{4}x^2}{1-x^2}}$, so that $E(\frac{1}{2}) = \int_\gamma R(x)\mathrm{d}x$.

1. We compute easily $R(x) = 1 + O(x^2)$.

Computing the perimeter, 3rd method

Goal: Compute $E(\frac{1}{2})$

Transcendental continuation, **inner** variant

Let $R(x) = \sqrt{\frac{1-\frac{1}{4}x^2}{1-x^2}}$, so that $E(\frac{1}{2}) = \int_\gamma R(x)\mathrm{d}x$.

1. We compute easily $R(x) = 1 + O(x^2)$.
2. We know the differential equation $(x^4 - 5x^2 + 4)R'(x) - 3xR(x) = 0$.

Computing the perimeter, 3rd method

Goal: Compute $E(\frac{1}{2})$

Transcendental continuation, inner variant

Let $R(x) = \sqrt{\frac{1-\frac{1}{4}x^2}{1-x^2}}$, so that $E(\frac{1}{2}) = \int_\gamma R(x)\mathrm{d}x$.

1. We compute easily $R(x) = 1 + O(x^2)$.
2. We know the differential equation $(x^4 - 5x^2 + 4)R'(x) - 3xR(x) = 0$.
3. Apply the continuation algorithm along γ to compute $E(\frac{1}{2})$.

Computing the perimeter, 3rd method

Goal: Compute $E(\frac{1}{2})$

Transcendental continuation, inner variant

Let $R(x) = \sqrt{\frac{1-\frac{1}{4}x^2}{1-x^2}}$, so that $E(\frac{1}{2}) = \int_\gamma R(x)\mathrm{d}x$.

1. We compute easily $R(x) = 1 + O(x^2)$.
2. We know the differential equation $(x^4 - 5x^2 + 4)R'(x) - 3xR(x) = 0$.
3. Apply the continuation algorithm along γ to compute $E(\frac{1}{2})$.

Computing the perimeter, 3rd method

Goal: Compute $E(\frac{1}{2})$

Transcendental continuation, inner variant

Let $R(x) = \sqrt{\frac{1-\frac{1}{4}x^2}{1-x^2}}$, so that $E(\frac{1}{2}) = \int_\gamma R(x)\mathrm{d}x$.

1. We compute easily $R(x) = 1 + O(x^2)$.
2. We know the differential equation $(x^4 - 5x^2 + 4)R'(x) - 3xR(x) = 0$.
3. Apply the continuation algorithm along γ to compute $E(\frac{1}{2})$.

∗ This is the “inner” method because to compute $E(\frac{1}{2})$ we work on ellipse, we don’t deform the ellipse.

Computing the perimeter, 3rd method

Goal: Compute $E(\frac{1}{2})$

Transcendental continuation, inner variant

Let $R(x) = \sqrt{\frac{1-\frac{1}{4}x^2}{1-x^2}}$, so that $E(\frac{1}{2}) = \int_\gamma R(x)\mathrm{d}x$.

1. We compute easily $R(x) = 1 + O(x^2)$.
2. We know the differential equation $(x^4 - 5x^2 + 4)R'(x) - 3xR(x) = 0$.
3. Apply the continuation algorithm along γ to compute $E(\frac{1}{2})$.

∗ This is the “inner” method because to compute $E(\frac{1}{2})$ we work on ellipse, we don’t deform the ellipse.

Initial conditions are simpler than what we want to compute.

Computing the perimeter, 3rd method

Goal: Compute $E(\frac{1}{2})$

Transcendental continuation, **inner** variant

Let $R(x) = \sqrt{\frac{1-\frac{1}{4}x^2}{1-x^2}}$, so that $E(\frac{1}{2}) = \int_\gamma R(x)\mathrm{d}x$.

1. We compute easily $R(x) = 1 + O(x^2)$.
2. We know the differential equation $(x^4 - 5x^2 + 4)R'(x) - 3xR(x) = 0$.
3. Apply the continuation algorithm along γ to compute $E(\frac{1}{2})$.

∗ This is the “inner” method because to compute $E(\frac{1}{2})$ we work on ellipse, we don't deform the ellipse.

Initial conditions are simpler than what we want to compute.

Needs more geometry, we need to figure out explicitely γ.

Computing the perimeter, 3rd method

Goal: Compute $E(\frac{1}{2})$

Transcendental continuation, inner variant

Let $R(x) = \sqrt{\frac{1-\frac{1}{4}x^2}{1-x^2}}$, so that $E(\frac{1}{2}) = \int_\gamma R(x)\mathrm{d}x$.

1. We compute easily $R(x) = 1 + O(x^2)$.
2. We know the differential equation $(x^4 - 5x^2 + 4)R'(x) - 3xR(x) = 0$.
3. Apply the continuation algorithm along γ to compute $E(\frac{1}{2})$.

- ∗ This is the "inner" method because to compute $E(\frac{1}{2})$ we work on ellipse, we don't deform the ellipse.
- Initial conditions are simpler than what we want to compute.
- Needs more geometry, we need to figure out explicitely γ.
- Quasi-linear complexity with respect to precision.

Computing the perimeter, 3rd method

Goal: Compute $E(\frac{1}{2})$

Transcendental continuation, inner variant

Let $R(x) = \sqrt{\frac{1-\frac{1}{4}x^2}{1-x^2}}$, so that $E(\frac{1}{2}) = \int_\gamma R(x)\mathrm{d}x$.

1. We compute easily $R(x) = 1 + O(x^2)$.
2. We know the differential equation $(x^4 - 5x^2 + 4)R'(x) - 3xR(x) = 0$.
3. Apply the continuation algorithm along γ to compute $E(\frac{1}{2})$.

∗ This is the "inner" method because to compute $E(\frac{1}{2})$ we work on ellipse, we don't deform the ellipse.

- Initial conditions are simpler than what we want to compute.
- Needs more geometry, we need to figure out explicitely γ.
- Quasi-linear complexity with respect to precision.

Computing the perimeter, 3rd method

Goal: Compute $E(\frac{1}{2})$

Transcendental continuation, inner variant

Let $R(x) = \sqrt{\frac{1-\frac{1}{4}x^2}{1-x^2}}$, so that $E(\frac{1}{2}) = \int_\gamma R(x)\mathrm{d}x$.

1. We compute easily $R(x) = 1 + O(x^2)$.
2. We know the differential equation $(x^4 - 5x^2 + 4)R'(x) - 3xR(x) = 0$.
3. Apply the continuation algorithm along γ to compute $E(\frac{1}{2})$.

- ∗ This is the "inner" method because to compute $E(\frac{1}{2})$ we work on ellipse, we don't deform the ellipse.
- Initial conditions are simpler than what we want to compute.
- Needs more geometry, we need to figure out explicitely γ.
- Quasi-linear complexity with respect to precision.

Demo!

Under the hood

How to compute 10000 decimal digits of

$$E(\tfrac{1}{2}) = 2\pi \sum_{n=0}^{\infty} \underbrace{\frac{64^{-n}}{1-2n}\binom{2n}{n}}_{=u_n}?$$

- $u_n \sim Cn^{-2}4^n$,
  so we need to sum 17 000 $\left(\approx \frac{\log(4)}{\log(10)} 10000\right)$ terms.
- $u_{n+1} = \frac{4n^2-1}{16(n+1)^2} u_n$, and $u_0 = 1$,
  that's an easy formula but computing all the u_n is quadratic time.

Under the hood: binary splitting

Define $U(a,b) = \dfrac{u_b}{u_a}$ and $S(a,b) = \dfrac{1}{u_a} \sum_{k=a}^{b-1} u_k$.

NB: We want to compute S(0, 17000).

Under the hood: binary splitting

Define $U(a,b) = \dfrac{u_b}{u_a}$ and $S(a,b) = \dfrac{1}{u_a}\displaystyle\sum_{k=a}^{b-1} u_k$.

NB: We want to compute S(0, 17000).

