Automated proofs of binomial identities

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The mother of all binomial sums

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$$\sum_{k=0}^{n} \binom{n}{k} 1^k 1^{n-k} = (1+1)^n$$

A binomial identities

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Proof. Let $[n] = \{1, \ldots, n\}$. There is a bijection

$$\{A, B \subseteq [n] \mid \#A + \#B = n\} \longrightarrow \{S \subseteq [2n] \mid \#S = n\}$$
$$(A, B) \longmapsto A \cup (B + n).$$

A slightly trickier one

$$\sum_{k=0}^{n} 2^{n-2k} \binom{n}{2k} \binom{2k}{k} = \binom{2n}{n}.$$

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A computer-aided proof (creative telescoping)

Let $u_n(k) = 2^{n-2k}\binom{n}{2k}\binom{2k}{k}$ and $v_n = \sum_{k=0}^{n} u_n(k)$.

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We look for $p_n \in \mathbb{Q}(n)$ and $R_n(k) \in \mathbb{Q}(n,k)$ such that

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Divide by $u_n(k)$ and we obtain

$$\frac{(n-2k)(n-2k-1)}{4(k+1)^2}R_n(k+1) - R_n(k) = \frac{2(n+1)}{n+1-2k} + p_n.$$

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There is a solution!

$$R_n(k) = -\frac{8k^2}{(n+1)(n+1-2k)} \text{ and } p_n = -\frac{2(2n+1)}{n+1}.$$

(Abramov, 1989; Gosper, 1978; Zeilberger, 1990b)

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Summing the relation

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The pole of R simplifies gracefully.

⚠ Major theoretical issue in general.

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$$\Longrightarrow \quad v_n = \binom{2n}{n}$$

More identities

$$\sum_{i,j} \binom{2n}{n+i}\binom{2n}{n+j}|i^3 - j^3| = \frac{2n^2(5n-2)}{4n-1}\binom{4n}{2n}$$

$$\sum_{i,j} \binom{2n}{n+i}\binom{2n}{n+j}|i^5 - j^5| = \frac{2n^2(43n^3-70n^2+36n-6)}{(4n-1)(4n-3)}\binom{4n}{2n}$$

$$\sum_{i,j} \binom{2n}{n+i}\binom{2n}{n+j}|i^7 - j^7| = \frac{2n^2(531n^5-1960n^4+2800n^3-1952n^2+668n-90)}{(4n-1)(4n-3)(4n-5)}\binom{4n}{2n}$$

$$\sum_{i,j} \binom{2n}{n+i}\binom{2n}{n+j}|ij(i^2 - j^2)| = \frac{2n^3(n-1)}{2n-1}\binom{2n}{n}^2$$

$$\sum_{i,j} \binom{2n}{n+i}\binom{2n}{n+j}|i^3j^3(i^2 - j^2)| = \frac{2n^4(n-1)(3n^2-6n+2)}{(2n-3)(2n-1)}\binom{2n}{n}^2$$

Brent, R. P., Ohtsuka, H., Osborn, J.-a. H., & Prodinger, H. (2014). Some binomial sums involving absolute values.

A complicated one (Le Borgne)

$$1 + F_n^{-1,-1} + 2F_n^{0,0} - F_n^{0,1} + F_n^{1,0} - 3F_n^{1,1} + F_n^{1,2} - F_n^{3,1} + 3F_n^{3,2}$$
$$- F_n^{3,3} - 2F_n^{4,2} + F_n^{4,3} - F_n^{5,2} = \sum_{m=0}^{n} \frac{\binom{n+2}{m}\binom{n+2}{m+1}\binom{n+2}{m+2}}{\binom{n+2}{1}\binom{n+2}{2}},$$
$$\text{where } F_n^{a,b} = \sum_{d=0}^{n-1} \sum_{c=0}^{d-a} \binom{d-a-c}{c}\binom{n}{d-a-c}\left(\binom{n+d+1-2a-2c+2b}{n-a-c+b} - \binom{n+d+1-2a-2c+2b}{n+1-a-c+b}\right).$$

Automation is nice to have...

Motivation from computer science

[50] Develop computer programs for simplifying sums that involve binomial coefficients.