Divide and conquer! For any $a \leq c \leq b$

$$U(a,b) = \underbrace{U(a,c)U(c,b)}_{\text{fast multiplication}} \text{ and } S(a,b) = S(a,c) + \underbrace{U(a,b)S(b,c)}_{\text{fast multiplication}}.$$

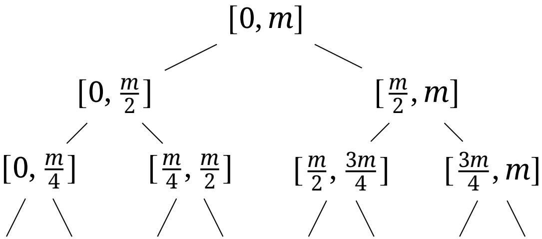
Under the hood: binary splitting

Define $U(a,b) = \dfrac{u_b}{u_a}$ and $S(a,b) = \dfrac{1}{u_a}\sum_{k=a}^{b-1} u_k$.

NB: We want to compute S(0, 17000).

Divide and conquer! For any $a \leq c \leq b$

$$U(a,b) = \underbrace{U(a,c)U(c,b)}_{\text{fast multiplication}} \text{ and } S(a,b) = S(a,c) + \underbrace{U(a,b)S(b,c)}_{\text{fast multiplication}}.$$



At depth k, there are $O(2^k)$ operations with integers with $\tilde{O}(m2^{-k})$ digits.

$\tilde{O}(m)$ complexity

Wrap up

* Transcendental functions arise from algebraic varieties and $\int$
* We can compute differential equations for integrals with a parameter
* We can compute numerically integrals (without parameter):
  - by the outer method, which introduces a parameter in the integral,
  - by the inner method, which uses the first integration variable as the parameter.
* We can compute to large precision thanks to quasilinear complexity.
  - But the precision is not the only large parameter, the size of the differential equations can grow big.

The endomorphism ring of an elliptic curve

Let $X = \left\{y^2 = x^3 + ax + b\right\} \subset \mathbb{P}^2(\mathbb{C})$ be an elliptic curve.

The endomorphism ring of an elliptic curve

Let $X = \left\{y^2 = x^3 + ax + b\right\} \subset \mathbb{P}^2(\mathbb{C})$ be an elliptic curve.

* X has the structure of an abelian group.
* $\mathrm{End}(X) = \{\text{regular maps } f : X \to X \text{ with } f(0) = 0\}$
  (they are automatically group endomorphisms).
* $\mathrm{End}(X)$ contains at least all the maps $p \in X \mapsto np$ with $n \in \mathbb{Z}$.

The endomorphism ring of an elliptic curve

Let $X = \{y^2 = x^3 + ax + b\} \subset \mathbb{P}^2(\mathbb{C})$ be an elliptic curve.

* X has the structure of an abelian group.
* $\mathrm{End}(X) = \{\text{regular maps } f : X \longrightarrow X \text{ with } f(0) = 0\}$ (they are automatically group endomorphisms).
* $\mathrm{End}(X)$ contains at least all the maps $p \in X \mapsto np$ with $n \in \mathbb{Z}$.

Problem

Is $\mathrm{End}(X)$ nontrivial ($\neq \mathbb{Z}$)?

Nature of the problem

Theorem

The set for all $a, b \in \mathbb{C}^2$ *such that the curve* $X = \left\{y^2 = x^3 + ax + b\right\}$ *has a nontrivial endomorphism is the union of countably many curves in* $\mathbb{C}^2$.

Nature of the problem

Theorem

The set for all $a, b \in \mathbb{C}^2$ *such that the curve* $X = \left\{y^2 = x^3 + ax + b\right\}$ *has a nontrivial endomorphism is the union of countably many curves in* $\mathbb{C}^2$.

- The problem does *not* reduce directly to polynomial system solving.

Nature of the problem

Theorem

The set for all $a, b \in \mathbb{C}^2$ such that the curve $X = \{y^2 = x^3 + ax + b\}$ has a nontrivial endomorphism is the union of countably many curves in $\mathbb{C}^2$.

* The problem does *not* reduce directly to polynomial system solving.
* *Most* elliptic curves does not have a nontrivial endomorphism.

Nature of the problem

Theorem

The set for all $a, b \in \mathbb{C}^2$ *such that the curve* $X = \left\{y^2 = x^3 + ax + b\right\}$ *has a nontrivial endomorphism is the union of countably many curves in* $\mathbb{C}^2$.

* The problem does *not* reduce directly to polynomial system solving.
* *Most* elliptic curves does not have a nontrivial endomorphism.
* But elliptic curves with a nontrivial endomorphism are *dense*!

Nature of the problem

Theorem

The set for all $a, b \in \mathbb{C}^2$ *such that the curve* $X = \left\{y^2 = x^3 + ax + b\right\}$ *has a nontrivial endomorphism is the union of countably many curves in* $\mathbb{C}^2$.

- The problem does *not* reduce directly to polynomial system solving.
- *Most* elliptic curves does not have a nontrivial endomorphism.
- But elliptic curves with a nontrivial endomorphism are *dense*!
- See Cremona and Sutherland (2023) for a recent progress on the question (algebraic approach).

Analytic approach

- There a meromorphic map $\wp : \mathbb{C} \longrightarrow \mathbb{C}$, *Weierstrass' function*, such that $z \longrightarrow (\wp(z), \wp'(z))$ is a sujective group homomorphism.
- It induces an isomorphism $X \simeq \mathbb{C}/\Lambda$, with $\Lambda = \mathbb{Z}\,\alpha_1 + \mathbb{Z}\,\alpha_2$. α_1 and α_2 are the *periods* of $\wp$.

Analytic approach

* There a meromorphic map $\wp : \mathbb{C} \longrightarrow \mathbb{C}$, *Weierstrass' function*, such that $z \longrightarrow (\wp(z), \wp'(z))$ is a sujective group homomorphism.
* It induces an isomorphism $X \simeq \mathbb{C}/\Lambda$, with $\Lambda = \mathbb{Z}\,\alpha_1 + \mathbb{Z}\,\alpha_2$. α_1 and α_2 are the *periods* of $\wp$.

Problem

Does $\mathbb{C}/\Lambda$ have a nontrivial *analytic* endomorphism?

Analytic approach

- There a meromorphic map $\wp : \mathbb{C} \longrightarrow \mathbb{C}$, *Weierstrass' function*, such that $z \longrightarrow (\wp(z), \wp'(z))$ is a sujective group homomorphism.
- It induces an isomorphism $X \simeq \mathbb{C}/\Lambda$, with $\Lambda = \mathbb{Z}\,\alpha_1 + \mathbb{Z}\,\alpha_2$. α_1 and α_2 are the *periods* of $\wp$.

Problem

Does $\mathbb{C}/\Lambda$ have a nontrivial *analytic* endomorphism?

- The continuous endomorphisms of $\mathbb{C}/\Lambda$ are induced by continuous endomorphisms of $\mathbb{C}$, that are $\mathbb{R}$-linear maps $\phi : \mathbb{C} \longrightarrow \mathbb{C}$ such that $\phi(\Lambda) \subseteq \Lambda$.

Analytic approach

- There a meromorphic map $\wp : \mathbb{C} \longrightarrow \mathbb{C}$, *Weierstrass' function*, such that $z \longrightarrow (\wp(z), \wp'(z))$ is a sujective group homomorphism.
- It induces an isomorphism $X \simeq \mathbb{C}/\Lambda$, with $\Lambda = \mathbb{Z}\,\alpha_1 + \mathbb{Z}\,\alpha_2$. α_1 and α_2 are the *periods* of $\wp$.

Problem

Does $\mathbb{C}/\Lambda$ have a nontrivial *analytic* endomorphism?

- The continuous endomorphisms of $\mathbb{C}/\Lambda$ are induced by continuous endomorphisms of $\mathbb{C}$, that are $\mathbb{R}$-linear maps $\phi : \mathbb{C} \longrightarrow \mathbb{C}$ such that $\phi(\Lambda) \subseteq \Lambda$.
- The analytic endomorphisms of $\mathbb{C}/\Lambda$ are induced by analytic endomorphisms of $\mathbb{C}$.

Analytic approach

- There a meromorphic map $\wp : \mathbb{C} \longrightarrow \mathbb{C}$, *Weierstrass' function*, such that $z \longrightarrow (\wp(z), \wp'(z))$ is a sujective group homomorphism.
- It induces an isomorphism $X \simeq \mathbb{C}/\Lambda$, with $\Lambda = \mathbb{Z}\,\alpha_1 + \mathbb{Z}\,\alpha_2$. α_1 and α_2 are the *periods* of $\wp$.

Problem

Does $\mathbb{C}/\Lambda$ have a nontrivial *analytic* endomorphism?

- The continuous endomorphisms of $\mathbb{C}/\Lambda$ are induced by continuous endomorphisms of $\mathbb{C}$, that are $\mathbb{R}$-linear maps $\phi : \mathbb{C} \longrightarrow \mathbb{C}$ such that $\phi(\Lambda) \subseteq \Lambda$.
- The analytic endomorphisms of $\mathbb{C}/\Lambda$ are induced by analytic endomorphisms of $\mathbb{C}$.
- The analytic endomorphisms of $\mathbb{C}$ are the maps $z \mapsto uz$, for $u \in \mathbb{C}$.

The endomorphism ring of a torus

Proposition

$\mathrm{End}(X) \simeq \{u \in \mathbb{C} \mid u\Lambda \subseteq \Lambda\}$.

The endomorphism ring of a torus

Proposition

$\mathrm{End}(X) \simeq \{u \in \mathbb{C} \mid u\Lambda \subseteq \Lambda\}$.

Corollary

$\mathrm{End}(X)$ is nontrivial if and only if the equation

$$\begin{cases} z\,\alpha_1 = a\,\alpha_1 + b\,\alpha_2 \\ z\,\alpha_2 = c\,\alpha_1 + d\,\alpha_2 \end{cases}$$

has a solution $z \in \mathbb{C}$ and $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathbb{Z}^{2\times 2}$ not scalar.

The endomorphism ring of a torus

Proposition

$\mathrm{End}(X) \simeq \{u \in \mathbb{C} \mid u\Lambda \subseteq \Lambda\}$.

Corollary

$\mathrm{End}(X)$ is nontrivial if and only if the equation

$$\begin{cases} u\,\alpha_1 = a\,\alpha_1 + b\,\alpha_2 \\ u\,\alpha_2 = c\,\alpha_1 + d\,\alpha_2 \end{cases}$$

has a solution $u \in \mathbb{C}$ and $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathbb{Z}^{2\times 2}$ not scalar.

The endomorphism ring of a torus

Proposition

$$\mathrm{End}(X) \simeq \{u \in \mathbb{C} \mid u\Lambda \subseteq \Lambda\}\,.$$

Corollary

$\mathrm{End}(X)$ is nontrivial if and only if the equation

$$\begin{cases} u\,\alpha_1 = a\,\alpha_1 + b\,\alpha_2 \\ u\,\alpha_2 = c\,\alpha_1 \end{cases}$$

has a solution $u \in \mathbb{C}$ and $\left(\begin{smallmatrix} a & b \\ c & 0 \end{smallmatrix}\right) \in \mathbb{Z}^{2\times 2}$ not zero.

The endomorphism ring of a torus

Proposition

$$\mathrm{End}(X) \simeq \{u \in \mathbb{C} \mid u\Lambda \subseteq \Lambda\}.$$

Corollary

$\mathrm{End}(X)$ is nontrivial if and only if the equation

$$b\,\alpha_2^2 + a\,\alpha_1\alpha_2 - c\,\alpha_1^2 = 0.$$

has a nonzero solution, $a, b, c \in \mathbb{Z}$.

The endomorphism ring of a torus

Proposition

$\mathrm{End}(X) \simeq \{u \in \mathbb{C} \mid u\Lambda \subseteq \Lambda\}.$

Corollary

$\mathrm{End}(X)$ is nontrivial if and only if the equation

$$b\,\tau^2 + a\,\tau - c = 0.$$

has a nonzero solution, $a, b, c \in \mathbb{Z}$, where $\tau = \alpha_2/\alpha_1$.

Recover exact data from approximate numbers?

Assume that we have computed τ with large precision.

Can we decide if there are nonzero integers a, b, and c such that $b\,\tau^2 + a\,\tau - c = 0$?

Recover exact data from approximate numbers?

Assume that we have computed τ with large precision.

Can we decide if there are nonzero integers a, b, and c such that $b\,\tau^2 + a\,\tau - c = 0$?

NO! (solutions appear or disappear with small perturbations)

Recover exact data from approximate numbers?

Assume that we have computed τ with large precision.

Can we decide if there are nonzero integers a, b, and c such that $b\,\tau^2 + a\,\tau - c = 0$?

NO! (solutions appear or disappear with small perturbations)

Yet, we do it every day. Which one of the following numbers is rational?

$$1.6180339887498948482045868343656381177203091798057628\\
62135448622705260462818902449707207204189391138\ldots$$

$$1.6153846153846153846153846153846153846153846153846153\\
84615384615384615384615384615384615384615384615\ldots$$

Recover exact data from approximate numbers?

Assume that we have computed τ with large precision.

Can we decide if there are nonzero integers a, b, and c such that $b\,\tau^2 + a\,\tau - c = 0$?

NO! (solutions appear or disappear with small perturbations)

Yet, we do it every day. Which one of the following numbers is rational?

$$1.61803398874989484820458683436563811772030917980576286213544862270526046281890244970720720418939113 8\ldots$$

$$1.61538461538461538461538461538461538461538461538461538461538461538461538461538461538461538461538461 5\ldots$$

Impossible question, but good practical answer: lattice reduction.

Computation of the periods

Recall that $\wp : \mathbb{C} \to \mathbb{C}$ is Weierstrass' functions and $(\wp(z), \wp'(z)) \in X$, that is

$$\wp'(z)^2 = \wp(z)^3 + a\wp(z) + b.$$

Computation of the periods

Recall that $\wp : \mathbb{C} \to \mathbb{C}$ is Weierstrass' functions and $(\wp(z), \wp'(z)) \in X$, that is

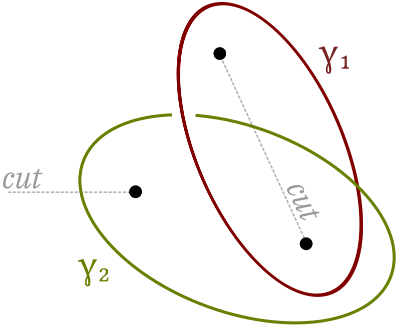
$$\wp'(z)^2 = \wp(z)^3 + a\wp(z) + b.$$

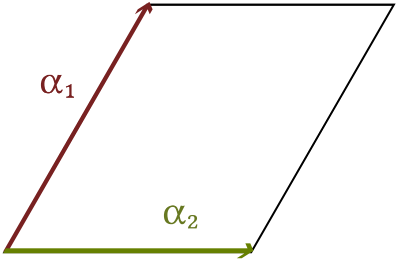
It follows that

$$\wp\left(\int_0^u \frac{\mathrm{d}x}{\sqrt{x^3 + ax + b}}\right) = u.$$

(Does it remind you of something?)

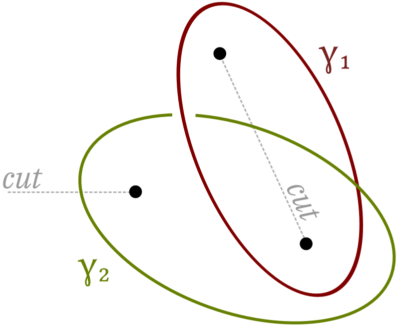
Computation of the periods

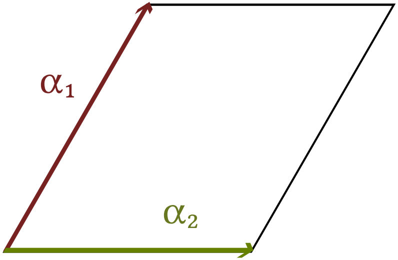




$$\alpha_i = \int_{\gamma_i} \frac{\mathrm{d}x}{\sqrt{x^3 + ax + b}}$$

Computation of the periods





$$\alpha_i = \int_{\gamma_i} \frac{\mathrm{d}x}{\sqrt{x^3 + ax + b}}$$

Demo!