Exercise 1.2.6.63
The Art of Computer Programming
Knuth (1968)

Motivation from number theory

Let $\zeta(3) = \sum_{n\geq 1} \frac{1}{n^3}$.
Can you prove that $\zeta(3) \notin \mathbb{Q}$? (Apéry, 1979)

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For $n \geq 0$, let

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There is some integer sequence (b_n) such that $b_n - 2l_n a_n \zeta(3) \longrightarrow 0$.

It implies that $\zeta(3) \notin \mathbb{Q}$.

Apéry, R. (1979). Irrationalité de $\zeta(2)$ et $\zeta(3)$. *Astérisque*, *61*, 11–13
van der Poorten, A. (1978–0079). A proof that euler missed...: Apéry's proof of the irrationality of $\zeta(3)$, an informal report. *Math. Intell.*, *1*(4), 195–203. https://doi.org/10/bkt9vb

Desired algorithms for binomial sums

Simplification

input $\sum_{i=0}^{n} \sum_{j=0}^{n} \binom{i+j}{i}^2 \binom{4n-2i-2j}{2n-2i}$

output $(2n+1)\binom{2n}{n}^2$

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Deciding equality

input $\sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 = \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} \sum_{j=0}^{k} \binom{k}{j}^3$

output *true*

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Computation of a recurrence relation

input $\sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$

output $n^3 u_n - (34n^3 - 51n^2 + 27n - 5) u_{n-1} - (n-1)^3 u_{n-2} = 0$

1. Introduction

2. The algebra of binomial sums

3. Coefficients of rational functions

4. Computing residues

The algebra of binomial sums

The formal grammar of binomial sums

$\bigcirc \longrightarrow$ integer linear combination of variables

$\square \longrightarrow \binom{\bigcirc}{\bigcirc}$ $\square \longrightarrow \mathrm{Cst}^{\bigcirc}$

$\square \longrightarrow \square + \square$ $\square \longrightarrow \square \cdot \square$

$$\square \longrightarrow \sum_{n=\bigcirc}^{\bigcirc} \square$$

The algebra of binomial sums

Let $\mathbb{S}$ be the algebra of functions $\mathbb{Z}^{(\mathbb{N})} \to \mathbb{C}$.
The algebra of binomial sums, denoted $\mathcal{B}$, is the smallest subalgebra of $\mathcal{S}$ such that

(a) The Kronecker delta sequence $n, \ldots \mapsto \delta_n$,
defined by $\delta_0 = 1$ and $\delta_n = 0$ if $n \neq 0$, is in $\mathcal{B}$.

(b) The geometric sequences $n, \ldots \mapsto C^n$, for all $C \in \mathbb{C} \setminus \{0\}$, are in $\mathcal{B}$.

(c) The binomial sequence $n, k, \ldots \mapsto \binom{n}{k}$ is in $\mathcal{B}$.

(d) If $\lambda : \mathbb{Z}^d \to \mathbb{Z}^e$ is an affine map and if $u \in \mathcal{B}$,
then $n_1, n_2, \ldots \mapsto u_{\lambda(n_1,\ldots,n_d),0,\ldots}$ is in $\mathcal{B}$.

(e) If $u \in \mathcal{B}$, then the following directed indefinite sum is in $\mathcal{B}$:

$$n_1, \ldots, n_d, m, \ldots \mapsto \sum_{k=0}^{m} u_{n_1,\ldots,n_d,k}.$$

Main result

Theorem

Let u be a binomial sum. Then $(u_n)_{n\in\mathbb{Z}}$ is P-recursive.
In other words, there are polynomials $p_0, \ldots, p_r$, not all zero, such that

$$p_0(n)u_n + p_1(n)u_{n+1} + \cdots + p_r(n)u_{n+r} = 0.$$

Moreover, this result is *effective*: there is an algorithm to compute a recurrence relation as above.

Corollary

Equality of binomial sums is decidable.