High precision quadrature uncovers the endomorphism ring of elliptic curves.

High precision quadrature uncovers the endomorphism ring of elliptic curves.

heuristic algorithm, only provides a safe bet.
No known way to trick the heuristic.

* Possibility to certify *a posteriori* (e.g. Costa, Mascot, Sijsling, & Voight, 2019), at the cost of simplicity of course
* For algorithms for computing periods of curves, see Deconinck and van Hoeij (2001), Neurohr (2018), and Molin and Neurohr (2019) for example.

Curves on a surface

Let $f \in \mathbb{C}[w,x,y,z]_4 \simeq \mathbb{C}^{35}$
such that $X = V(f) \subseteq \mathbb{P}^3$ is smooth.

* X contains algebraic curves.
* *Trivial* curves are those obtained by intersecting X with another surface. (Every curve is included in the intersection with another surface, but may not be equal.)

Problem

Does X contain a nontrivial curve?

Curves on a surface

Let $f \in \mathbb{C}[w,x,y,z]_4 \simeq \mathbb{C}^{35}$
such that $X = V(f) \subseteq \mathbb{P}^3$ is smooth.

- X contains algebraic curves.
- *Trivial* curves are those obtained by intersecting X with another surface. (Every curve is included in the intersection with another surface, but may not be equal.)

Problem

Does X contain a nontrivial curve?

Noether-Lefschetz theorem (Lefschetz, 1924)

Let $f \in \mathbb{C}[w,x,y,z]_4 \setminus$ (countable union of algebraic hypersurfaces).
Then X_f contains only trivial curves.

Example of a nontrivial curve

Let $X = V(w^4 - x^4 + y^4 - z^4)$ in $\mathbb{P}^3$.

* X contains the line $L = \left\{[u : u : v : v] \,\middle|\, [u : v] \in \mathbb{P}^1\right\}$.
* This line is a curve of degree 1 in $\mathbb{P}^3$: it has exactly one intersection point with a generic plane.

Example of a nontrivial curve

Let $X = V(w^4 - x^4 + y^4 - z^4)$ in $\mathbb{P}^3$.

- X contains the line $L = \left\{[u : u : v : v] \,\middle|\, [u : v] \in \mathbb{P}^1\right\}$.
- This line is a curve of degree 1 in $\mathbb{P}^3$: it has exactly one intersection point with a generic plane.

Let $S = V(g)$ be another surface in $\mathbb{P}^3$.

- The curve $X \cap S$ has degree $4\deg(g)$, because the intersection points of X, S and a generic hyperplane are the solutions of a polynomial system with an equation of degree 4, and equation of degree $\deg(g)$, and an equation of degree 1.

$\rightsquigarrow$ The line L is not the intersection of X with another surface.

Findind hay in a haystack

Theorem (Terasoma, 1985)

There is a smooth $f \in \mathbb{Q}[w,x,y,z]_4$
such that X_f contains only trivial curves.

Findind hay in a haystack

Theorem (Terasoma, 1985)

There is a smooth $f \in \mathbb{Q}[w,x,y,z]_4$ such that X_f contains only trivial curves.

Theorem (van Luijk, 2007)

Let $f = 2w^4 + w^3z + w^2x^2 + 2w^2xy + 2w^2xz - w^2y^2 + w^2z^2 + wx^3 - wx^2y - wx^2z - wxy^2 - wxyz + wxz^2 + wy^3 + wy^2z + wyz^2 - 3x^2y^2 - xy^2z - 4xyz^2 - 2xz^3 - 5yz^3 - z^4$. Then X_f contains only trivial curves.

Findind hay in a haystack

Theorem (Terasoma, 1985)

There is a smooth $f \in \mathbb{Q}[w,x,y,z]_4$
such that X_f contains only trivial curves.

Theorem (van Luijk, 2007)

Let $f = 2w^4+w^3z+w^2x^2+2w^2xy+2w^2xz-w^2y^2+w^2z^2+wx^3-wx^2y-wx^2z-wxy^2-wxyz+wxz^2+wy^3+wy^2z+wyz^2-3x^2y^2-xy^2z-4xyz^2-2xz^3-5yz^3-z^4$.
Then X_f contains only trivial curves.

Theorem (Lairez & Sertöz, 2019)

Let $f = wx^3 + w^3y + xz^3 + y^4 + z^4$. Then X_f contains only trivial curves.

Nature of the problem

Reduction to countably many polynomial systems.

$$
\begin{aligned}
\{\text{lines in } X\} &= \left\{(u,v) \in (\mathbb{C}^4)^2 \;\middle|\; u \wedge v \neq 0 \text{ and } \forall t, f(u+tv) = 0\right\} /\!\sim \\
\{\text{conic curves in } X\} &= \Big\{(u,v,w) \in (\mathbb{C}^4)^3 \;\Big| \\
&\qquad u \wedge v \wedge w \neq 0 \text{ and } \forall t, f(u+tv+t^2 w) = 0\Big\}/\!\sim \\
\{\text{twisted cubics in } X\} &= \Big\{(u_0,\dots,u_3) \in (\mathbb{C}^4)^4 \;\Big| \\
&\qquad u_0 \wedge \cdots \wedge u_3 \neq 0 \text{ and } \forall t, f\left(\sum_{i=0}^{3} u_i t^i\right) = 0\Big\}/\!\sim \\
\{\text{deg. 4 gen. 1 c. in } X\} &= \Big\{(g_1,g_2,h_1,h_2) \in (\mathbb{C}[\mathbf{x}]_2)^4 \;\Big| \\
&\qquad g_1 \text{ and } g_2 \text{ generic and } f = h_1 g_1 + h_2 g_2\Big\}/\!\sim
\end{aligned}
$$

The structure of curves on a surface

Let X be a smooth quartic complex surface.

Consider the 2nd singular homology group of X:

$$H_2(X, \mathbb{Z}) = \frac{\text{sum of triangles in } X \text{ with no boundary}}{\text{sum of boundaries of 3-simplices in } X} \simeq \mathbb{Z}^{22}.$$

The structure of curves on a surface

Let X be a smooth quartic complex surface.

Consider the 2nd singular homology group of X:

$$H_2(X,\mathbb{Z}) = \frac{\text{sum of triangles in } X \text{ with no boundary}}{\text{sum of boundaries of 3-simplices in } X} \simeq \mathbb{Z}^{22}.$$

A curve $C \subset X$ can be triangulated, so we can consider the Néron-Severi group

$$\mathrm{NS}(X) = \{[C] \in H_2(X) \mid C \text{ is a curve on } X\}.$$

The structure of curves on a surface

Let X be a smooth quartic complex surface.

Consider the 2nd singular homology group of X:

$$H_2(X,\mathbb{Z}) = \frac{\text{sum of triangles in } X \text{ with no boundary}}{\text{sum of boundaries of 3-simplices in } X} \simeq \mathbb{Z}^{22}.$$

A curve $C \subset X$ can be triangulated, so we can consider the Néron-Severi group

$$\mathrm{NS}(X) = \{[C] \in H_2(X) \mid C \text{ is a curve on } X\}.$$

Noether-Lefschetz theorem (Lefschetz, 1924)

Let $f \in \mathbb{C}[w,x,y,z]_4 \setminus$ (countable union of algebraic hypersurfaces). Then $\mathrm{NS}(X_f) = \mathbb{Z}$.

Periods of a quartic surface

Let $f \in \mathbb{C}[w,x,y,z]_4 \simeq \mathbb{C}^{35}$
such that $X = V(f) \subseteq \mathbb{P}^3$ is smooth.

Let $\gamma_1, \dots, \gamma_{22}$ be a basis of $H_2(X, \mathbb{Z})$,
and let $\omega_X \in \Omega^2(X)$ be the unique holomorphic 2-form on X.

The *periods* of X are the complex numbers $\alpha_1, \dots, \alpha_{22}$ defined – up to scaling and choice of basis – by

$$\alpha_i \stackrel{\text{def}}{=} \oint_{\gamma_i} \omega_X = \frac{1}{2\pi i} \oint_{\text{Tube}(\gamma_i)} \frac{\mathrm{d}x\mathrm{d}y\mathrm{d}z}{f|_{w=1}}$$

Periods determine the Néron-Severi group

The Néron-Severi group of X (a smooth quartic surface) is the sublattice of $H_2(X, \mathbb{Z})$ generated by the classes of algebraic curves on X.

Theorem (Lefschetz, 1924)

$$\mathrm{NS}(X) = \left\{ \gamma \in H_2(X, \mathbb{Z}) \,\middle|\, \int_\gamma \omega_X = 0 \right\}$$

In coordinates, $\mathrm{NS}(X) \simeq \left\{ \mathbf{u} \in \mathbb{Z}^{22} \,\middle|\, u_1\alpha_1 + \cdots + u_{22}\alpha_{22} = 0 \right\}$.
This is the lattice of *integer relations between the periods*.

The NS group determine the possible degree and genus of all the algebraic curves lying on X.

The Fermat hypersurface

Let $f = w^4 + x^4 + y^4 + z^4$.
The vector of periods is

$$\big(1 \quad i \quad i \quad i \quad i \quad -1 \quad -1 \quad -1 \quad -1 \quad -1 \quad -1 \quad -1 \quad -1 \quad -1 \quad -1 \quad -i \quad -i \quad -i \quad -i \quad -i \quad -i$$

$$\operatorname{rank} \mathrm{NS}(X_f) = 22 - \dim \mathrm{Vect}_{\mathbb{Q}} \{\text{periods}\} = 20.$$

Indeed there are 48 lines on X_f spanning a sublattice of $H_2(X, \mathbb{Z})$ of rank 20.

The outer method for computing periods (Sertöz, 2019)

Let $f \in \mathbb{C}[w,x,y,z]_4$
and let $f_t = (1-t)f + t(w^4 + x^4 + y^4 + z^4) \in \mathbb{C}(t)[w,x,y,z]_4$.

The outer method for computing periods (Sertöz, 2019)

Let $f \in \mathbb{C}[w,x,y,z]_4$
and let $f_t = (1-t)f + t(w^4 + x^4 + y^4 + z^4) \in \mathbb{C}(t)[w,x,y,z]_4$.

1. The periods of X_t satisfy a Picard–Fuchs linear differential equation (Picard, 1902).

The outer method for computing periods (Sertöz, 2019)

Let $f \in \mathbb{C}[w,x,y,z]_4$
and let $f_t = (1-t)f + t(w^4+x^4+y^4+z^4) \in \mathbb{C}(t)[w,x,y,z]_4$.

1. The periods of X_t satisfy a Picard–Fuchs linear differential equation (Picard, 1902).
2. The initial conditions are (generalized) periods of the Fermat quartic, studied by Pham (1965).

The outer method for computing periods (Sertöz, 2019)

Let $f \in \mathbb{C}[w, x, y, z]_4$
and let $f_t = (1-t)f + t(w^4 + x^4 + y^4 + z^4) \in \mathbb{C}(t)[w, x, y, z]_4$.

1. The periods of X_t satisfy a Picard–Fuchs linear differential equation (Picard, 1902).
2. The initial conditions are (generalized) periods of the Fermat quartic, studied by Pham (1965).
3. Numerical analytic continuation provides quasilinear-time algorithms for computing the periods.

The outer method for computing periods (Sertöz, 2019)

Let $f \in \mathbb{C}[w, x, y, z]_4$
and let $f_t = (1 - t)f + t(w^4 + x^4 + y^4 + z^4) \in \mathbb{C}(t)[w, x, y, z]_4$.

1. The periods of X_t satisfy a Picard–Fuchs linear differential equation (Picard, 1902).
2. The initial conditions are (generalized) periods of the Fermat quartic, studied by Pham (1965).
3. Numerical analytic continuation provides quasilinear-time algorithms for computing the periods.

The outer method for computing periods (Sertöz, 2019)

Let $f \in \mathbb{C}[w,x,y,z]_4$
and let $f_t = (1-t)f + t(w^4+x^4+y^4+z^4) \in \mathbb{C}(t)[w,x,y,z]_4$.

1. The periods of X_t satisfy a Picard–Fuchs linear differential equation (Picard, 1902).
2. The initial conditions are (generalized) periods of the Fermat quartic, studied by Pham (1965).
3. Numerical analytic continuation provides quasilinear-time algorithms for computing the periods.

⚠ Afflicted by the size of the PF equation (generically order 21 and degree ≥ 1000), the algorithm does not always terminate in reasonnable time.

Computation of the lattice of integer relations

We have the periods $\alpha_1, \ldots, \alpha_{22}$ with high precision (hundreds of digits); we want a basis of

$$\Lambda = \left\{ \mathbf{u} \in \mathbb{Z}^{22} \,\middle|\, u_1\alpha_1 + \cdots + u_{22}\alpha_{22} = 0 \right\}.$$

Computation of the lattice of integer relations

We have the periods $\alpha_1, \ldots, \alpha_{22}$ with high precision (hundreds of digits); we want a basis of

$$\Lambda = \left\{ \mathbf{u} \in \mathbb{Z}^{22} \,\middle|\, u_1\alpha_1 + \cdots + u_{22}\alpha_{22} = 0 \right\}.$$

It is an application of the Lenstra–Lenstra–Lovász algorithm:

1. For $1 \leq i \leq 22$, compute the Gaussian integer $[10^{1000}\alpha_i]$.

Computation of the lattice of integer relations

We have the periods $\alpha_1, \ldots, \alpha_{22}$ with high precision (hundreds of digits); we want a basis of

$$\Lambda = \left\{ \mathbf{u} \in \mathbb{Z}^{22} \,\middle|\, u_1\alpha_1 + \cdots + u_{22}\alpha_{22} = 0 \right\}.$$

It is an application of the Lenstra–Lenstra–Lovász algorithm:

1. For $1 \le i \le 22$, compute the Gaussian integer $[10^{1000}\alpha_i]$.
2. Let $L = \left\{ (\mathbf{u}, x, y) \in \mathbb{Z}^{22+2} \,\middle|\, \sum_i u_i[10^{1000}\alpha_i] = x + y\sqrt{-1} \right\}$,
   this is a rank 22 lattice. Short vectors are expected to come from integer relations between the periods.

Computation of the lattice of integer relations

We have the periods $\alpha_1, \dots, \alpha_{22}$ with high precision (hundreds of digits); we want a basis of

$$\Lambda = \left\{ \mathbf{u} \in \mathbb{Z}^{22} \,\middle|\, u_1\alpha_1 + \cdots + u_{22}\alpha_{22} = 0 \right\}.$$

It is an application of the Lenstra–Lenstra–Lovász algorithm:

1. For $1 \le i \le 22$, compute the Gaussian integer $[10^{1000}\alpha_i]$.
2. Let $L = \left\{ (\mathbf{u}, x, y) \in \mathbb{Z}^{22+2} \,\middle|\, \sum_i u_i[10^{1000}\alpha_i] = x + y\sqrt{-1} \right\}$,
   this is a rank 22 lattice. Short vectors are expected to come from integer relations between the periods.
3. Compute a LLL-reduced basis of L

Computation of the lattice of integer relations

We have the periods $\alpha_1, \ldots, \alpha_{22}$ with high precision (hundreds of digits); we want a basis of

$$\Lambda = \left\{ \mathbf{u} \in \mathbb{Z}^{22} \;\middle|\; u_1\alpha_1 + \cdots + u_{22}\alpha_{22} = 0 \right\}.$$

It is an application of the Lenstra–Lenstra–Lovász algorithm:

1. For $1 \leq i \leq 22$, compute the Gaussian integer $[10^{1000}\alpha_i]$.
2. Let $L = \left\{ (\mathbf{u}, x, y) \in \mathbb{Z}^{22+2} \;\middle|\; \sum_i u_i [10^{1000}\alpha_i] = x + y\sqrt{-1} \right\}$,
   this is a rank 22 lattice. Short vectors are expected to come from integer relations between the periods.
3. Compute a LLL-reduced basis of L
4. Output the *short* vectors

What is a short vector?