Zeilberger, D. (1990a). A holonomic systems approach to special functions identities. *J. Comput. Appl. Math.*, *32*(3), 321–368. https://doi.org/10/ctbwnk

Bostan, A., Lairez, P., & Salvy, B. (2017). Multiple binomial sums. *J. Symb. Comput.*, *80*, 351–386. https://doi.org/10/ggck6p

Deciding equality for P-recursive sequences

input Two sequences (u_n) and (v_n) defined by linear recurrence relations with polynomial coefficients and initial conditions

output *true* if and only if $u_n = v_n$ for all $n \in \mathbb{Z}$.

1. Compute a common recurrence relation R.
2. Compute u_n and v_n for all n such that R does not impose the value at n.
3. Check that $u_n = v_n$ for all these *critical* indices.

Laurent series

For a field K, let $K(\!(x)\!) \doteq \cup_{N\geq 0} x^{-N} K[\![x]\!]$, the field of *Laurent series over* K.
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We will work in the field of *iterated Laurent series*

$$\mathbb{C}(\!(x_1, \ldots, x_r)\!) \doteq \mathbb{C}(\!(x_r)\!)(\!(x_{n-1})\!) \cdots (\!(x_1)\!).$$

It means: expand first with respect to x_1, then x_2, etc.

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For a monomial $\mathbf{x}^\alpha = x_1^{\alpha_1} \cdots x_r^{\alpha_r}$ and $R \in \mathbb{C}((x_1, \ldots, x_r))$ we denote $[\mathbf{x}^\alpha] R$ the coefficient of $x_r^{\alpha_r}$ in the coefficient of $x_{r-1}^{\alpha_{r-1}}$ of [...] the coefficient of $x_1^{\alpha_1}$ in R.

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Coefficient of a monomial

$\mathbb{C}(x_1,\ldots,x_r) \subset \mathbb{C}((x_1,\ldots,x_r))$, so we now know what is the coefficient of a monomial in a rational function!

Exercise

What is the coefficient of 1 in $\frac{x_1}{x_1+x_2}$?

$$[1]\frac{x_1}{x_1 + x_2} = [1]\left(\frac{1}{x_2}\frac{x_1}{1 + \frac{x_1}{x_2}}\right) = [1]\left(\frac{x_1}{x_2} - \frac{x_1^2}{x_2^2} + \cdots\right) = 0$$

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What is the coefficient of 1 in $\frac{x_2}{x_1+x_2}$?

$$[1]\frac{x_2}{x_1+x_2} = [1]\left(\frac{1}{1+\frac{x_1}{x_2}}\right) = [1]\left(1 - \frac{x_1}{x_2} + \frac{x_1^2}{x_2^2} + \cdots\right) = 1$$

An intermediary representation

Lemma

Every binomial sum is a linear combination of sequences of the form $n_1, n_2, \dots \mapsto [1] R_0 R_1^{n_1} \cdots R_d^{n_d}$, *for some* $R_0, \dots, R_d \in \mathbb{C}(x_1, \dots, x_r, \dots)$.

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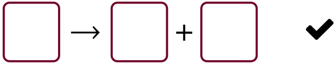

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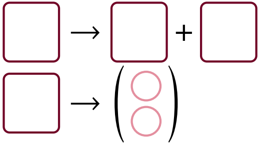

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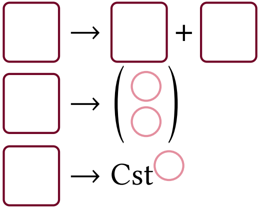

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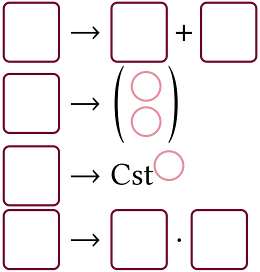

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$$([1] R_0 R_1^n)([1] S_0 S_1^n) = [1] R_0 S_0|_{x \leftarrow y} \left(R_1 S_1|_{x \leftarrow y} \right)^n$$

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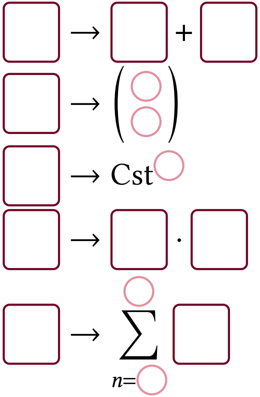

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$([1]R_0R_1^n)([1]S_0S_1^n) = [1]R_0S_0|_{x\leftarrow y}\left(R_1S_1|_{x\leftarrow y}\right)^n$

$\sum_{k=a}^{b}[1]R_0R_1^n = [1]R_0\frac{R_1^a - R_1^{b+1}}{1-R_1}$

Residues

For $R \in \mathbb{C}((t, x_1, \ldots, x_r))$, let $\operatorname{res}_{x_1,\ldots,x_r} R \doteq \sum_{k\in\mathbb{Z}} \left([x_1^{-1} \cdots x_r^{-1} t^k] R \right) t^k$.