Let $f = 3x^3z - 2x^2y^2 + xz^3 - 8y^4 - 8w^4$.
With 100 digits of precision on the periods, here is a LLL-reduced basis of the lattice L (last 5 columns omitted).

0	0	0	0	0	0	0	0	0	0	0	0	0	0	−1669083212117905913652734	0	1937019641160560221317687	..
0	0	0	0	0	0	0	0	0	0	0	0	0	0		1669083212117905913652734	1937019641160560221317687	..
1	0	0	−1	0	0	0	1	1	0	0	0	0	0	−1465118299011954436717 89	844784290445878224678 23	−365980228690630104919296	..
0	0	0	0	1	0	0	0	0	0	0	0	0	0	−337167720252678310258177	224110151973403946221421	−743116955936487279910552	..
0	0	0	0	0	0	0	0	0	0	0	0	1	−1	357031479253522311483650	768066337666351099432748	940525994719391079998435	..
0	0	0	0	0	1	0	0	1	0	1	0	0	0	−552756671828854153114905	−126018248279583585486071	535095811953165917210863	..
0	−1	1	0	0	0	0	0	1	0	0	−1	0	0	104335431129908645825133	−231616284585318363570849	502730408585962411025306	..
0	0	0	0	0	0	0	0	0	0	0	0	0	−1	−649159586430203173692632	770784867967071100945665	−215201446973799931553127 2	..
0	0	0	0	0	0	0	0	0	1	1	0	0	0	277747983934797690835205	−286257398730613729663 84	−638732179408358479990097	..
1	0	0	0	0	0	0	0	0	0	0	1	0	0	146511829901195443671790	−844784290445878224678 23	365980228690630104919296	..
0	0	0	0	0	0	0	0	0	0	0	−1	1	1	250899146775406645936761	575615030011256031395007	−114830012426104078247291	..
0	1	0	0	0	0	0	1	0	0	−1	0	0	0	104335431129908645825133	−231616284585318363570849	502730408585962411025307	..
0	0	0	0	0	0	−1	0	0	0	0	0	1	−1	−140644950443454586919439	−393058206212350140614235	429933080833930208291557	..
0	0	0	0	0	0	0	0	1	0	0	0	0	0	594933070600140950961561	273156103820314126589096	−671845991848498223316874	..
0	0	0	0	1	0	0	−1	0	0	0	0	0	0	337167720252678310258177	−224110151973403946221421	743116955936487279910552	..
0	0	0	0	0	0	0	0	0	0	0	0	0	1	−824317154838996681984621	177119763197465887754938	−236792300924643740702432	..
0	0	0	0	0	0	0	1	0	0	1	0	0	0	379344119023965108104833	−769722964326734051183 95	606366776041154973804541	..
0	0	0	0	0	1	0	0	0	0	0	0	0	0	552756671828854153114905	126018248279583585486070	−535095811953165917210864	..
0	0	0	0	0	0	1	0	0	0	0	0	0	−1	−140644950443454586919440	−393058206212350140614234	429933080833930208291557	..
0	0	1	0	0	0	0	0	0	0	0	0	0	0	−104335431129908645825133	231616284585318363570849	−502730408585962411025307	..
0	0	0	0	0	0	0	0	0	0	0	0	1	0	−467285675585474370500971	−950623161465256990213520	−125562906312721721004270 2	..
0	0	0	1	0	0	0	0	0	0	0	0	0	0	−146511829901195443671790	844784290445878224678 23	−365980228690630104919296	..
0	0	0	0	0	0	0	0	0	1	0	−1	0	0	−277747983934797690835206	286257398730613729663 84	638732179408358479990097	..
0	0	0	0	0	0	0	0	0	0	0	1	0	0	−69025235930677842745100	457102914343586863258366	660652346877586707848817	..

A triple alternative

- ϟ Certified error bounds!
- ∗ assume that the periods are known $\pm\beta^{-1}$ (Remember that typically $\beta = 10^{1000}$).

Lemma

If the heuristic algorithm succeeds then one of the following holds:

A triple alternative

- ⚡ Certified error bounds!
- ∗ assume that the periods are known $\pm\beta^{-1}$
  (Remember that typically $\beta = 10^{1000}$).

Lemma

If the heuristic algorithm succeeds then one of the following holds:

1. The lattice computed is correct.

A triple alternative

- ϟ Certified error bounds!
- ∗ assume that the periods are known $\pm\beta^{-1}$
  (Remember that typically $\beta = 10^{1000}$).

Lemma

If the heuristic algorithm succeeds then one of the following holds:

1. The lattice computed is correct.
2. The NS group is not generated by curves of degree $\sim \beta^{O(1)}$.

A triple alternative

- ϟ Certified error bounds!
- ∗ assume that the periods are known $\pm\beta^{-1}$ (Remember that typically $\beta = 10^{1000}$).

Lemma

If the heuristic algorithm succeeds then one of the following holds:

1. The lattice computed is correct.
2. The NS group is not generated by curves of degree $\sim \beta^{O(1)}$.
3. There is a rare numerical coincidence.

A triple alternative

- ϟ Certified error bounds!
- ∗ assume that the periods are known $\pm\beta^{-1}$ (Remember that typically $\beta = 10^{1000}$).

Lemma

If the heuristic algorithm succeeds then one of the following holds:

1. The lattice computed is correct.
2. The NS group is not generated by curves of degree $\sim \beta^{O(1)}$.
3. There is a rare numerical coincidence.

A triple alternative

- ϟ Certified error bounds!
- ∗ assume that the periods are known $\pm\beta^{-1}$
  (Remember that typically $\beta = 10^{1000}$).

Lemma

If the heuristic algorithm succeeds then one of the following holds:

1. The lattice computed is correct.
2. The NS group is not generated by curves of degree $\sim \beta^{O(1)}$.
3. There is a rare numerical coincidence.

I do not know how to deal with 2, there are quartic surfaces with NS group minimaly generated by arbitrary large elements (Mori, 1984). (But we don't know how to do it over $\mathbb{Q}$.)

But we can do something about 3.

Separation of periods

Let $f \in \mathbb{Q}[w, x, y, z]_4$
and let $\alpha_1, \ldots, \alpha_{22}$ be the periods.

Theorem (Lairez & Sertöz, 2022)

There exist a computable constant $c > 0$ depending only on f and the choice of the homology basis, such that for any $\mathbf{u} \in \mathbb{Z}^{22}$,

$$|u_1\alpha_1 + \cdots + u_{22}\alpha_{22}| < 2^{-c^{\max_i |u_i|^9}} \Rightarrow u_1\alpha_1 + \cdots + u_{22}\alpha_{22} = 0.$$

Separation of periods

Let $f \in \mathbb{Q}[w,x,y,z]_4$
and let $\alpha_1, \dots, \alpha_{22}$ be the periods.

Theorem (Lairez & Sertöz, 2022)

There exist a computable constant $c > 0$ depending only on f and the choice of the homology basis, such that for any $\mathbf{u} \in \mathbb{Z}^{22}$,

$$|u_1\alpha_1 + \cdots + u_{22}\alpha_{22}| < 2^{-c^{\max_i |u_i|^9}} \Rightarrow u_1\alpha_1 + \cdots + u_{22}\alpha_{22} = 0.$$

✖ Not useful in practice...

An inner method for computing periods?

- Sertöz' algorithm is very indirect.
- Can we directly compute

$$\alpha_i = \oint_{\gamma_i} \omega_X?$$

An inner method for computing periods?

* Sertöz' algorithm is very indirect.
* Can we directly compute

$$\alpha_i = \oint_{\gamma_i} \omega_X?$$

* That's a *double* integral.

An inner method for computing periods?