Proposition

For any binomial sum $(u_n)_{n\geq 0}$, there is a rational function $R \in \mathbb{C}(t, x_1, \ldots, x_r)$ such that

$$\sum_{n\geq 0} u_n t^n = \operatorname{res}_{x_1,\ldots,x_r}(R).$$

Proof. We may assume that $u_n = [1]RS^n$ for some rational functions R and S. Then

$$\sum_{n\geq 0} u_n t^n = \sum_n ([1]RS^n) t^n = \sum_n \left[\mathbf{x}^{-1} t^n\right] \left(\mathbf{x}^{-1} R(tS)^n\right) t^n$$
$$= \sum_n \left[\mathbf{x}^{-1} t^n\right] \left(\frac{\mathbf{x}^{-1} R}{1 - tS}\right) t^n = \operatorname{res}_{\mathbf{x}} \left(\frac{\mathbf{x}^{-1} R}{1 - tS}\right)$$

Characterisation of binomial sums

Theorem (Bostan, Lairez, & Salvy, 2017)

Let $(u_n)_{n\geq 0}$ be a sequence and let $f(t) = \sum_n u_n t^n$ be its generating function. The following are equivalent:

1. (u_n) is a binomial sum;
2. $f(t) = \operatorname{res}_{x_1,\dots,x_r} R$, for some $R \in \mathbb{C}(t, x_1, \dots, x_r)$;

Example.

$$\sum_{n\geq 0} \left(\sum_{k=0}^{2n} (-1)^k \binom{2n}{k}^3 \right) t^n = \operatorname{res}_{x_1,x_2} \frac{(1-x_2)(1-x_1)x_1x_2}{x_1^2 x_2^2 (1-x_2)^2 (1-x_1)^2 - (1-x_1-x_2)^2 t}$$

The only thing we need to know about residues

Residues of derivatives

For any $A_1, \dots, A_n \in \mathbb{C}(t, x_1, \dots, x_n)$,

$$\operatorname{res}_{x_1,\dots,x_n}\left(\frac{\partial A_1}{\partial x_1} + \cdots + \frac{\partial A_n}{\partial x_n}\right) = 0.$$

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Corollary (creative telescoping)

Let $R \in \mathbb{C}(t, x_1, \dots, x_n)$ and $f(t) = \operatorname{res}_{x_1,\dots,x_n} R$.
Let $p_0, \dots, p_r \in \mathbb{C}(t)$ and $A_1, \dots, A_n \in \mathbb{C}(t, x_1, \dots, x_n)$.

$$\sum_{k=0}^{r} p_k(t)\frac{\partial^k R}{\partial t^k} = \sum_{i=1}^{n}\frac{\partial A_i}{\partial x_i} \Rightarrow \sum_{k=0}^{r} p_k(t) f^{(k)}(t) = 0.$$

Residues of rational functions are D-finite

Theorem (Grothendieck, 1966)

Let K be a characteristic-zero field (for example $K = \mathbb{C}(t)$).
Let $P \in \mathbb{K}[x_1, \ldots, x_r]$ and let $\mathcal{O}_P = \mathbb{K}[x_1, \ldots, x_r, P^{-1}]$.
Then the quotient space

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Let $R \in \mathcal{O}_P$ and $f(t) = \operatorname{res}_{\mathbf{x}}(R)$.
The derivatives $\frac{\partial^k R}{\partial t^k}$ form an infinite family in $\mathcal{O}_P$ with residues $f(t), f'(t)$, etc., so there is a linear dependency relation

$$p_r(t) f^{(r)}(t) + \cdots + p_1(t) f'(t) + p_0(t) f(t) = 0.$$

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