- Sertöz’ algorithm is very indirect.
- Can we directly compute

$$\alpha_i = \oint_{\gamma_i} \omega_X?$$

- That’s a *double* integral.
- How do we get γ_i?
  How do we compute a basis of the singular homology group $H_2(X)$?

Double integrals *via* Fubini

- $f \in \mathbb{C}[w, x, y, z]_4$ (generic coordinates)
- $X \triangleq V(f) \subseteq \mathbb{P}^3(\mathbb{C})$
- $X_t \triangleq X \cap \left\{\frac{w}{x} = t\right\}$ (hyperplane section)
- Consider the surface as a family of curves

Double integrals *via* Fubini

* $f \in \mathbb{C}[w,x,y,z]_4$ (generic coordinates)
* $X \triangleq V(f) \subseteq \mathbb{P}^3(\mathbb{C})$
* $X_t \triangleq X \cap \left\{\frac{w}{x} = t\right\}$ (hyperplane section)
* Consider the surface as a family of curves

Main idea

$$\int_\gamma \omega_X = \oint_{\text{loop in } \mathbb{C}} \mathrm{d}t \underbrace{\oint_{\text{cycle in } X_t} \frac{\omega_X}{\mathrm{d}t}}\,.$$

ϟ satisfies a Picard–Fuchs equation!

Double integrals *via* Fubini

- $f \in \mathbb{C}[w,x,y,z]_4$ (generic coordinates)
- $X \triangleq V(f) \subseteq \mathbb{P}^3(\mathbb{C})$
- $X_t \triangleq X \cap \left\{\frac{w}{x} = t\right\}$ (hyperplane section)
- Consider the surface as a family of curves

Main idea

$$\int_\gamma \omega_X = \oint_{\text{loop in } \mathbb{C}} \mathrm{d}t \underbrace{\oint_{\text{cycle in } X_t} \frac{\omega_X}{\mathrm{d}t}}_{\text{satisfies a Picard–Fuchs equation!}} .$$

satisfies a Picard–Fuchs equation!

Requires a concrete description of γ to be implemented.
We need to *compute* $H_2(X, \mathbb{Z})$

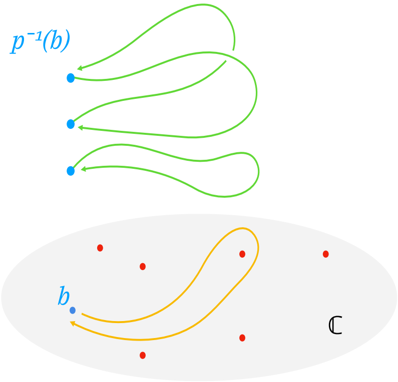
The homology of curves (Tretkoff & Tretkoff, 1984)

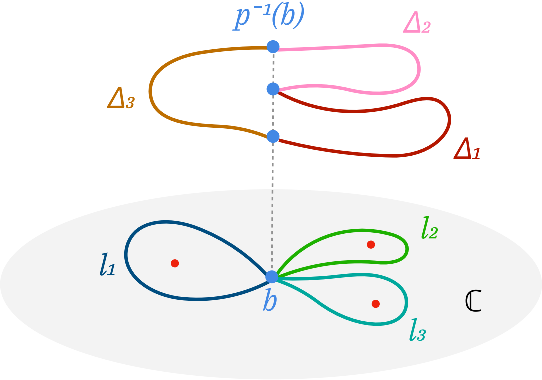
- X a complex algebraic curve
- $p : X \longrightarrow \mathbb{P}^1(\mathbb{C})$ nonconstant map
- $\Sigma \triangleq \{\text{critical values}\}$

- Given a loop in $\mathbb{P}^1(\mathbb{C}) \setminus \Sigma$, starting from a base point b, and a point in the fiber $p^{-1}(b)$, the loop lifts in X uniquely.
- Compute loops in $\mathbb{P}^1(\mathbb{C})$ that lift in a basis of $H_1(X, \mathbb{Z})$

(Deconinck & van Hoeij, 2001, see also)

Principle of the method

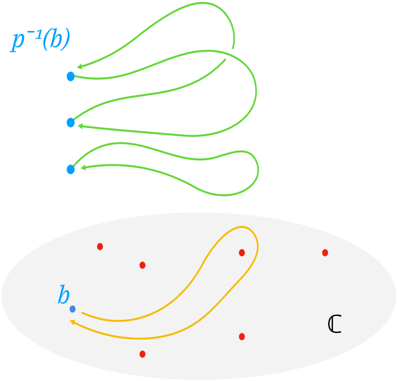


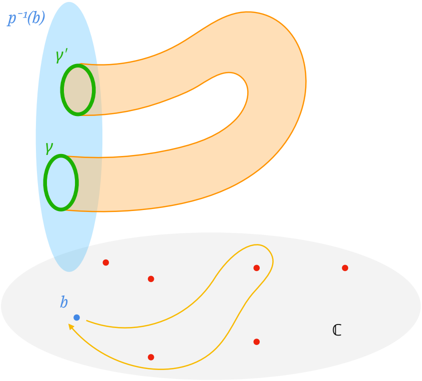


1. compute pieces of paths in X by lifting loops
2. connect them to form loops

Homology of surfaces

	dimension 1	dimension 2
monodromy action	permute the fiber	linear action on $H_1(X)$
lift in X	path	*hosepipe*
computable with	path tracking	numerical ODE solving





Homology of surface from the monodromy

* X a complex algebraic curve
* $p : X \longrightarrow \mathbb{P}^1(\mathbb{C})$ nonconstant map, define $X_t = p^{-1}(t)$
* $\Sigma \triangleq \{\text{critical values}\}$

* Given a loop γ in $\mathbb{P}^1 \setminus \Sigma$ starting from a base point b, and a cycle $c \in H_1(X_b)$, the cycle deforms as t runs along γ.
* This defines the monodromy action $\gamma_* : H_1(X_b) \longrightarrow H_1(X_b)$.
* Compute the monodromy action of generators or $\pi_1(\mathbb{P}^1 \setminus \Sigma)$ to construct elements of $H_2(X)$.

(Lefschetz, 1924; Lamotke, 1981; Lairez, Pichon-Pharabod, & Vanhove, 2024; Pichon-Pharabod, 2024)

Monodromy computation in higher dimension

De Rham duality

The monodromy action on $H_1(X_t)$ is dual to the monodromy action on the solutions of the Picard–Fuchs equation of the periods of X_t.

- We can connect hosepipes by integrating a Picard–Fuchs differential equation.

Monodromy computation in higher dimension

De Rham duality

The monodromy action on $H_1(X_t)$ is dual to the monodromy action on the solutions of the Picard–Fuchs equation of the periods of X_t.

- We can connect hosepipes by integrating a Picard–Fuchs differential equation.

Concretely,

- let α_t be a cycle varying continuously in X_t,
- and let $\phi(t) = \int_{\alpha_t} \frac{\omega_X}{\mathrm{d}t}$ in a neighborhood of b.

$\phi(t)$ is solution of a Picard–Fuchs differential equation, we can compute its analytic continuation along γ.

Monodromy computation in higher dimension

De Rham duality

The monodromy action on $H_1(X_t)$ is dual to the monodromy action on the solutions of the Picard–Fuchs equation of the periods of X_t.

- ϟ We can connect hosepipes by integrating a Picard–Fuchs differential equation.

Concretely,

- $*$ let α_t be a cycle varying continuously in X_t,
- $*$ and let $\phi(t) = \int_{\alpha_t} \frac{\omega_X}{\mathrm{d}t}$ in a neighborhood of b.

$\phi(t)$ is solution of a Picard–Fuchs differential equation, we can compute its analytic continuation along γ.We obtain another solution $\tilde{\phi}(t)$ of the Picard–Fuchs equations. We check that

$$\tilde{\phi}(t) = \int_{\gamma_* \alpha_t} \frac{\omega_X}{\mathrm{d}t}.$$

High precision quadrature

uncovers non trivial curves

on quartic surfaces.

heuristic algorithm, only provides a safe bet...

* ... but there is no alternative at the moment.
* Based on results by Sertöz (2019), Lairez and Sertöz (2019), Lairez, Pichon-Pharabod, and Vanhove (2024).

Faster symbolic integration over domains with boundaries

(joint work in progress with Hadrien Brochet and Frédéric Chyzak)

Question

How to compute differential equations for integrals in the form

$$\phi(t) = \int_{X_t} F_t(x_1, \ldots, x_n) \mathrm{d}x_1 \cdots \mathrm{d}x_n,$$

where X_t is a semialgebraic set that depends on t, and F_t an algebraic function?

- These are *holonomic* integrals (Oaku, 2013).
- Many applications (e.g. Feynman integrals) but current integration algorithms are too slow.

Numerical computation of homology spaces

(joint work in progress with Alexandre Guillemot and Eric Pichon-Pharabod)

Question

Given a smooth complex projective variety X, how to compute *directly* the homology spaces $H_\bullet(X, \mathbb{Z})$?
(Sufficiently explicitely to be able the integration pairing.)

- We gave an overview of an algorithm relying on the action of the monodromy on the homology of a hyperplane section.
- We computed the monodromy action using transcendental methods.
- But homology (and its monodromy action) is purely topological, it is not a “fine invariant”!
- Can we compute it directly with classical numerical algebraic geometry?

Applications in experimental mathematics

- Investigate the geometry of Feynman integrals.
  Complicated varieties may hide simpler building blocks. How to uncover them? Period computation is one of the tools (Doran, Harder, Pichon-Pharabod, & Vanhove, 2023).

- Numerical checks of Deligne's conjecture in higher dimension.
  Generalizing the Birch and Swinnerton-Dyer conjecture, Deligne conjectures a relation between values of some L-functions and periods. (work in progress by Nutsa Gegelia, Eric Pichon-Pharabod and Duco van Straten)

Thank you!

References I

Bostan, A., Chyzak, F., Lairez, P., & Salvy, B. (2018).Generalized Hermite reduction, creative telescoping and definite integration of D-finite functions. *Proc. ISSAC 2018*, 95–102. https://doi.org/10/ddv8

Chen, S., van Hoeij, M., Kauers, M., & Koutschan, C. (2018).Reduction-based creative telescoping for Fuchsian D-finite functions. *J. Symb. Comput.*, *85*, 108–127. https://doi.org/10/ggck9k

Chudnovsky, D. V., & Chudnovsky, G. V. (1990). Computer algebra in the service of mathematical physics and number theory. In *Computers in mathematics (Stanford, CA, 1986)* (pp. 109–232, Vol. 125). Dekker.

Chyzak, F. (2000).An extension of Zeilberger’s fast algorithm to general holonomic functions. *Discrete Math.*, *217*(1-3), 115–134. https://doi.org/10/drkkn6

References II

Costa, E., Mascot, N., Sijsling, J., & Voight, J. (2019).Rigorous computation of the endomorphism ring of a Jacobian. *Math. Comput.*, *88*(317), 1303–1339. https://doi.org/10/ggck8g

Cremona, J. E., & Sutherland, A. V. (2023). *Computing the endomorphism ring of an elliptic curve over a number field*. arXiv: 2301.11169. https://doi.org/10.48550/arXiv.2301.11169

Deconinck, B., & van Hoeij, M. (2001).Computing Riemann matrices of algebraic curves. *Phys. Nonlinear Phenom.*, *152–153*, 28–46. https://doi.org/10/c95vnb

Doran, C. F., Harder, A., Pichon-Pharabod, E., & Vanhove, P. (2023). *Motivic geometry of two-loop Feynman integrals*. arXiv: 2302.14840.

Euler, L. (1733).Specimen de constructione aequationum differentialium sine indeterminatarum separatione. *Comment. Acad. Sci. Petropolitanae*, *6*, 168–174.

References III

Koutschan, C. (2010).A fast approach to creative telescoping. *Math. Comput. Sci.*, *4*(2-3), 259–266. https://doi.org/10/bhb6sv

Lairez, P. (2016).Computing periods of rational integrals. *Math. Comput.*, *85*(300), 1719–1752. https://doi.org/10/ggck95

Lairez, P., Pichon-Pharabod, E., & Vanhove, P. (2024).Effective homology and periods of complex projective hypersurfaces. *Math. Comp.* https://doi.org/10.1090/mcom/3947

Lairez, P., & Sertöz, E. C. (2019).A numerical transcendental method in algebraic geometry: Computation of Picard groups and related invariants. *SIAM J. Appl. Algebra Geom.*, *3*(4), 559–584. https://doi.org/10/ggck6n

Lairez, P., & Sertöz, E. C. (2022).Separation of periods of quartic surfaces. *Algebra Number Theory*
To appear.

Lamotke, K. (1981).The topology of complex projective varieties after S. Lefschetz. *Topology*, *20*(1), 15–51. https://doi.org/10/dw8m2q

References IV

Lefschetz, S. (1924). *L'analysis situs et la géométrie algébrique*. Gauthier-Villars.

Liouville, J. (1834).Sur les Transcendantes Elliptiques de première et de seconde espèce, considérées comme fonctions de leur amplitude. *J. LÉcole Polytech.*, *14*(23), 73–84.

Mezzarobba, M. (2010).NumGFun: A package for numerical and analytic computation with D-finite functions. *Proc. ISSAC 2010*, 139–146. https://doi.org/10/cg7w72

Molin, P., & Neurohr, C. (2019).Computing period matrices and the Abel-Jacobi map of superelliptic curves. *Math. Comp.*, *88*(316), 847–888. https://doi.org/10/ggck8t

Mori, S. (1984).On degrees and genera of curves on smooth quartic surfaces in $\mathbb{P}^3$. *Nagoya Math. J.*, *96*, 127–132. https://doi.org/10/grk9rj

Neurohr, C. (2018). *Efficient integration on Riemann surfaces and applications*. http://oops.uni-oldenburg.de/3607/1/neueff18.pdf

References V

Oaku, T. (2013).Algorithms for integrals of holonomic functions over domains defined by polynomial inequalities. *J. Symb. Comput.*, *50*, 1–27. https://doi.org/10/f4hpj4

Oaku, T., & Takayama, N. (2001).Algorithms for D-modules — restriction, tensor product, localization, and local cohomology groups. *J. Pure Appl. Algebra*, *156*(2), 267–308. https://doi.org/10/bct97n

Pham, F. (1965).Formules de Picard-Lefschetz généralisées et ramification des intégrales. *B. Soc. Math. Fr.*, *79*, 333–367. https://doi.org/10/ggck9f

Picard, É. (1902).Sur les périodes des intégrales doubles et sur une classe d’équations différentielles linéaires. *Comptes Rendus Hebd. Séances Académie Sci.*, *134*, 69–71. http://gallica.bnf.fr/ark:/12148/bpt6k3085b/f539.image

References VI

Pichon-Pharabod, E. (2024).A semi-numerical algorithm for the homology lattice and periods of complex elliptic surfaces over $\mathbb{P}^1$. *Journal of Symbolic Computation*, 102357. https://doi.org/10.1016/j.jsc.2024.102357

Sertöz, E. C. (2019).Computing periods of hypersurfaces. *Math. Comput.*, *88*(320), 2987–3022. https://doi.org/10/ggck7t

Terasoma, T. (1985).Complete intersections with middle Picard number 1 defined over $\mathbb{Q}$. *Math. Z.*, *189*(2), 289–296. https://doi.org/10/bhf8gv

Tretkoff, C. L., & Tretkoff, M. D. (1984). Combinatorial group theory, Riemann surfaces and differential equations. In *Contributions to group theory* (pp. 467–519, Vol. 33). AMS. https://doi.org/10.1090/conm/033/767125

van der Hoeven, J. (1999).Fast evaluation of holonomic functions. *Theoret. Comput. Sci.*, *210*(1), 199–215. https://doi.org/10/b95scc

References VII

van Luijk, R. (2007).K3 surfaces with Picard number one and infinitely many rational points. *Algebra Number Theory*, *1*(1), 1–15. https://doi.org/10/dx3cmr